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Some Sufficient Conditions of Generalized Distribution Series on Univalent Functions

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Abstract— The purpose of the present paper is to obtain some sufficient conditions for the generalized distribution series belonging to the certain classes of analytic univalent functions.

Keywords— *Analytic, Univalent functions, generalized Distribution.*

I. INTRODUCTION

Suppose that $U = \{z : z \in \mathbb{C} \text{ and} |z| < 1\}$ denotes the unit disc. A single valued function *f(z)* is said to be univalent in U, if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in U$ and $z_1 \neq z_2$. Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

which are analytic in the open unit disc U and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by S the subclass of A consisting of functions of the form (1) which are also univalent in U.
Further T denotes the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \qquad (2)$$

A function *f* of the form (1) is said to be starlike of order α if and only if

$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \ z \in U$, and is said to be convex of order α if it satisfies the following condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ z \in U.$$

The classes of all starlike and convex functions of order α are denoted by $S^*(\alpha)$ and $K(\alpha)$, respectively, studied by Robertson [1] and Silverman [2].
The classes of $S^*(\alpha)$ and $K(\alpha)$ were unified in to the classes $P_\lambda(\alpha)$ and $D_\lambda(\alpha)$ by Altintas *et al.* [3] as

A function $f \in S$ is said to be in the class $P_\lambda(\alpha)$ if it satisfies the following condition

$$\Re\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)}\right\} > \alpha, \ z \in U$$

and is said to be in the class $D_\lambda(\alpha)$ if it satisfies the following condition

$$\Re\left\{\frac{\lambda z^3 f'''(z) + (1+2\lambda) z^2 f''(z) + zf'(z)}{(1-\lambda) f(z) + \lambda z f'(z)}\right\} > \alpha, \ z \in U.$$

Further, we denote by $P_\lambda^*(\alpha) = P_\lambda(\alpha) \cap T$ and $D_\lambda^*(\alpha) = D_\lambda(\alpha) \cap T$.
It is worthy to note $P_0(\alpha) \equiv S^*(\alpha)$ and $P_1(\alpha) \equiv D_0(\alpha) = K(\alpha)$.
In 1995, Dixit and Pal [4] introduced the class $R^\tau(A, B)$ consisting of functions f of the form (1) if it satisfy the following condition

$$\left|\frac{f'(z) - 1}{(A-B)\tau - B(f'(z) - 1)}\right| < 1,$$

where $\tau \in C \setminus \{0\}, -1 \le B < A \le 1, z \in U.$

By using Poisson distribution, Porwal [5] introduce Poisson distribution series and give a nice application of it in Geometric Function Theory. It opens up a new and interesting direction of research in G.F.T. After the appearance of this paper some other distribution series e.g. Hypergeometric distribution series [6], confluent hypergeometric distribution series [7], Bionomial distribution series [8], generalized distribution series [9] are introduced and mapping properties of these series are investigated.

Very recently Porwal [9] (see also [10]) introduced a generalized discrete probability distribution and give a nice application on certain analytic univalent functions. Now we recall the definition of generalized distribution. The probability mass function of the generalized distribution is given as

$$p(n)=\frac{t_n}{S}, \qquad n=0,1,2,...$$

where $t_n \ge 0$ and the series $\sum_{n=0}^{\infty} t_n$ is convergent and

$$S=\sum_{n=0}^{\infty} t_n . \qquad (3)$$

Also we introduce the series

$$\phi(x)=\sum_{n=0}^{\infty} t_n x^n. \qquad (4)$$

From (3) we have the series given by (4) is convergent for $|x|<1$ and for $x=1$, it is also convergent.

Now we introduce generalized distribution series as

$$K_\phi(z)=z+\sum_{n=2}^{\infty}\frac{t_{n-1}}{S}z^n. \qquad (5)$$

The convolution (or Hadamard product) of two power series $F(z)=\sum_{n=0}^{\infty} A_n z^n$ and $G(z)=\sum_{n=0}^{\infty} B_n z^n$ is defined as the power series

$$(F*G)(z)=\sum_{n=0}^{\infty} A_n B_n z^n. \qquad (6)$$

Next, we introduce the convolution operator $K_\phi(f,z)$ for functions *f* of the form (1) as follows

$$K_\phi(f,z)=K_\phi(z)*f(z)$$

or $K_\phi(f,z)=z+\sum_{n=2}^{\infty}\frac{a_n t_{n-1}}{S}z^n. \qquad (7)$

In the present paper, we obtain some sufficient conditions for the functions $K_\phi(z)$ belonging to the classes $P_\lambda(\alpha)$ and $D_\lambda(\alpha)$ and connections of these subclasses with $R^\tau(A,B)$.

II. MAIN RESULTS

In our investigation, we shall require the following lemmas.

Lemma 1 ([4]) If $f \in R^\tau(A,B)$ is of the form (1) then

$$|a_n| \le \frac{(A-B)|\tau|}{n},\ (n \in N\ ,\ \{1\}). \qquad (8)$$

The bounds given in (8) are sharp.

Lemma 2. ([3]) Let $f \in A$ be of the form (1) then $f \in P_\lambda(\alpha)$, if

$$\sum_{n=2}^{\infty}(n\lambda-\lambda+1)(n-\alpha)|a_n| \le 1-\alpha. \qquad (9)$$

Lemma 3. ([3]) Let $f \in A$ be of the form (1) then $f \in D_\lambda(\alpha)$, if

$$\sum_{n=2}^{\infty}n(n\lambda-\lambda+1)(n-\alpha)|a_n| \le 1-\alpha. \qquad (10)$$

Remark 1. Let $f \in A$ be of the form (2) then $f \in P_\lambda^*(\alpha)$, if and only if (9) is satisfied.

Remark 2. Let $f \in A$ be of the form (2) then $f \in D_\lambda^*(\alpha)$, if and only if (10) is satisfied.

By specializing the parameter $\lambda=0$ and $\lambda=1$ in the Lemmas 2 and 3, we obtain the results of Silverman [2].

Theorem 1. If the function $K_\phi(z)$ is defined in (5) and the inequality

$$\lambda\phi''(1)+(1+2\lambda-\alpha\lambda)\phi'(1)\le(1-\alpha)\phi(0), \qquad (11)$$

is satisfied then $K_\phi(z)\in P_\lambda(\alpha)$.

Proof. From Lemma 2, we have to prove that

$$P_1=\sum_{n=2}^{\infty}(n\lambda-\lambda+1)(n-\alpha)\frac{t_{n-1}}{S}\le 1-\alpha.$$

Now

$$P_1=\sum_{n=2}^{\infty}(n\lambda-\lambda+1)(n-\alpha)\frac{t_{n-1}}{S}$$

$$=\frac{1}{S}\left[\sum_{n=2}^{\infty}\left(\lambda(n-1)(n-2)+(1+2\lambda-\alpha\lambda)(n-1)+(1-\alpha)\right)t_{n-1}\right]$$

$$=\frac{1}{S}\left[\lambda\sum_{n=1}^{\infty}n(n-1)t_n+(1+2\lambda-\alpha\lambda)\sum_{n=1}^{\infty}nt_n+(1-\alpha)\sum_{n=1}^{\infty}t_n\right]$$

$$=\frac{1}{S}\left[\lambda\phi''(1)+(1+2\lambda-\alpha\lambda)\phi'(1)+(1-\alpha)\{\phi(1)-\phi(0)\}\right]$$

$$\le 1-\alpha,$$

by the given hypothesis.

This completes the proof of Theorem 1.

Theorem 2. If the function $K_\phi(z)$ is defined in (5) and the inequality

$$\lambda\phi'''(1)+(1+5\lambda-\alpha\lambda)\phi''(1) +(3+4\lambda-2\alpha\lambda-\alpha)\phi'(1)\le(1-\alpha)\phi(0), \quad (12)$$

is satisfied then $K_\phi(z)\in D_\lambda(\alpha)$.

Proof. The proof of this theorem is much akin to that of Theorem 1. Therefore we omit the details involved.

Theorem 3. The function $G_\phi(z)=\int_0^z \frac{K_\phi(t)}{t}dt$ is in the class $D_\lambda(\alpha)$, if (11) holds.

Proof. Here $G_\phi(z)=\int_0^z \frac{K_\phi(t)}{t}dt$ which can be re-written as

$$G_\phi(z)=z+\sum_{n=2}^{\infty}\frac{t_{n-1}}{nS}z^n.$$

To prove that $G_\phi(z)\in D_\lambda(\alpha)$, from Lemma 3 we have to prove that

$$\sum_{n=2}^{\infty}n(n\lambda-\lambda+1)(n-\alpha)\frac{t_{n-1}}{nS}\le 1-\alpha.$$

The proof of the rest part is similar to the proof of Theorem 1, therefore we omit the details involved.

3. Inclusion Properties

Theorem 4. If $f\in R^\tau(A,B)$ and the inequality

$$\frac{(A-B)|\tau|}{S}\begin{bmatrix}\lambda\phi''(1)+(1+2\lambda-\alpha\lambda)\phi'(1)\\ +(1-\alpha)(\phi(1)-\phi(0))\end{bmatrix}\le 1-\alpha$$

is satisfied then $K_\phi(f,z)\in D_\lambda(\alpha)$.

Proof. Let *f* be of the form (1) belong to the class $R^\tau(A,B)$. To show that $K_\phi(f,z)\in D_\lambda(\alpha)$, we have to prove that

$$\sum_{n=2}^{\infty}n(n\lambda-\lambda+1)(n-\alpha)\frac{t_{n-1}}{S}|a_n|\le 1-\alpha.$$

Since $f\in R^\tau(A,B)$, then by Lemma 1, we have

$$|a_n|\le\frac{(A-B)|\tau|}{n},\ (n\in N,\ \{1\}).$$

Hence

$$\sum_{n=2}^{\infty}n(n\lambda-\lambda+1)(n-\alpha)\frac{t_{n-1}}{S}|a_n|$$

$$\le\frac{(A-B)|\tau|}{S}\sum_{n=2}^{\infty}(n\lambda-\lambda+1)(n-\alpha)t_{n-1}$$

$$\le\frac{(A-B)|\tau|}{S}\begin{bmatrix}\lambda\sum_{n=2}^{\infty}(n-1)(n-2)t_{n-1}\\ +(1+2\lambda-\alpha\lambda)\sum_{n=2}^{\infty}(n-1)t_{n-1}+(1-\alpha)\sum_{n=2}^{\infty}t_{n-1}\end{bmatrix}$$

$$\le\frac{(A-B)|\tau|}{S}\begin{bmatrix}\lambda\phi''(1)+(1+2\lambda-\alpha\lambda)\phi'(1)\\ +(1-\alpha)(\phi(1)-\phi(0))\end{bmatrix}$$

$$\le 1-\alpha,$$

by the given hypothesis.

This completes the proof of Theorem 4.

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III CONCLUSION AND FUTURE SCOPE

The main conclusion of this paper is to obtain some sufficient conditions of generalized distribution series for belonging to certain classes of univalent functions. By giving specific values on generalized distribution series we obtain sufficient conditions for Poisson distribution series, Hypergeometric distribution series, confluent hypergeometric distribution series and Binomial distribution series.

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