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Dynamics of a Discrete Duffing Equation with Fractional Order

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Abstract - This Paper considers a discrete fractional order 2-Dimensional system for a damped and driven Duffing Oscillator with linear and cubic term. Stability of the system is analyzed using the fixed points of the system. Time plot and Phase portraits are provided for different values of the parameters. Chaotic behavior of the system is studied with Bifurcation diagram. Numerical simulations are performed supporting the theoretical analysis.

Keywords- Duffing equation, Fixed Points, Stability, Bifurcation.

I INTRODUCTION

In recent years, mathematicians are very much attracted towards modeling and solving various problems in applied sciences with fractional calculus. Though the origin dates back to 17^{th} century, the development fractional calculus took place during the end of 19^{th} involving the likes of great mathematicians Liouville, Grünwald, Letnikov and Riemann. Chemical and biological systems and Mechanical devices with non-linear springs, electric circuits etc are among the systems exhibiting chaotic motions [3]. Duffing equation used in modeling of damped and driven oscillators is named after Georg Duffing (1861-1944). Comparing with the equation representing simple harmonic motions more complex potential in damped oscillatory motion are described by the Duffing equations.

The paper is organized with descrption of the mathematical model in Section 2 followed by the analysis of the stability of the system in Section 3. Numerical simulations for the examples and Bifurcations of the system are presented in Section 4 and Section 5 respectively. Section 6 gives the conclusion of the paper.

II. MATHEMATICAL MODEL

A discrete fractional order 2-Dimensional system of Duffing equation with damping and driving force terms is considered in this paper. The fractional-order Duffing oscillator with non linearity is described by

$$D^{\beta}x(t) = y(t) \ ; \quad D^{\beta}y(t) = (a + l\sin(wt))x(t) - qy(t) - d(x(t))^3; \qquad (1)$$

Using Piecewise constant arguments method (1) is discretized [[1],[2],[7]]. The Discrete fractional order system of Duffing equation is given by

$$x(n+1) = x(n) + \frac{\delta^{\beta}}{\Gamma(1+\beta)}[y(n)] \ ; \quad y(n+1) = y(n) + \frac{\delta^{\beta}}{\Gamma(1+\beta)}[(a + l\sin(wn))x(n) - qy(n) - d(x(n))^3]; \qquad (2)$$

where a is a constant, β denotes the fractional order and δ denotes the step size of discretization. The damping of the oscillator is being controlled by q. The coefficient of the linear term a controls the linear stiffness. The non-linearity coefficient d is 1 for if d vanishes, damped and driven simple harmonic oscillator is described by the duffing equation. The symbols l and w represent the amplitude and the angular frequency of the driving force respectively.

III. STABILITY ANALYSIS OF THE SYSTEM (2)

The fixed points of the system (2) are

- $F_0 = (0,0)$ (trivial)
- $F_1 = (\sqrt{a + l\sin(wn)}, 0)$ (Axial Point)

The variational matrix [6] of (2) for equilibrium $E = (x^*, y^*)$ is

$$V(x^*, y^*) = \begin{bmatrix} 1 & H \\ H[a + l\sin(wn) - 3(x^*)^2] & 1 - Hq \end{bmatrix} \qquad (3)$$

where $H = \frac{\delta^\beta}{\Gamma(1+\beta)}$.

The characteristics equation of the matrix (3) is given by

$$C(p) = p^2 - Trace(V(x^*, y^*))\ p + Det(V(x^*, y^*)) \quad (4)$$

where $Trace(V(x^*, y^*)) = 2 - Hq$ and the Determinant of (2) is $1 - Hq - H^2[(a + l\sin(wn)) - 3(x^*)^2]$. The eigen values $p_{1,2}$ are obtained by solving $C(p) = 0$. The local conditions of stability for the fixed points are obtained using Jury's conditions.

Proposition 1 *The trivial fixed point F_0 of the system (2) is*

1. a sink if $-H(a + l\sin(wn)) < q < \frac{4-H^2(a+l\sin(wn))}{H}$.
2. a source if $q < Min\left\{\frac{4-H^2(a+l\sin(wn))}{H}, -H(a + l\sin(wn))\right\}$.
3. saddle if $q > \frac{4-H^2(a+l\sin(wn))}{H}$

Proof. At F_0, (3) becomes

$$V(F_0) = \begin{bmatrix} 1 & H \\ H[a + l\sin(wn)] & 1 - Hq \end{bmatrix}$$

By Jury's condition and (4), it is clear that the fixed point

- F_0 is a sink if $-H(a + l\sin(wn)) < q < \frac{4-H^2(a+l\sin(wn))}{H}$.
- F_0 is a source if $q < Min\left\{\frac{4-H^2(a+l\sin(wn))}{H}, -H(a + l\sin(wn))\right\}$
- Fixed point is saddle if $q > \frac{4-H^2(a+l\sin(wn))}{H}$.

Proposition 2 *The Axial fixed point F_1 of the system (2) is*

1. a sink if $-(\frac{2}{H^2} + l\sin(wn)) < a < \frac{q}{2H} - l\sin(wn)$.
2. a source if $a > Max\left\{-(\frac{2}{H^2} + l\sin(wn)), \frac{q}{2H} - l\sin(wn)\right\}$.
3. saddle if $q < -(\frac{2}{H^2} + l\sin(wn))$

Proof. The matrix (3) at the axial fixed point F_1 is

$$V(F_1) = \begin{bmatrix} 1 & H \\ -2\ H(a + l\sin(wn)) & 1 - Hq \end{bmatrix}.$$

Applying the Jury's condition to the characteristic equation (4) and the determinant of the matrix $V(F_1)$, the conditions for the fixed point F_1 to be sink, source and saddle are obtained.

IV. NUMERICAL EXAMPLES

Example 1 *For the values $\beta = 0.85, \delta = 0.01, a = 0.99, w = 0.5, l = 0.5$ and $q = 0.3$ with the initial condition $(0.03, 0.05)$, the stability of the system (2) is illustrated with time plot and phase portrait in Figure 1.* Starting from the initial conditions there is disturbance in the system which is explained by the oscillations in the time plots A & B. The phase portrait with curve moving towards origin ensures the stability of the system.

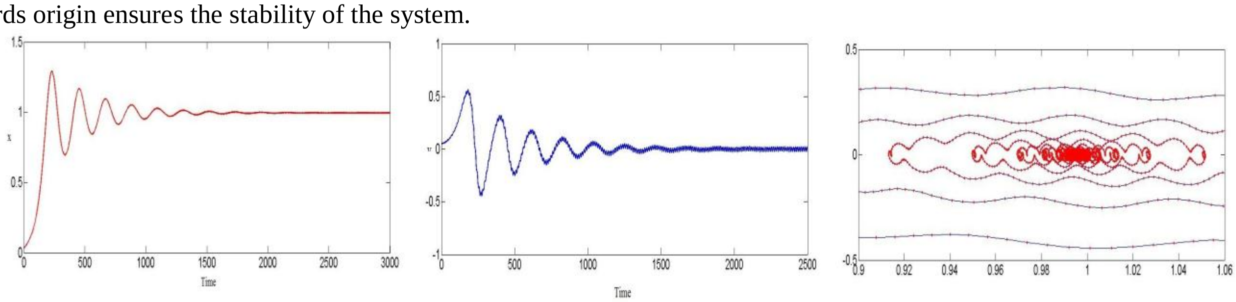

Figure 1: Stability of the system (2)

Example 2 *Phase portrait in Figure - 2 presents the limit cycle formation of the system with order 0.75, $stepsize = 0.09$ with* *other parameters taking the values* $a = 0.97, w = 1, l = 0.5, q = 0.3$ *and the initial condition* $(0.03, 0.05)$. The phase portrait is the closed curve moving inwards starting from the initial points,thus a stable limit cycle is formed for the above given parameter values.

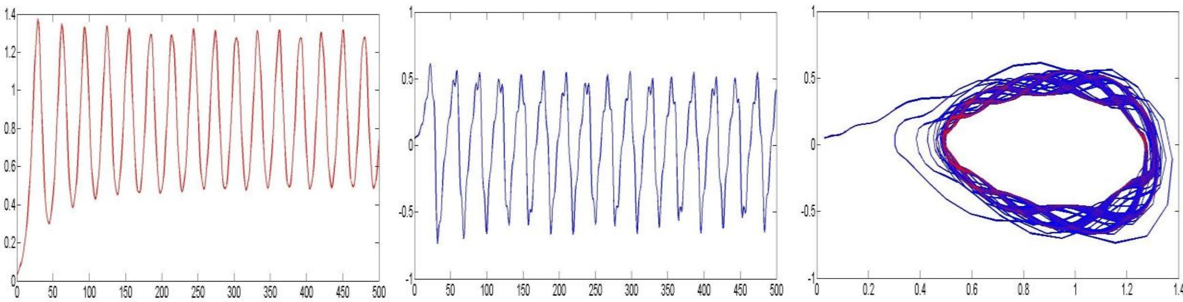

Figure 2: Limit Cycle of the system (2)

Example 3 *Chaotic behavior of the system (2) is given in 2-D Phase portrait in Figure - 3. The non-uniform oscillations of the time plots explain the unstable behavior of the system for the parametric values* $\beta = 0.75, \delta = 0.2, a = 0.07, w = 0.1, l = 0.5$ *and* $q = 0.3$ *with the initial condition* $(0.03, 0.05)$. A & B of the Figure 3 is time plots with non uniform oscillations and phase portrait gives the chaotic motion of the system.

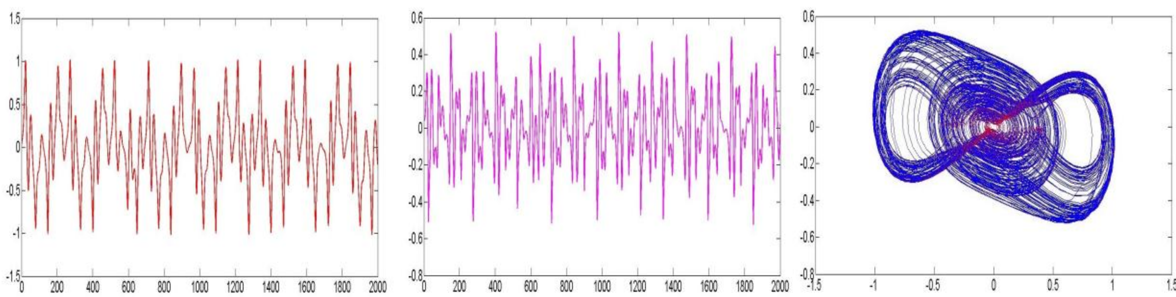

Figure 3: Unstability of the system (2)

V. BIFURCATION OF THE SYSTEM (2)

Bifurcation diagrams are used in predicting the dynamic behavior of the real process that has been modeled. The structural change with varying parametric value is study of Bifurcation.[5] The last three decades has seen rapid development in the theory of Bifurcation and also been widely used in applied and non-linear sciences. The Bifurcation point is the point at which fixed point behavior and trajectory's nature exhibits dramatic changes. Occurrence of the bifurcation results in alteration in attractor and repellor characters.

With fractional order β as the bifurcation parameter in the range 0 to 1.5, the bifurcation diagram Figure 4 is simulated for parametric values $\delta = 0.2, l = 0.02, a = 0.01, w = 0.1, q = 0.07$ and initial conditions $0.06, 0.05$

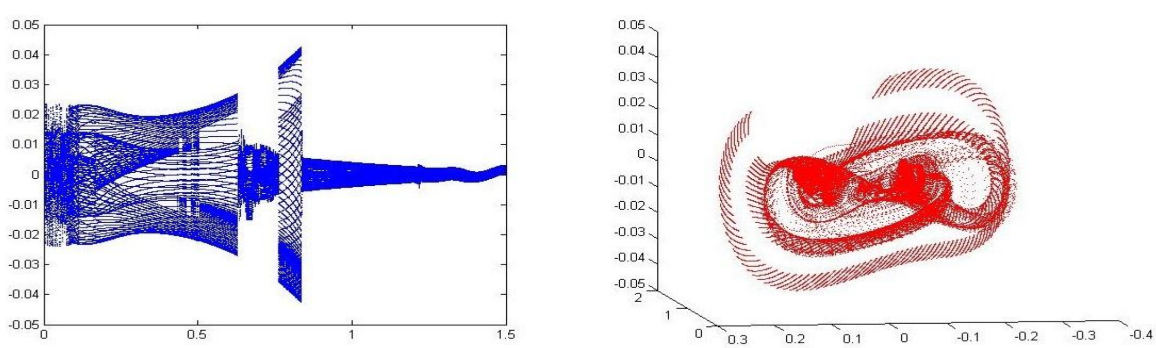

Figure 4: Bifurcation for β

VI. CONCLUSION

The conditions for the stability of discrete order fractional system are investigated. Time plots and Phase plane diagrams are provided to explain the stability and limit cycle formation of the system. Further the chaotic behavior of the Duffing equation are illustrated with bifurcation diagrams. Numerical simulations are carried out describing the dynamic nature of the system.

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