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Zero-divisor graphs of finite direct products of finite Semirings

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Abstract - In this paper we establish a connection between graph theory and Semiring theory. To relate the graph theory and ring theory, we define the zero divisor graph of Semiring. The main objective of this paper is to determine a formula to find the number of edges of the zero-divisor graph of a direct product of Semirings. Then by using the formula we prove some results.

Key words - Zero-divisor, Graph, Semiring, Direct product, Card (n).

I. INTRODUCTION

A Semiring is a set S with binary operations + and $\cdot$ such that $(S, +)$ is monoid with identity element 0 and $(S, \cdot)$ is a monoid with identity element 1. In addition, operations + and $\cdot$ are connected by distributivity and 0 annihilates S. A Semiring is commutative if $ab = ba$ for all $a, b \in S$. The Semiring S is additively cancellative if $a + c = b + c$ implies that $a = b$ for all $a, b, c \in S$.

For any Semiring S, we denote by $Z^*(S)$ the set of non-zero zero–divisors, $Z^*(S) = \{x \in S$; there exists $0 \neq y \in S$ such that $xy = 0$ or $yx = 0\}$. The zero-divisor graph of S denoted by $\Gamma(S)$, is a undirected graph whose vertices are labeled by the elements of $Z^*(S)$. Let $x, y \in \Gamma(S)$, there is an edge from x to y if and only if $xy = 0$. Here we say that x and y are adjacent to each other. By the definition of graph theory the vertex set $V(\Gamma(S))$ of $\Gamma(S)$ is the set of elements in $Z^*(S)$ and an unordered pair of vertices $x, y \in V(\Gamma(S))$, $x \neq y$ and is an edge $x - y$ in $\Gamma(S)$ if $xy = 0$ or $yx = 0$. For general background of graph theory, we can see Chartrand, Lesniak, and Chang [1].

The zero–divisor graphs of commutative rings have been first introduced by Beck in [2] in the study of graph coloring. Anderson and Naseer [3] continued working with Beck's definition. David F. Anderson and Philip S. Livingston [4] proposed different method associating to commutative ring and later studied by various authors. The graph of Semiring have been first introduced by Y.F. Lin and J.S. Ratti [5]. Dolzan et.al has studied zero-divisor graphs of Semirings as well as those of rings [6].

In this paper, we determine a formula for the number of edges of the zero-divisor graph of a direct product of Semirings $S_1 \times \dots \times S_t$, given the zero-divisor graphs of each S_i. This problem was solved for finite commutative rings without nonzero nilpotent elements by Lagrange [7]. Redmond uses a technique to count the same [8]. L M. Birch **et.al** [9] found the zero divisor graphs of finite direct product of finite rings. Ryan L. Miller and others [10] proved the same for non-commutative rings and semigroups in zero divisor graphs of finite direct products of finite non-commutative rings and semigroups. We apply the formula in this paper to describe completely the zero-divisor graph of any direct product of Z_m's.

The results of above paper are holding true for Semirings. Here every element in a Semiring is either a zero-divisor or not. For any set X, let |X| denote the cardinality of X. Let U denote the set of non zero-divisors of S and Z^* denote the set of non zero zero-divisors of S. Then $|S| - 1 = |U| + |Z^*|$. We will use this fact without explicit mention when needed.

The paper has three sections. In section 2 we describe the zero-divisor graph for an arbitrary direct product of Semirings. In section 3 we describe completely the zero-divisor graph for Z_p^k, where p is a prime number. We indicate how the formulas in section 2 and 3 can be used to describe completely the zero-divisor graph of any finite direct product of Z_m's.

II. THE ZERO-DIVISOR GRAPH OF A DIRECT PRODUCT OF SEMIRINGS

In this section, we determine a formula for the number of edges in the zero-divisor graph of a direct product $S_1 \times \ldots. \times S_t$ of Semirings, given complete information about each $\Gamma(S_i)$ and each S_i. We develop a recursive formula for an arbitrary direct product and then we derive a non-recursive version of this formula. To develop a formula we prove the following lemma.

Lemma 2.1. Let S be a Semiring and let x be the number of nilpotent elements of index 2 in S. Then the number of neighbor in $\Gamma(S)$ is $2|E| - x$.

Proof: A vertex r in the zero-divisor graph S has a loop if and only if $r^2 = 0$. Hence, if we want to count each loop once and each non-loop twice, we obtain the result $2|E| - x$.

Lemma 2.2. Let $S = S_1 \times \ldots. \times S_t$ be a Semiring, let x_i be the number of non-zero nilpotent elements of index 2 in each S_i, and let $x_{1,2,\ldots,t}$ denote the number of non-zero nilpotent elements of index 2 in S. Then $x_{1,2,\ldots,t} = -1 + \prod_{i=1}^{t}(x_i + 1)$.

Proof: An element $r = r_1, r_2, \ldots., r_i \epsilon S$ is nilpotent index less than or equal to 2 if and only if each r_i is nilpotent of index less than or equal to 2. If we count 0, then there are $x_i + 1$ possible entries for each position in r. We subtract 1 to avoid counting the zero element of S.

Corollary 2.3. Let $S = S_1 \times \ldots. \times S_t$ then the number of neighbors in $\Gamma(S)$ is $n_{1,2,\ldots,t} = 2|E| - x_{1,2,\ldots.,t}$.

Let $S = S_1 \times S_2$. Let E be the set of edges of $\Gamma(S)$. For $i = 1, 2$, let Z_i^* be the set of non-zero zero-divisors of S_i, let E_i denote the set of edges in $\Gamma(S_i)$, and let U_i be the set of non zero-divisors of S_i except 0.

In order to count the number of edges in $\Gamma(S_1 \times \ldots. \times S_t)$, we first count the number of edges in $\Gamma(S_1 \times S_2)$, and then we extend this result to $\Gamma(S_1 \times \ldots. \times S_t)$ by induction. Since any Semiring consists of non zero-divisors and zero-divisors, the set of non-zero elements of $S_1 \times S_2$ is $U(A_1, A_2)$, where $A_i = Z_i^*$ or $A_i = U_i$ or $A_i = \{0\}$ for $i = 1, 2$ and either $A_1 = \{0\}$ or $A_2 = \{0\}$. To count the number of edges in $\Gamma(S_1 \times S_2)$. We construct the graph in figure 1.

The numbers on the edges are labels. The vertices of this graph are the sets. Let $A_1 = \{Z_1^*, U_1, \{0\}\}$ and $A_2 = \{Z_2^*, U_2, \{0\}\}$, then the vertices are the set $(A_1, A_2) = \left\{\begin{matrix}(Z_1^*, Z_2^*), (Z_1^*, U_2), (Z_1^*, 0), (U_1, Z_1^*), (U_1, 0), \\ (0, Z_2^*), (0, U_2)\end{matrix}\right\}$. Since A_1 and A_2 not contain zero and (U_1, U_2) cannot be a zero divisor. Therefore there is no edge between these two elements. Hence $(0,0), (U_1, U_2) \notin (A_1, A_2) \subseteq S_1 \times S_2$. We draw an edge from (A_1, A_2) to $(A'_1, A'_2) \subseteq S_1 \times S_2$ precisely when there are elements $(0,0) \neq (a_1, a_2) \in (A_1, A_2)$ and $(0,0) \neq (a'_1, a'_2) \in (A'_1, A'_2)$ such that $(a_1, a_2)(a'_1, a'_2) = (0,0)$ which means each edge in Figure 1 between (A_1, A_2) and (A'_1, A'_2) represents the set of all edges in $\Gamma(S_1 \times S_2)$ between elements of (A_1, A_2) and (A'_1, A'_2).

If S_1 is a domain, then $Z_1^* = \emptyset$ and hence the vertices (Z_1^*, Z_2^*), $(Z_1^*, 0)$, (Z_1^*, U_2) do not appear in the graph. Likewise, if S_2 is a domain, then $Z_2^* = \emptyset$ and hence the vertices $(Z_1^*, Z_2^*), (U_1, Z_2^*), (0, Z_2^*)$ do not appear in the graph.

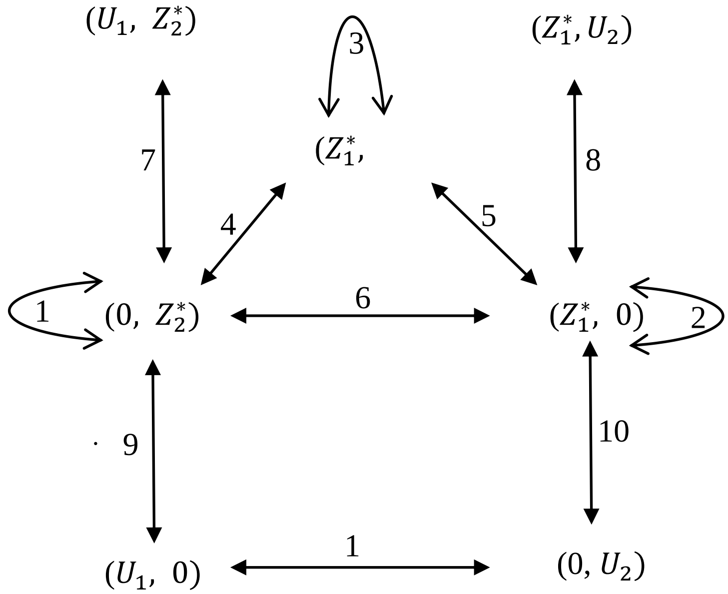


Figure 1. Sets of zero-divisors in $S_1 \times S_2$.

Lemma 2.4. For each edge labeled by n, $1 \leq n \leq 11$, in Figure 1, let Card (n) denote the number of edges in $\Gamma(S_1 \times S_2)$ represented by this edge. Then the values of Card (n) are given as follows:

n	1	2	3
Card (n)	$\lvert E_2\rvert$	$\lvert E_1\rvert$	$(\lvert E_1\rvert - x_1)(\lvert E_2\rvert - x_2)$ $+\lvert E_1\rvert\lvert E_2\rvert$

n	4	5	6	7
Card(n)	$n_2\lvert Z_1^*\rvert$	$n_1\lvert Z_2^*\rvert$	$\lvert Z_1^*\rvert\lvert Z_2^*\rvert$	$n_2\lvert U_1\rvert$

n	8	9	10	11
Card(n)	$n_1\lvert U_2\rvert$	$\lvert U_1\rvert\lvert Z_2^*\rvert$	$\lvert Z_1^*\rvert\lvert U_2\rvert$	$\lvert U_1\rvert\lvert U_2\rvert$

Proof:

Card (1): It is a cardinality of edge (1). It is a loop $(0, Z_2^*) \leftrightarrow (0, Z_2^*)$ and it is the number of edges in $\Gamma(S_2) = |E_2|$.

Card (2): It is a cardinality of edge (2). It is a loop $(Z_1^*, 0) \leftrightarrow (Z_1^*, 0)$ and it is the number of edges in $\Gamma(S_1)$ $=|E_1|$.

Card (3), It is a cardinality of edge (3). It is a loop $(Z_1^*, Z_2^*) \to (Z_1^*, Z_2^*)$, let $z_i, z_i' \in Z_i^*$ for i = 1, 2, where $z_i z_i' = 0$. These elements give rise to the edge$(z_1,\ z_2) \leftrightarrow (z_1',\ z_2')$.

The number of such edges is $(|E_1| - x_1)(|E_2| - x_2) + |E_1||E_2|$, where x_1, x_2 be the number of nilpotent elements of index 2 in *S*.

Card (4): It is a cardinality of edge (4). It is an edge between the two vertices $(0, Z_2^*) \to (Z_1^*, Z_2^*)$. Let $z_1 \in Z_1^*$ and let$z_2, z_2' \in Z_2^*$, where $z_2 z_2' = 0$. Then, there are distinct edges $(0,\ z_2') \leftrightarrow (z_1,\ z_2)$ and $(0,\ z_2) \leftrightarrow (z_1,\ z_2')$. Therefore, Card (4) $= n_2|Z_1^*|$.

Card (5): It is a cardinality of edge (5). It is an edge between the two vertices$(Z_1^*, Z_2^*) \to (Z_1^*, 0)$. Let $z_2 \in Z_2^*$ and let $z_1, z_1' \in Z_1^*$, where $z_1 z_1' = 0$. Then, there are distinct edges $(z_1,\ z_2) \leftrightarrow (z_1',\ 0)$ and$(z_1',\ z_2) \leftrightarrow (z_1,\ 0)$. Therefore, Card (5) $= n_1|Z_2^*|$.

Card (6): It is a cardinality of edge (6). It is an edge between the two vertices$(0, Z_2^*) \to (Z_1^*, 0)$. Let $z_1 \in Z_1^*$ and let $z_2 \in Z_2^*$, then there are edges $(0,\ z_2) \leftrightarrow (z_1,\ 0)$. Therefore Card (6) $=|Z_1^*||Z_2^*|$.

Card (7): It's a cardinality of edge (7). It is an edge between the two vertices$(0, Z_2^*) \to (U_1, Z_2^*)$. let $u_1 \in U_1$, $z_2, z_2' \in Z_2^*$, where $z_2\, z_2' = 0$. These elements give rise to the edges $(u_1,\ z_2) \leftrightarrow (0,\ z_2')$ and $(0,\ z_2) \to (u_1,\ z_2')$. The number of such edges is $n_2|U_1|$ where n_2 the number of neighbours in zero divisor graph of S2 is.

Card (8): It is a cardinality of edge (8). It is an edge between the two vertices$(Z_1^*, 0) \to (Z_1^*, U_2)$. let $z_1, z_1' \in Z_1^*$, $u_2 \in U_2$, where $z_1\, z_1' = 0$. These elements give rise to the edges $(z_1, 0) \leftrightarrow (z_1',\ u_2)$ and$(z_1,\ u_2) \leftrightarrow (z_1', 0)$. The number of such edges is $n_1|U_2|$. Where n_1 is the number of neighbours in zero divisor graph of *S1*.

Card (9): It's a cardinality of edge (9). It is an edge between the two vertices$(U_1, 0) \to (0, Z_2^*)$. let $u_1 \in U_1$, $z_2 \in Z_2^*$. These elements give rise to the edges$(u_1, 0) \to (0,\ z_2)$. The number of such edges is$|U_1||Z_2^*|$.

Card (10): It is a cardinality of edge (10). It is an edge between the two vertices$(0, U_2) \to (Z_1^*, 0)$. let $u_2 \in U_2$, $z_1 \in Z_1^*$. These elements give rise to the edges$(0, u_2) \to (z_1, 0)$. The number of such edges is $|Z_1^*||U_2|$.

Card (11): It is a cardinality of edge (11). It is an edge between the two vertices$(U_1, 0) \to (0, U_2)$. let $u_1 \in U_1, u_2 \in U_2$. These elements give rise to the edges$(u_1, 0) \to (0,\ u_2)$. The number of such edges is $|U_1||U_2|$.

Proposition 2.5. The number of edges in $\Gamma(S_1 \times S_2)$ is

$$|E| = (2|S_1| + |E_1| - 1)|E_2| +$$

$$(2|S_2|-1)|E_1| + 2(|S_1| - 1)(|S_2| - 1) \ldots. (1)$$

Proof: From lemma 2.4,

Add Card (1) + Card (2) + Card (3) we get

$$|E_1| + |E_2| + (|E_1| - x_1)$$

$$(|E_2| - x_2) + |E_1||E_2|$$

$$= |E_1| + |E_2| - x_2|E_1| + |E_1||E_2|$$

$$-x_1|E_2| + x_1x_2 + |E_1||E_2|$$

$$= |E_2| - x_1|E_2| + |E_1| - x_2|E_1|$$

$$+2|E_1||E_2| + x_1x_2$$

$$= |E_2|(1 - x_1) + |E_1|(1 - x_2 + 2|E_2|)$$

$$+x_1x_2$$

Now add Card (5) + Card (8)

$$n_1|Z_2^*| + n_1|U_2| = n_1(|Z_2^*| + |U_2|)$$

$$= n_1(|S_2| - 1)$$

$$= (|E_1| - x_1)\ (|S_2| - 1)$$

$$= |E_1||S_2| - 2|E_1| - x_1|S_2| + x_1$$

$$= |E_1|(|S_2| - 2) - x_1(|S_2| - 1)$$

Add Card (4) + Card (7) + Card (6) + Card (9) + Card (10) + Card (11), we get

$$2|Z_1^*||E_2| + |E_1||E_2| + 2|U_1||E_2|$$

$$= |E_2| + 2|E_2|(|Z_1^*|+|U_1|)$$

$$+ |E_1||E_2|$$

$$= |E_2| + 2(|S_1|-1)|E_2| + |E_1||E_2|$$

$$= (2|S_1| + |E_1| - 1)|E_2|$$

In a similar manner, from Lemma 2.4, add Card (4) + Card (5) + Card (7) to obtain$(2|S_2| - 1)|E_1|$. Finally, add the remaining terms in Lemma 2.4 to obtain$(2|S_1| - 1)(|S_2| - 1)$.

Example. The zero divisor graph of $Z_9 \times Z_6$ is $\Gamma(Z_9 \times Z_6)$

Let $A_1 = \{Z_1^*, U_1, \{0\}\}$ and $A_2 = \{Z_2^*, U_2, \{0\}\}$ and $Z_1 = Z_9$, $Z_2 = Z_6$

Here $Z_1^* = \{3,6\}$, $Z_2^* = \{2,3,4\}$,

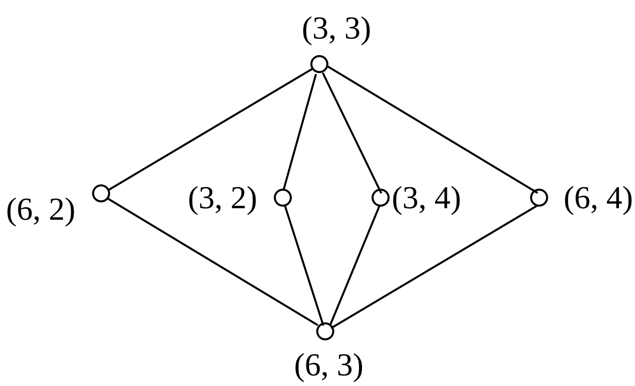
$U_1 = \{1,2,4,5,7,8\}$, $U_2 = \{1,5\}$

$x_1 = 2$ and $x_2 = 0$,

$n_1 =$ The number of neighbors in $\Gamma(Z_9)$

$n_2 =$ The number of neighbors in $\Gamma(Z_6)$

$$Z_9 \times Z_6 = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5),$$
$$(1,0), (1,1), (1,2), (1,3), (1,4), (1,5),$$
$$(2,0), (2,1), (2,2), (2,3), (2,4), (2,5),$$
$$(3,0), (3,1), (3,2), (3,3), (3,4), (3,5)$$
$$(4,0), (4,1), (4,2), (4,3), (4,4), (4,5),$$
$$(5,0), (5,1), (5,2), (5,3), (5,4), (5,5)$$
$$(6,0), (6,1), (6,2), (6,3), (6,4), (6,5),$$
$$(7,0), (7,1), (7,2), (7,3), (7,4), (7,5)$$
$$(8,0), (8,1), (8,2), (8,3), (8,4), (8,5)\}$$



The graph of $\Gamma(Z_9 \times Z_6)$

We now describe the general case. Let $S_{1,2,\dots,t} = S_1 \times S_2 \times \dots \dots \times S_t$ for some t and let $S = S_1 \times S_2 \times \dots \dots \times S_t \times S_{t+1}$. Let $E_{1,2,\dots,t}$ denote the edges of $\Gamma(S_{1,2,\dots,t})$ and let E denote the edges of $\Gamma(S_{1,2,\dots,t+1})$. Suppose that we know$|E_{1,2,\dots,t}|$, $|E_{t+1}|$ and $|S_i|$ for each$1 \le i \le t+1$.

Proposition 2.6. The number of edges in $\Gamma(S)$ is

$$|E| = \left(2\left|S_{1,\dots,t}\right| + \left|E_{1,\dots,t}\right| - 1\right)|E_{t+1}| + (2|S_{t+1}| - 1)|E_{1,\dots,t}| + 2(|S_{1,\dots,t}| - 1)(|S_{t+1}| - 1). \quad (2)$$

Proof: In the Equation (1) replace E_1 and S_1 by $E_{1,\dots,t}$

And $S_{1,\dots,t}$ respectively and replace E_2 and S_2 by E_{t+1} and S_{t+1} respectively.

We now present a non-recursive version of proposition 2.6.

Theorem 2.7. Let$c_i = 2|S_{t+1}| + |E_{t+1}| - 1$, and let $b_i = 2\left|S_{1,\dots,t}\right|(|S_{t+1}| + |E_{t+1}| - 1) - 2|S_{t+1}| - |E_{t+1}| + 2$. Then the Equation of Proposition 2.6 becomes

$$|E_{1,2,\dots,t+1}| = \left[\prod_{i=1}^{t} c_i\right] E_1 + b_t + \sum_{i=1}^{t-1}\left(b_i \prod_{j=i+1}^{t} c_j\right) \quad (3)$$

Proof: We know the equation (2)

$$|E| = \left(2\left|S_{1,\dots,t}\right| + \left|E_{1,\dots,t}\right| - 1\right)|E_{t+1}| + (2|S_{t+1}| - 1)|E_{1,\dots,t}| + 2(|S_{1,\dots,t}| - 1)(|S_{t+1}| - 1).$$

Write the above equation as $a_{i+1} = c_i a_i + b_i$, where $a_{i+1} = \left|E_{1,\dots,t+1}\right|$, $a_i = \left|E_{1,\dots,t}\right|$ and are given above.

III. CONCLUSION

We determine a formula for counting the number of edges of the zero-divisor graph of the direct product of finite Semirings.

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