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Stochastic Analysis of a Discrete-time queuing model for a working Vacations

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Abstract- In this paper, we analysis the discrete time queuing model with working vacation. during the vacation period, server completely stopping and taken the original work at the lower rate .we have obtained the closed property of conditional probability for negative binomial distribution. finally two special model model are presented.

Keywords**:** *Discrete-time, Working vacations, Matrix-geometric approach, Closed property of conditional probability , Waiting time distribution*

I. INTRODUCTION

During the last two decades, the queuing systems with server vacations have been well investigated. In the models with various vacation policies, the server completely stops service in the vacation period, but he can take the assistant work. Research work of the vacation queues has been extensively used in computer networks, communication systems and production management et al. we consider a discrete-time system with *working vacations*. This vacation policy is more general than most classical vacation policies in the sense that the well known multiple vacation policy and single vacation policy become two extreme cases of this policy. With this vacation model, we can study many different vacation policies between these two extremes for the purpose of better allocation of the server's time to doing primary jobs (serving queue) and doing secondary jobs (vacations). Furthermore, we use this model to discuss the optimal control issue of vacation policies under a given cost structure.

In this paper, we will extend the queue with working vacations in to the discrete-time queue with working vacations (WV). In discrete-time epochs, the customer is served at the lower rate during the vacation period. Such model with working vacations has some certain implications in practice: (1) Servi and Finn [14] illustrated that the model with working vacations can be used to analyze the performance of a WDM optical access network. But, in the digital communication systems, the messages are always divided into units and their arrivals and departures occur at certain fixed time epochs. Therefore,. Meanwhile, in the cyclic service queue model which is always used to reconfigure the communication network, we also can apply the working vacation policy to model; (2) The essence of the working vacation policy is that, when the number of customers is less relatively, a "lower speed period" is established to economize the operational cost in the system. The analysis of the model with working vacations can provide the theory and analysis method to design the optimal lower speed period`

This paper analyzes GI/Geo/1 queue with multiple working vacation policy. The model was previously analyzed by Baba [10] for infinite buffer queue considering multiple working vacation policy. For the sake of notational convenience the model is denoted by, where MWV stands for 'multiple working vacation policy'. One final comment on the model is that by equating working vacation parameter (g) equal to zero one can get the results for queue with multiple vacations. The paper organizes as follows

II. MODEL DESCRIPTION AND ITS STRUCTURAL MATRIX

Assume that customer arrivals can only occur at discrete time instants $t = n-$, $n = 0, 1 \ldots$ The inter-arrival times $\{Tn, n\geq 1\}$ are independent each other and has a general discrete distribution:

$$P\{Tn = j\} = gj, \quad j\geq 1;$$

$E(Tn) = \lambda - 1$; $a(z) = zjgj$.

The service starting and service ending can only occur at discrete time instants $t = n +$, $n = 1, 2, \ldots$. The service times $\{Sn, n \geq 1\}$ are independent each other and follow a discrete geometric distribution

$$P\{Sn = j\} = \mu(1 - \mu)j-1 \quad , j \geq 1; \qquad E(Sn) = \mu - 1,\ 0 < \mu < 1.$$

Define that the arrival and the service starting occur at instants $n -$ and $n +$, respectively in order to make the system state description more clear. The inter-arrival times and service times are mutually independent and the service discipline is FCFS. The model is often called a "late-arrival" system and is denoted by *GI/Geo/*1 with vacations. The classical *GI/Geo/*1 queue without vacations has been studied by Hunter [4]. We consider an exhaustive service and multiple vacation policy which requires the server to take a vacation if and only if the system is empty at a service completion instant or at a vacation completion instant. The vacation time is an independent and identically distributed (i.i.d.) random variable and is denoted by *V*. In this analysis, we assume that *V* follows a geometric distribution with parameter θ.

$$P\{V = j\} = \theta(1 - \theta)j-1, \quad j \geq 1; \qquad E(V) = \theta - 1,\ 0 < \theta < 1.$$

To be precise, the vacation can only start or end at discrete time instants $t = n +$, $n = 1, 2 \ldots$ Let Ln be the number of customers in the system at nth arrival instant $t = n -$. Define

$$Jn = \begin{cases} 0, & \text{the } n\text{th arrival occurs in a server's vacation period} \\ 1, & \text{the } n\text{th arrival occurs in a server's busy period.} \end{cases}$$

Then $\{(Ln, Jn), n \geq 1\}$ is a Markov chain with state space

$$\Omega=\{(0,0)\} \cup \{(k,j), k \geq 1, j = 0,1\}.$$

According to the late arrival system definition, the nth arrival occurs at time instant $t = n -$ and it is possible that a service completion or a vacation completion occurs at time instant $t = n +$. However, Ln represents the number of customers just before the nth arrival so any change occuring at time instant $t = n +$ will not affect Ln. Note that if we discuss the waiting time of this nth arriving customer then the state change at time instant $t = n +$ will certainly affect the his or her waiting time (see section 4). For any real number $0 < x < 1$, we denote $.x = 1 - x$. To express the transition probability, we use the following symbols:

$$a_j = \sum_{k=j}^{\infty} gk \binom{k}{j} \mu^{j-k-j} \mu, \qquad j \geq 1,$$

$$v_j = \sum_{k=j+1}^{\infty} gk \ \sum_{i=0}^{k-j-1} \vartheta^{-1} \vartheta \binom{k-i-1}{j} \mu^{j-k-j-1-i}, \qquad j \geq 1$$

aj is the probability that the server completes j customer services in a inter-arrival time period and its probability generating function (p.g.f.) or z-transform is $a(1 - \mu(1 - z))$ and its mean equals $\lambda - 1\mu = \rho - 1$, where ρ is the traffic intensity. vj is the probability that the server completes j customer services in a residual inter-arrival time given that a residual vacation period has elasped. Note that

$$\sum_{j=0}^{\infty} v_j = \sum_{k=j+1}^{\infty} gk \ \sum_{i=0}^{k-j-1} \vartheta^{-1} \vartheta \binom{k-i-1}{j} \mu^{j-k-j-1-i}$$

$$= \sum_{k=1}^{\infty} gk \ \sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} \binom{k-i-1}{j} \mu^{j-k-j-1-i}$$

$$= \sum_{k=1}^{\infty} g_k (1-\vartheta^k)$$

$$= 1-a(\vartheta).$$

Hence, $\{vj, j \geq 0\}$ is not a complete probability distribution. The Markov chain process $\{(Ln, Jn), n \_ 1\}$ has the transition probability denoted by

P(h,i)(k,j) = P{Ln+1 = k, Jn+1 = j | Ln = h , Jn = i}.

Given that the server is busy, the transition probability of this Markov chain is the same as that of a classical *GI/Geo/*1 system. It is as follows:

P (i, 1) (j, 1) = ai+1−j, i≥1, 1 ≥ j ≤ i + 1.

Transition from *(i,* 0*)* to *(i* +1, 0*)* means that the inter-arrival time period is smaller than
a residual vacation period. We have

$$P\ (i,1)(j,1,0)\ = \sum_{k=1}^{\infty} g_k\text{-}\vartheta^k = 1\text{-a}(\vartheta), \quad i \geq 1$$

The transition from *(i,* 0*)* to *(j,* 1*)* represents that the process changes from the state in which a customer arrives during the vacation period and there are *i* customers are waiting to the state that the next arrival occurs after $i + 1 - j$ customer services are completed from the ending of the vacation. Hence, we have

P(i,0)(j,1) = vi+1−j, i≥ 0, 1 ≤ j ≤ i + 1.

Similarly, we can obtain

$$P_{(I,1)(0,0)} = 1\text{-}\sum_{k=1}^{i} a_{k,} \qquad i \geq 1,$$

$$P_{(I,1)(0,0)} = 1\text{-a}(\vartheta) - \sum_{k=1}^{i} v_{k,} \qquad i \geq 0,$$

Using the lexigraphical sequence to order the states as (0, 0), {(k, 1), (k, 0)}, k ≥ 1, the transition probability matrix of the Markov process {(Ln, Jn), n ≥ 1} can be written as

$$P=\begin{pmatrix} B_{00} & A_{01} & 0 & 0 & 0 \\ B_1 & A_1 & A_0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{1}$$

where B00 = 1−a(θ)−v0, A01 = (v0, a(θ)), all Ak’s, k ≥0, are 2×2 square matrices; and all Bk’s, k ≥1, are 2 × 1 column vectors as follows:

$$A_0=\begin{pmatrix} a_0 & 0 \\ v_0 & a(\theta) \end{pmatrix}, \qquad A_k=\begin{pmatrix} a_k & 0 \\ v_k & 0 \end{pmatrix}, \qquad k \geq 1.$$

$$B_k= \begin{pmatrix} 1-\sum_{j=0}^{k} a_j \\ 1-a(\hat{\theta})-\sum_{j=0}^{k} v_j \end{pmatrix}, \qquad k \geq 1.$$

The probability transition matrix in (1) is a *GI/M/*1 type matrix (see [7]). To analyze the *GI/M/*1 type system, the minimal non-negative solution *R* of the matrix equation below is called the rate matrix,

$$R= \sum_{k=0}^{\infty} R^k\ A_k\ . \tag{2}$$

To find the explicit expression of *R*, we let

$$\beta = \frac{\theta}{\mu(1-a(\theta))-\theta}\,.$$

Based on the classical GI/Geo/1 queue analysis (see [4]), we know that the equation z = a(1−μ(1−z)) has a unique solution z = ξ in (0, 1) and the stationary distribution of the queue length at the arrival instant is

P{L = j} = (1 – ξ)ξj, j ≥ 0. (3)

Waiting time distribution

Conditional probability for negative binomial distributions To analyze the steady-state waiting time, we firstly demonstrate the closed property of conditional probability for negative binomial distributions.

Assume that *X* follows the negative binomial distribution with parameters *r* and *p*, and *V* follows the geometric distribution with the parameter *θ,* i.e.,

$$P=\{X=m\}\binom{m-1}{n-1}p^r\,(1-p)^{m-n}\,,\qquad m\geq r;$$

$$P\{v=k\}=\theta(1-\theta)^k,\qquad k\geq 1;$$

We have two lemmas to present the closed property of conditional probability for negative binomial distributions

Lemma 4.1 If X and V are mutually independent, under the condition X ≤ V, X follows the negative binomial distribution with parameters r and θ +P (1 −θ).

Proof First, we compute the conditional probability

$$P\{X\leq V\}=\sum_{m=r}^{\infty}P\{X=M\}\,P\{V\geq m\}$$

$$=\sum_{m=r}^{\infty}\binom{m-1}{r-1}p^r(1-p)^{m-r}(1-\theta)^m,\qquad (k=m-r+1)$$

$$(\therefore m=k+r-1)$$

$$=\sum_{k=1}^{\infty}\binom{k+r-1-1}{r-1}\quad p^r(1-p)^{k+r-1-r}(1-\theta)^{k+r-1}$$

$$=\sum_{k=1}^{\infty}\binom{k+r-1-1}{r-1}\quad p^r(1-p)^{k-1}(1-\theta)^r$$

$$=[P(1-\theta)]^r\quad\sum_{k=1}^{\infty}\binom{k+r-1-1}{r-1}\ [(1-p)(1-\theta)]^{k-1}$$

$$=\left[\frac{p(1-\theta)}{\theta+p(1-\theta)}\right]^r$$

Thus,

$$P\{X=k/X\leq v\}=\frac{P\{X=k,X\geq V\}}{P\{X\leq V\}}$$

$$=\left[\frac{\theta+p(1-\theta)}{p(1-\theta)}\right]^r\binom{k-1}{r-1}\quad p^r(1-p)^{k-r}(1-\theta)^k$$

$$=\binom{k-1}{r-1}\quad\theta+P(1-\theta)^r(1-\theta)^k\,\{1-[\theta+p(1-\theta)]\}^{k-r}$$

Thus, we can find that under the condition X ≤ V, X follows negative binomial distribution with parameters r and θ + p (1−θ).

So, we obtain the PGF of X under the condition X ≤ V,

$$\psi_r(z)=\left(\frac{z[\theta+p(1-\theta)]}{1-z(1-[\theta+p(1-\theta)])}\right)^r.$$

We present that, under the condition $X \leq V$, X also follows the negative binomial distribution and we call it the closed property of the conditional probability of negative binomial distributions.

Meanwhile, assuming Y follows the geometric distribution with the parameter p, i.e. ,

P{Y = j} = p(1 −p)j−1, j≥ 1,

we can verify the closed property of another conditional probability.

Lemma 4.2 If X and V are mutually independent, under the condition X ≤V <X+ Y , V can be composed into the sum of two independent random variables X*,Y*: X* follows the geometric distribution with the parameter θ + p(1−θ); Y * follows the negative binomial distribution with parameters r and θ +p(1 −θ).

Proof Similarly, we have

P{$X \leq V < X+Y$}

$=\sum_{k=r}^{\infty}\binom{k-1}{r-1} p^r(1-p)^{k-r}\sum_{j=k}^{\infty}\theta(1-\theta)^j\,(1-p)^{j-k}$

$=\sum_{k=r}^{\infty}\binom{k-1}{r-1} p^r(1-p)^{k-r}(1-\theta)^k\sum_{j=k}^{\infty}\theta(1-\theta)^{j-k}\,(1-p)^{j-k}$

$=\frac{\theta}{\theta+p(1-\theta)}\left[\frac{p(1-\theta)}{\theta+p(1-\theta)}\right]^r$

And,

P{$V = k | X \leq V < X+Y$}

= P{$V = k, X \leq V < X+Y$}/P{$X \leq V < X+Y$}

$=[\theta\ +p(1\ -\theta)]^{r+1}\sum_{j=r}^{k}\binom{j-1}{r-1}(1-p)^{j-r}(1-\theta)^{k-r}(1-p)^{k-j}$

For the above equation, we obtain the generating function

$$\phi_r(z) = [\theta\ +p(1\ -\theta)]^{r+1}\sum_{k=r}^{\infty} z^k\sum_{j=r}^{\infty}(1-p)^{j-r}(1-\theta)^{k-r}(1-p)^{k-j}$$

$$=[\theta\ +p(1\ -\theta)]^{r+1}\sum_{j=r}^{\infty}(1-p)^{j-r}(1-p)^{k-j}\sum_{k=j}^{\infty}\theta(1-\theta)^{j-k}\,(1-p)^{j-k}$$

$=\left[\frac{\theta+p(1-\theta)}{1-[1-(\theta+p(1-\theta))]z}\right]\frac{[\theta+p(1-\theta)]z}{1-[1-(\theta+p(1-\theta))]z}$

Thus, we get the result and such two lemmas demonstrate the closed property of conditional probability for negative binomial distributions

Waiting time distribution

Let W and $W^*(z)$ be the steady-state waiting time and it PGF, respectively. we can obtain the distribution of W. The service discipline is first in first out (FIFO). Meanwhile, denote $H1$ be the probability that the server is in the service period when the new customer arrives, and $H0$ be the probability that the server is in the vacation period and the new customer should wait. We can easily compute

$H1 = \sum_{k=1}^{\infty} \pi_k$

$= \frac{\beta(\gamma-\xi)+w0\xi(1-\gamma)}{(1-\xi)(1-\eta\delta(1-\gamma))+\beta(\gamma-\xi)+w0\xi(1-\gamma)}$

$H0 = \sum_{k=1}^{\infty} \pi_k$

$= \frac{(1-\xi)\gamma}{(1-\xi)(1-\eta\delta(1-\gamma))+\beta(\gamma-\xi)+w0\xi(1-\gamma)}$

Theorem 4 If $\rho < 1$ and $\theta > 0, \eta \leq \mu$, the PGF of stationary waiting time W is

$= 1 - H0 - H1 + H0 \frac{[1-(1-\theta)(\eta\gamma+1-\eta)]z}{1-(1-\theta)(\eta\gamma+1-\eta)z} \{1-q+q\frac{\mu(1-\gamma)}{1-[1-\mu(1-\gamma)]z}\} + H1 \frac{[1-(1-\theta)(\eta\gamma+1-\eta)]z}{1-(1-\theta)(\eta\gamma+1-\eta)z} +$

$p\frac{\mu(1-\gamma)}{1-[1-\mu(1-\gamma)]z} + 1\text{-}p\frac{\mu z}{1-[1-\mu z}\}$

Where

$P = \frac{\beta(\gamma-\xi)}{\beta(\gamma-\xi)+w0\xi(1-\gamma)}$;

$Q = \frac{\theta}{1-(1-\theta)(\eta\gamma+1-\eta)} = \frac{\theta}{\theta-\eta(1-\theta)(1-\gamma)}$

Proof Firstly, we easily obtain the probability that a new customer should not wait.

$P\{W=0\} = \pi 00 = \sigma(1-\xi)(1-\eta\delta) = 1-H0-H1$. (9)
When a new customer arrives at the instant $t = mv-$ (LAS) or $t = m+$ (EAS), if there are k customers and the server is in the busy period, the waiting time equals k services by the rate μ.

Then, we easily have

$= \sum_{k=1}^{\infty} \pi_k \; W^*{}_{k1}(z)$

$= \sigma(1-\xi) \sum_{k=1}^{\infty} [\beta(\gamma\, k - \xi\, k) + w0\xi\, k] \left(\frac{\mu z}{1-(1-\mu)z}\right)$

$= \sigma(1-\xi)\beta(\gamma-\xi)\frac{1-(1-\mu)z}{1-[1-\mu(1-\xi)]z}\left(\frac{\mu z}{1-(1-\mu)z}\right) + w0\xi\frac{\mu z}{1-[1-\mu(1-\xi)]z}$

$= \frac{(1-\xi)[1-(1-\mu)z]}{1-[1-\mu(1-\xi)]z}\; \sigma\left|\beta(\gamma-\xi)\frac{\mu z}{1-[1-\mu(1-\gamma])z}\right|$

$= H1\,\frac{(1-\xi)[1-(1-\mu)z]}{1-[1-\mu(1-\xi)]z}\left|p\frac{\mu(1-\gamma)z}{1-[1-\mu(1-\gamma)]z} + (1-p)\frac{\mu z}{1-(1-\mu)z}\right|$

Denote $S(j)v$ the sum of j service times Sv with the rate η, i.e., j -dimensional convolution of Sv, and evidently it follows negative binomial distribution with parameters j and η. If there are k customers and the server is in the vacation period when the new customer arrives, there are two cases. Case 1: if there are j customer service completions when the vacation ends, i.e., $S(j)v \leq V < S(j+1)v$, $0 \leq j \leq k-1$, the waiting time is the sum of the vacation time under the above condition and $k-j$ service times by the rate μ; case 2: if at least k customers are served when the vacation ends, i.e., $V \geq Sk\,v$, the waiting time is k service times with the rate η under the above condition. From Lemmas 4.1 and 4.2, the PGF of the waiting time is

$W_{k0}^{*}(z)=p\{V\geq S_V^K\}\ \Psi_k(z) + \sum_{j=0}^{k-1} p\left\{s_v^{(j+1)}\right\}\phi_j(z)\left[\frac{\mu z}{1-(1-\mu)z}\right]^{k-j}$

Then

$$=\sum_{k=1}^{\infty}\pi_{k0}\ W^{*}{}_{k0}(z)=\ \sigma(1-\xi)\sum_{k=1}^{\infty}\gamma^k\left\{\left[\frac{\eta(1-\theta)z}{1-(1-\theta)(1-\eta)z}\right]^k+\sum_{j=0}^{k-1}\frac{\theta}{1-(1-\theta)(1-\eta)z}\left[\frac{\eta(1-\theta)z}{1-(1-\theta)(1-\eta)z}\right]^k\left[\frac{\mu z}{1-(1-\mu)z}\right]\right\}$$

Thus ,we easily

$W^{*}(z)=\pi 00+\sum_{k=1}^{\infty}\pi_k\ W^{*}{}_{k1}(z)+\sum_{k=1}^{\infty}\pi_{k0}\ W^{*}{}_{k0}(z)$

we can easily get the expected waiting time

$$E(W)=H1\left[\frac{\xi}{\mu(1-\xi)}+p\frac{1}{\mu(1-\gamma)}+(1-p)\frac{1}{\mu}\right]+H0\frac{1}{\theta+\eta(1-\theta)(1-\gamma)}\left[1+\theta\frac{1-\mu(1-\gamma)}{\mu(1-\gamma)}\right]$$

we consider the waiting time, it is possible that a vacation ends at the arrival instant, so we assume that the vacation time can be 0. Meanwhile, the waiting time of an arbitrary customer has the special probability explanation. The waiting time equals 0 with the probability $1-H0-H1$; with the probability $H1$, it equals the sum of one geometric random variable with the rate $\mu(1-j)$ and one modified geometric random variable with the rate $\mu(1-\xi)$; with the probability $H0$, it equals the sum of one geometric random variable with the rate $[1-(1-\theta)(\eta\gamma+1-\eta)]$ and one random variable which is the mixture of two geometric random variables with parameter $\mu(1-\gamma)$ and μ, respectively.

For the waiting time, we can easily verify that there is no complete stochastic decomposition property, but under some conditions, we can obtain the conditional stochastic decomposition structures.

First, denoting $W1$ the conditional waiting time when the server is in the busy period, i.e., $J = 1$, we obtain.
Thus taking different values of θ or η, we can obtain the results of the special G1/Geo/1 queue with working vocations under EAS and LAS schemes.

III. CONCLUSION

In this paper, we have study the results of the G1/Geo/1 queue with working vacations under the EAS and LAS schemes. Several G1/Geo/1 models studied before there are some special examples of the model we consider here. Similarly important is that we also find the closed property of conditional probability for negative bionomial distributions. This result makes the computation of the distribution for the waiting time easy and the expression concise and specific.

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