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On three Parameter Weighted Quasi Lindley Distribution: Properties and Applications

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Abstract- In this paper, we have introduced a weighted model of the Quasi Lindley Distribution (QLD) as a new generalization of QLD. Statistical properties of this new distribution are derived and the model parameters are estimated by Maximum Likelihood (ML) estimation technique. Finally, the model is examined with an application to real life data.

Keywords: *Quasi Lindley Distribution, Weighted Probability Models, Reliability and Order Statistics, Strength Data.*

I. INTRODUCTION

Shanker and Mishra (2013) introduced a two-parameter Quasi Lindley distribution (QLD), of which the Lindley distribution (LD) is a particular case. Its moments, failure rate function, mean residual life function and stochastic orderings have been discussed.

Probability density function (pdf) of Quasi Lindley Distribution (QLD) is given by

$$f(x;\alpha,\theta)=\frac{\theta(\alpha+\theta x)}{\alpha+1}e^{-\theta x} \qquad x>0,\alpha>0,\theta>0 \tag{1.1}$$

The corresponding cdf of (1.1) is given by

$$F(x;\alpha,\theta)=1-\left[\frac{1+\alpha+\theta x}{\alpha+1}\right]e^{-\theta x}\ ,\ x>0,\theta>0 \tag{1.2}$$

The rth moment of (1.1) is given as

$$\mu_r{}'=\frac{(\alpha+r+1)\Gamma(r+1)}{\theta^r(\alpha+1)},\quad r=1,2,3,\ldots$$

II. WEIGHTED QUASI LINDLEY DISTRIBUTION (WQLD)

Weighted technique is one of the prominent techniques for generalizing the probability models using the weight functions. The concept of weighted distributions can be traced to the work of Fisher (1934), in connection with his studies, on how methods of ascertainment can influence the form of distribution of recorded observations. Later it was introduced and formulated in general terms by Rao (1965). Many researchers developed some important weighted probability models with their significant role in handling data sets from various practical fields. Gove (2003) studied some of the more recent results on weighted distributions pertaining to parameter estimation in forestry. Gupta and Tripathi (1996) studied the weighted version of the bivariate three parameter logarithmic series distribution which has applications in many fields such as ecology, social and behavioral sciences and species abundance studies. Warren (1975) was the first to apply the weighted distributions in connection with sampling wood cells. Recently Para and jan (2018) introduced weighted Pareto type II distribution with applications in medical sciences.

If X is a non negative random variable with probability density function (pdf) $f(x)$. Let $w(x)$ be the weight function which is a non negative function, then the probability density function of the weighted random variable X_w is given by:

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))}, \quad x > 0,$$

where $w(x)$ be a non-negative weight function and $E(w(x)) = \int w(x)f(x)dx < \infty$.

In this paper, we have considered the weight function as $w(x) = x^c$ to obtain the weighted Quasi Lindley distribution. The probability density function (pdf) of weighted Quasi Lindley distribution is given as:

$$f_w(x;c,\alpha,\theta) = \frac{x^c f(x,\theta)}{E[x^c]},$$

$$f_w(x;c,\alpha,\theta) = \frac{x^c \theta^{(c+1)}(\alpha+\theta x)e^{-\theta x}}{(\alpha+c+1)\Gamma(c+1)}, \qquad x>0, c>0, \ \alpha>0, \ \theta>0, \tag{2.1}$$

where $E(x^c) = \frac{(\alpha+c+1)\Gamma(r+1)}{\theta^c(\alpha+1)}$.

The corresponding cumulative distribution function (cdf) of weighted Quasi Lindley Distribution (WQLD) is obtained as

$$F_w(x;c,\alpha,\theta) = \int_0^x f_w(x;c, \ \alpha, \ \theta)dx$$

$$= \int_0^x \frac{x^c \theta^{(c+1)}(\alpha+\theta x)e^{-\theta x}}{(\alpha+c+1)\Gamma(c+1)}dx \qquad \text{, put } \theta x = t, \ \ \theta\, dx = dt,$$

$as\, x \to 0, t \to 0 \ \ and \ \ x \to x, t \to \theta x$, after simplification

$$F_w(x;c,\alpha,\theta) = \frac{1}{(\alpha+c+1)\Gamma(c+1)}(\alpha\gamma(c+1,\theta x) + \gamma(c+2,\theta x)), \quad x>0, c>0, \ \alpha>0, \ \theta>0, \tag{2.2}$$

where θ and c are positive parameters and $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ is a lower incomplete gamma function.

The graphs of probability density function and cumulative distribution function are plotted for different values of parameters θ and c given in Fig.1 and Fig. 2 respectively. Fig. 1 gives the description of some of the possible shapes of weighed Quasi Lindley distribution for different values of the parameters θ and c. It illustrates that the density function of weighted Quasi Lindley distribution is positively skewed, for fixed θ it becomes more and more flatter as the value of c is increased. Fig. 2 shows the graph of distribution function which is an increasing function.

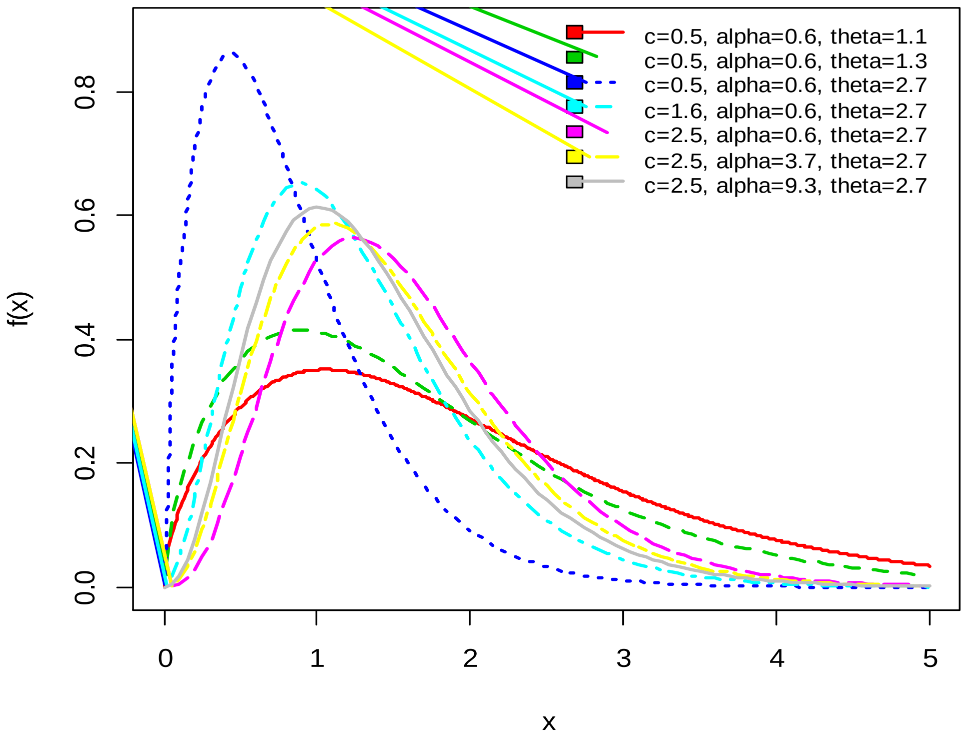
III. SPECIAL CASES

Case I: If we put $c = 0$, then weighted Quasi Lindley distribution (2.1) reduces to Quasi Lindley distribution with probability density function as:

$$f(x;\alpha,\theta) = \frac{\theta(\alpha+\theta x)}{\alpha+1}e^{-\theta x} \qquad x>0, \ \alpha>0, \theta>0,$$

Case II: For $\alpha = 0$, then weighted Quasi Lindley distribution (2.1) reduces to two parameter gamma distribution with parameters θ and $c+2$.

Fig.1: pdf plot of Weighted Quasi Lindley Distribution



IV. RELIABILITY ANALYSIS

In this section, we have obtained the reliability, hazard rate, reverse hazard rate of the proposed weighted Quasi Lindley Distribution.

4.1 Reliability function R(x)

The reliability function is defined as the probability that a system survives beyond a specified time. It is also referred to as survival or survivor function of the distribution. It can be computed as complement of the cumulative distribution function of the model. The reliability function or the survival function of weighted Quasi Lindley distribution is calculated as:

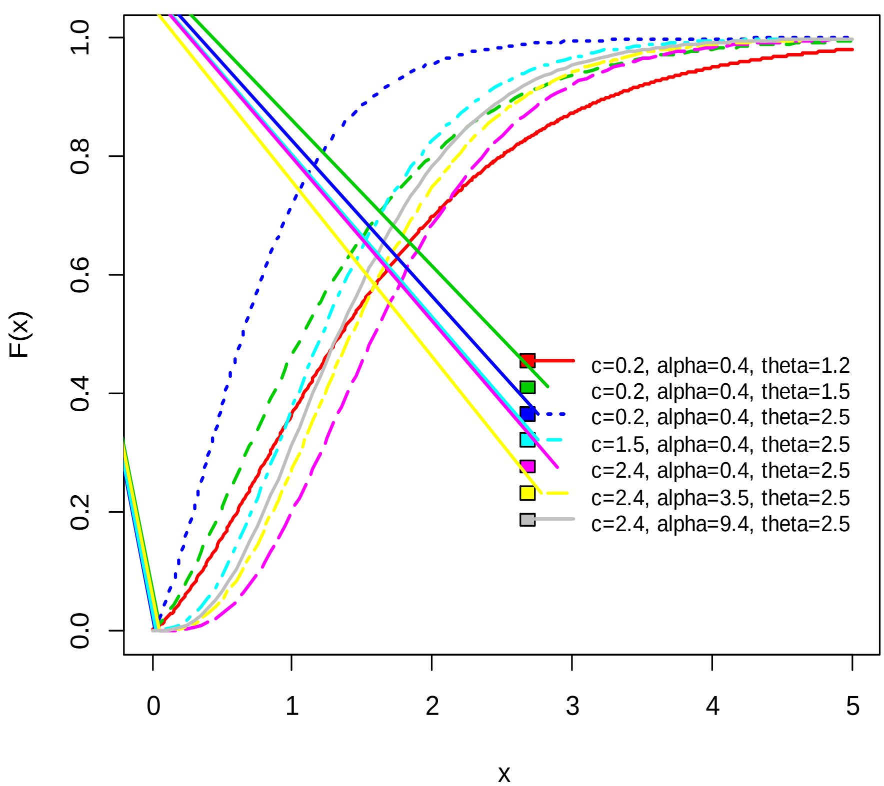
$$R_w(x,c,\theta) = 1 - \frac{1}{(\alpha+c+1)\Gamma(c+1)}(\alpha\gamma(c+1,\theta x) + \gamma(c+2,\theta x)), \quad x>0, c>0,\ \alpha>0,\ \theta>0, \tag{4.1}$$

The graphical representation of the reliability function for the weighted Quasi Lindley distribution is shown in fig. 3.

4.2 Hazard Function:

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality is given as:

$$H.R = h(x;c,\theta) = \frac{f_w(x,\theta)}{R_w(x,\theta)} = \frac{x^c\theta^{c+1}(\alpha+x\theta)e^{-\theta x}}{(\alpha+c+1)\Gamma(c+1) - \{\alpha\gamma(c+1,\theta x) + \gamma(c+2,\theta x)\}}, \qquad x>0, c>0,\ \alpha>0,\ \theta>0,$$



4.3 Reverse Hazard Rate:

The reverse hazard rate of the weighted quasi Lindley distribution are respectively given as:

$$R.H.R = h_r(x,c,\theta) = \frac{f_w(x,\theta)}{F_w(x,\theta)} = \frac{x^c\,\theta^{c+1}(\alpha+\theta x)e^{-\theta x}}{(\alpha\gamma(c+1,\theta x)+\gamma(c+2,\theta x))}. \qquad x>0, \theta>0, c>0.$$

V. STATISTICAL PROPERTIES

In this section, the different structural properties of the proposed weighted quasi Lindley model have been evaluated. These include moments, mode, harmonic mean, moment generating function and characteristic function.

5.1 Moments: Suppose X is a random variable following weighted Quasi Lindley distribution with parameter θ, and then the rth moment for a given probability distribution is given by

$$\mu_r{}' = E\left(X_w{}^r\right) = \int_0^\infty x^r f_w(x,c,\theta)dx$$

$$= \int_0^\infty x^r \frac{x^c\theta^{c+1}(\alpha+\theta x)e^{-\theta x}}{(\alpha+c+1)\Gamma(c+1)}dx$$

$$\mu_r{}' = \frac{(c+r)!(\alpha+c+r+1)}{\theta^r(\alpha+c+1)\Gamma(c+1)} \tag{5.1}$$

$$\mu_1{}' = \frac{(c+1)(\theta^3+(c+2)(c+3))}{(\theta^3+(c+1)(c+2)}.$$

Which is mean of the weighted quasi Lindley distribution

Put r=2 in equation (5.1) we get

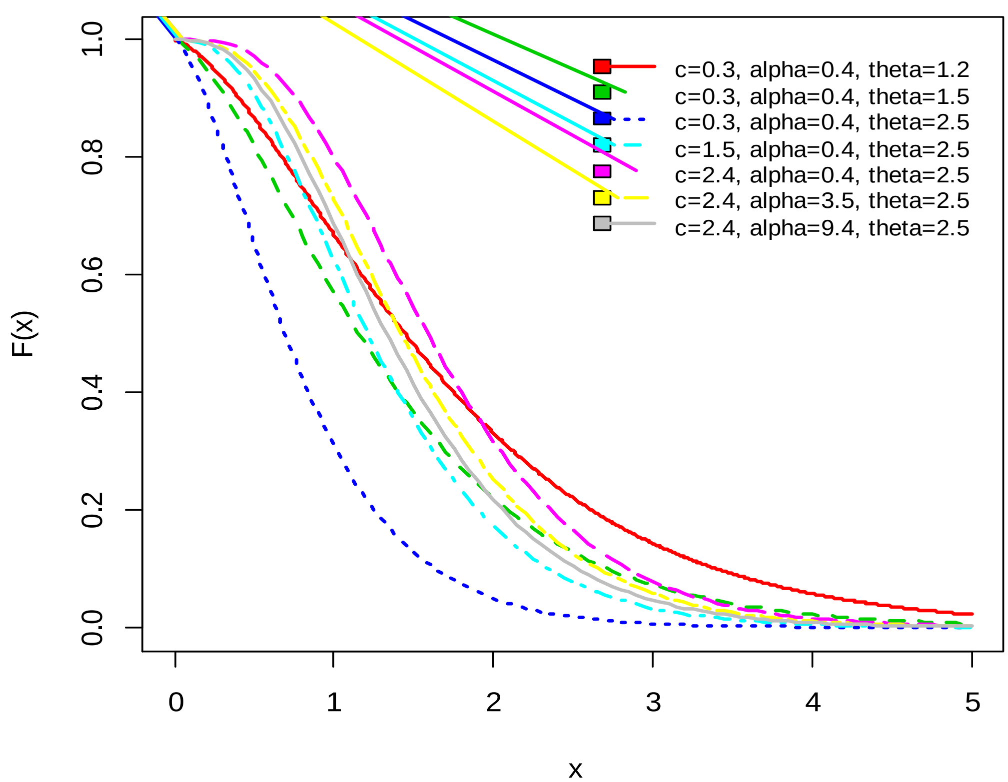
$$\mu_2{}' = \frac{(c+1)(c+2)(\alpha+c+3)}{\theta^2(\alpha+c+1)}$$

And variance of weighted quasi Lindley distribution is

$$V(X) = \frac{(c+1)(c+2)(\alpha+c+3)}{\theta^2(\alpha+c+1)} - \frac{(c+1)^2(\alpha+c+2)^2}{\theta^2(\alpha+c+1)^2}$$

$$= \frac{(c+1)}{\theta^2(\alpha+c+1)^2}\left\{(\alpha+c+1)(c+2)(\alpha+c+3)-(c+1)(\alpha+c+2)^2\right\}$$

Fig.3: Survival function plot of Weighted Quasi Lindley Distribution



5.2 Harmonic mean

The harmonic mean for the proposed model is computed as:

$$H.M = E\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} f_w(x;c,\theta)dx$$

$$= \int_0^\infty \frac{1}{x} \frac{x^c \theta^{c+1}(\alpha + x\theta)e^{-\theta x}}{(\alpha + c + 1)\Gamma(c+1)} dx$$

$$= \frac{c(\alpha + c + 1)}{\theta(c+\alpha)}. \qquad \theta > 0, c > 0,$$

5.3 Moment Generating Function and Characteristic Function of Weighted Quasi Lindley Distribution (WQLD)

We will derive moment generating function and characteristic function of WQLD in this section.

Theorem 1.1: If X has the WQLD (c,θ,α), then the moment generating function $M_X(t)$ and the characteristic function $\psi_X(t)$ are $\frac{\theta^{c+1}\left[\alpha(\theta-t)+\theta(c+1)\right]}{(\theta-t)^{c+2}(\alpha+c+1)}$ and $\frac{\theta^{c+1}\left[\alpha(\theta-it)+\theta(c+1)\right]}{(\theta-it)^{c+2}(\alpha+c+1)}$ respectively.

And hence show that two parameter gamma distribution is a particular case of weighted quasi Lindley distribution.

Proof: We begin with the well known definition of the moment generating function given by

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x;\alpha,\lambda)\, dx$$

$$= \int_0^\infty \frac{e^{tx} x^c \theta^{c+1}(\alpha + \theta x)e^{-\theta x}}{(\alpha + c + 1)\Gamma(c+1)} dx$$

$$= \frac{\theta^{c+1}}{(\alpha + c + 1)\Gamma(c+1)} \theta^{c+1} \int_0^\infty e^{-x(\theta - t)} x^{c+1-1}(\alpha + \theta x)dx$$

$$\Rightarrow M_X(t) = \frac{\theta^{c+1}\left[\alpha(\theta-t)+\theta(c+1)\right]}{(\theta-t)^{c+2}(\alpha+c+1)} \qquad (5.3.1)$$

For $\alpha = 0$ in equation (5.3.1) we get

$$M_X(t) = \left[\frac{\theta}{(\theta - t)}\right]^{c+2}$$

Which is mgf of two parameter gamma distribution with parameters θ and c+2.

Also we know that $\psi_X(t) = M_X(it)$

Therefore, $\psi_X(t) = \frac{\theta^{c+1}\left[\alpha(\theta-it)+\theta(c+1)\right]}{(\theta-it)^{c+2}(\alpha+c+1)}$

5.4 Quantile and Random Number Generation from WQLD

Inverse CDF Method is one of the methods used for the generation of random numbers from a particular distribution. In this method the random numbers from a particular distribution are generated by solving the equation obtained on equating the CDF of a distribution to a number u. The number u is itself being generated from $U(0,1)$. Thus following the same procedure for the generation of random numbers from the WQLD we will proceed as

$$F_w(x\,;c,\alpha,\theta)=u$$

$$\frac{1}{(\alpha+c+1)\Gamma(c+1)}\alpha\gamma(c+1,\theta x)+\gamma(c+2,\theta x)=u \tag{5.4.1}$$

On solving the equation (5.4.1) for x, we will obtain the required random number from the WQLD. Main problem, which is being faced while using this method of generating the random numbers is to solve the equations which are usually complex and complicated. In order to overcome such hindrance, we use softwares like MATLAB, Mathematica or R for solving such a complex equation.

VI. ORDER STATISTICS

Let $X_{(1)},X_{(2)},X_{(3)}\ldots,X_{(n)}$ be the ordered statistics of the random sample $X_1,X_2,X_3,\ldots X_n$ drawn from the continuous distribution with cumulative distribution function $F_X(x)$ and probability density function $f_X(x)$, then the probability density function of rth order statistics $X_{(r)}$ is given by:

$$f_{x(r)}(x,c,\theta)=\frac{n!}{(r-1)!(n-r)!}f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}. \qquad r=1,2,3,\ldots,n$$

Using the equations (2.1) and (2.2), the probability density function of rth order statistics of weighted quasi Lindley distribution is given by:

$$f_{w(r)}(x,c,\theta)=\frac{n!}{(r-1)!(n-r)!}\frac{x^c\theta^{c+3}(\theta+x^2)e^{-\theta x}}{c!(\theta^3+(c+1)(c+2))}\left[\frac{1}{(\alpha+c+1)\Gamma(c+1)}(\alpha\gamma(c+1,\theta x)+\gamma(c+2,\theta x))\right]^{r-1}$$
$$\left[1-\frac{1}{(\alpha+c+1)\Gamma(c+1)}(\alpha\gamma(c+1,\theta x)+\gamma(c+2,\theta x))\right]^{n-r}.$$

Then, the pdf of first order $X_{(1)}$ weighted quasi Lindley distribution is given by:

$$f_{w(1)}(x,c,\theta)=n\frac{x^c\theta^{c+1}(\alpha+\theta x)e^{-\theta x}}{(\alpha+c+1)\Gamma(c+1)}\left[1-\frac{1}{(\alpha+c+1)\Gamma(c+1)}(\alpha\gamma(c+1,\theta x)+\gamma(c+2,\theta x))\right]^{n-1}.$$

and the pdf of nth order $X_{(n)}$ weighted quasi Lindley model is given as:

$$f_{w(n)}(x,c,\theta)=n\frac{x^c\theta^{c+1}(\alpha+x\theta)e^{-\theta x}}{(\alpha+c+1)\Gamma(c+1)}\left[\frac{1}{(\alpha+c+1)\Gamma(c+1)}(\alpha\gamma(c+1,\theta x)+\gamma(c+2,\theta x))\right]^{n-1}.$$

VII. MAXIMUM LIKELIHOOD ESTIMATION OF WEIGHTED QUASI LINDLEY DISTRIBUTION

Let $x_1, x_2, \ldots, x_n$ be the random sample of size n drawn from weighted Quasi Lindley Distribution and let f_i be the observed frequency in the sample corresponding to $X = x_i (i = 1,2,\ldots,n)$, such that $\sum f_i = n$, where k is the largest observed value having non-zero frequency then the likelihood function of Weighted Quasi Lindley Distribution is given as:

$$L(x \mid c, \theta, \alpha) = \prod_{i=1}^{n} f(x; c, \theta) = \prod_{i=1}^{n} \frac{x^c \theta^{c+1} (\alpha + x\theta) e^{-\theta x}}{(\alpha + c + 1)\Gamma(c+1)}$$

$$L(x \mid c, \theta, \alpha) = \prod \left[(x_i)^{cf_i} \right] \theta^{n(c+1)} e^{-n\theta\bar{x}} \prod \left[(\alpha + \theta x_i)^{f_i} \right] \frac{1}{(\Gamma(c+1))^n (\alpha + c + 1)^n}$$

The log likelihood function becomes:

$$\log L(x \mid c, \theta, \alpha) = \sum (cf_i \log x_i) + n(c+1)\log\theta - n\theta\bar{x} + \sum f_i (\alpha + \theta x_i) - n \log\Gamma(c+1)) - n\log(\alpha + c + 1)$$

The three log likelihood equations are thus obtained as

$$\frac{\partial \log L}{\partial \theta} = \frac{n(c+1}{\theta}.+ \frac{\sum f_i x_i}{\alpha + \theta x_i} - n\bar{x} = 0 \qquad (7.1)$$

$$\frac{\partial \log L}{\partial c} = \frac{\sum f_i \log xli}{1} + n\log\theta - \frac{n}{\alpha + c + 1} - n\log(c+1) + \frac{n}{2(c+1)} = 0 \qquad (7.2)$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{\sum f_i}{\alpha + \theta x_i} - \frac{n}{\alpha + c + 1} = 0 \qquad (7.3)$$

The three equations (7.1), (7.2) and (7.3) don't seem to be solved directly. However the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n(c+1}{\theta^2}.+ \frac{\sum f_i x_i^2}{(\alpha + \theta x_i)^2}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{n}{(\alpha + c + 1)^2}.- \frac{\sum f_i}{(\alpha + \theta x_i)^2}$$

$$\frac{\partial^2 \log L}{\partial c^2} = \frac{n}{(\alpha + c + 1)^2} - \frac{n}{(c+1)} - \frac{n}{2(c+1)^2}$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = -\frac{\sum f_i x_i}{(\alpha + \theta x_i)^2}; \quad \frac{\partial^2 \log L}{\partial \theta \partial c} = \frac{n}{\theta}; \quad \frac{\partial^2 \log L}{\partial \alpha \partial c} = \frac{n}{(\alpha + c + 1)^2}$$

The following equations for $\hat{\theta}$, $\hat{\alpha}$ and $\hat{c}$ can be solved as

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta \partial c} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \theta} & \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial c} \\ \frac{\partial^2 \log L}{\partial c \partial \theta} & \frac{\partial^2 \log L}{\partial c \partial \alpha} & \frac{\partial^2 \log L}{\partial c^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0 \\ \hat{c}=c_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \\ \hat{c}-c_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \theta} \\ \frac{\partial \log L}{\partial \alpha} \\ \frac{\partial \log L}{\partial c} \end{bmatrix},$$

where θ_0, α_0 and c_0 are the initial values of θ, α and c respectively. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$, $\hat{\alpha}$ and $\hat{c}$ are obtained.

7.1 Simulation Study of ML estimators of WQLD

In this section, we study the performance of ML estimators for different sample sizes (n=25,75,100,150,200,300). We have employed the inverse CDF technique for data simulation for WQLD using R software. The process was repeated 1000 times for calculation of bias, variance and MSE as given values in table 1. For two parameter combinations of WQLD, decreasing trend is being observed in average bias, variance and MSE as we increase the sample size. Hence, the performance of ML estimators is quite well, consistent in case of WQLD.

Table 1: Simulation Study of ML estimators for weighted Quasi Lindley Distribution

Parameter	n	$c = 1.2, \alpha = 0.8, \theta = 0.5$			$c = 2.3, \alpha = 1.5, \theta = 0.9$		
		Bias	Variance	MSE	Bias	Variance	MSE
α	25	0.698985	0.111666	0.600246	0.794085	0.331057	0.961628
β		1.474035	0.111255	2.284034	1.2268	0.351011	1.856049
θ		-0.03908	0.013196	0.014723	0.288817	0.010859	0.094274
α	75	0.453829	0.020782	0.226743	0.412148	0.026023	0.195889
β		1.125111	0.101674	1.367549	0.78452	0.573856	1.189328
θ		-0.03658	0.012579	0.013917	0.207594	0.005101	0.048196
α	100	0.415148	0.0168	0.189148	0.33123	0.012468	0.122181
β		0.739167	0.094289	0.640657	0.602663	0.216566	0.579769
θ		-0.05377	0.010805	0.013696	0.182033	0.004131	0.037267
α	150	0.400494	0.011392	0.171787	0.292492	0.007101	0.092653
β		0.69953	0.049215	0.538557	0.439972	0.128085	0.32166
θ		-0.07008	0.00601	0.010921	0.151623	0.00527	0.02826
α	200	0.21848	0.009064	0.056798	0.21823	0.005912	0.053536
β		0.630678	0.015107	0.412862	0.225452	0.056483	0.107311
θ		-0.07488	0.004829	0.010436	0.111407	0.003061	0.015472
α	300	0.165541	0.00728	0.034684	0.138878	0.003637	0.022924
β		0.513483	0.006986	0.270651	0.180969	0.053695	0.086445
θ		-0.05221	0.000143	0.002869	0.102064	0.002081	0.012498

VIII. MODEL COMPARISON BASED ON SIMULATED DATA FROM WQLD

In order to compare the Weighted Model with the base model on the basis of simulated data. In this section, we proceed by simulating a data from WQLD using inverse CDF technique discussed in the section 5.4. Two sets of random parameter combinations with sample sizes (n=10, 25, 100,300,500) have been taken into consideration for data generation. It is evident from the table 2 and table 3, that weighted parameter plays a highly significant role for large samples. Log-likelihood ratio test reveals that the role of weighted parameter exhibits a highly significant role in case of large samples only. LR statistic for testing H_0 versus H_1 is $\psi = 2(L(\hat{\Theta}) - L(\hat{\Theta}_0))$, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under H_1 and H_0. The statistic ψ is asymptotically $(as\ n \to \infty)$ distributed as χ_k^2, with k degrees of freedom which is equal to the difference in dimensionality of $\hat{\Theta}$ and $\hat{\Theta}_0$. H_0 will be rejected if the LR-test p-value is <0.01 (or LR Statistic value >6.635) at 99% confidence level.

Table 2: Model Comparison Based On Simulated Data from WQLD.

$\hat{c} = 0.8, \hat{\alpha} = 0.1, \hat{\theta} = 0.2$				Parameter Estimates		Likelihood Ratio Statistic
Criterion	WQLD	QLD	Sample Size (n)	WQLD	QED	
-logL	33.14778	33.32745	10	$\hat{c}$ =1.24 (0.97) $\hat{\alpha}$ =0.96 (0.87) $\hat{\theta}$ =0.213 (0.06)	$\hat{\alpha}$ =0.10 (1.25) $\hat{\theta}$ =0.16 (0.89)	0.359
AIC	72.29556	70.65489				
AICC	76.29556	72.36918				
BIC	73.20331	71.26006				
-logL	85.35855	87.47477	25	$\hat{c}$ =1.86 (1.78) $\hat{\alpha}$ =1.37 (0.847) $\hat{\theta}$ =0.23 (0.06)	$\hat{\alpha}$ =0.10 (0.98) $\hat{\theta}$ =0.12 (0.012)	4.232
AIC	176.7171	178.9495				
AICC	177.86	179.495				
BIC	180.3737	181.3873				
-logL	345.7888	350.2193	100	$\hat{c}$ =1.20 (0.57) $\hat{\alpha}$ =0.78 (0.65) $\hat{\theta}$ =0.20 (0.03)	$\hat{\alpha}$ =0.11 (1.52) $\hat{\theta}$ =0.13 (0.07)	8.861
AIC	697.5776	704.4386				
AICC	697.8276	704.5623				
BIC	705.3931	709.6489				
-logL	994.6124	1012.538	300	$\hat{c}$ =1.56 (0.60) $\hat{\alpha}$ = 1.39 (0.87) $\hat{\theta}$ = 0.25 (0.02)	$\hat{\alpha}$ = 0.10 (0.54) $\hat{\theta}$ = 0.15 (0.003)	35.851
AIC	1995.225	2029.076				
AICC	1995.306	2029.116				
BIC	2006.336	2036.483				
-logL	1694.878	1719.592	500	$\hat{c}$ =0.87 (0.23) $\hat{\alpha}$ =0.06 (0.13) $\hat{\theta}$ =0.21 (0.01)	$\hat{\alpha}$ =0.12 (0.46) $\hat{\theta}$ =0.14 (0.0007)	49.428
AIC	3395.757	3443.185				
AICC	3395.805	3443.209				
BIC	3408.4	3451.614				

Table 3: Model Comparison Based On Simulated Data from WQLD.

$\hat{c} = 1.5, \hat{\alpha} = 1.2, \hat{\theta} = 1.3$				Parameter Estimates		Likelihood Ratio Statistic
Criterion	WQLD	QLD	Sample Size (n)	WQLD	QED	
-logL	18.90418	18.96473	10	$\hat{c}$ =0.549 (1.12) $\hat{\alpha}$ =0.821 (1.39) $\hat{\theta}$ = 0.812 (0.335)	$\hat{\alpha}$ =0.10 (0.45) $\hat{\theta}$ =0.70 (0.21)	0.1210
AIC	43.80836	41.92945				
AICC	47.80836	43.64374				
BIC	44.71612	42.53462				
-logL	44.00957	44.60075	25	$\hat{c}$ =1.03 (1.02) $\hat{\alpha}$ =1.18 (1.01) $\hat{\theta}$ =1.04 (0.28)	$\hat{\alpha}$ =0.13 (0.69) $\hat{\theta}$ =0.75 (0.05)	1.1823
AIC	94.01913	93.2015				
AICC	95.16199	93.74695				
BIC	97.67576	95.63925				
-logL	174.5136	180.4392	100	$\hat{c}$ =1.49 (0.65) $\hat{\alpha}$ =1.21 (0.985) $\hat{\theta}$ =1.18 (0.163)	$\hat{\alpha}$ =0.11 (0.41) $\hat{\theta}$ =0.72 (0.008)	11.851
AIC	355.0273	364.8784				
AICC	355.2773	365.0022				
BIC	362.8428	370.0888				
-logL	517.4113	535.7299	300	$\hat{c}$ =1.46 (0.54) $\hat{\alpha}$ =0.904 (0.89) $\hat{\theta}$ =1.21 (0.108)	$\hat{\alpha}$ =0.10 (0.38) $\hat{\theta}$ =0.73 (0.009)	36.637
AIC	1040.823	1075.46				
AICC	1040.904	1075.5				
BIC	1051.934	1082.867				
-logL	858.2261	892.628	500	$\hat{c}$ =1.47 (0.31) $\hat{\alpha}$ =0.39 (0.48) $\hat{\theta}$ =1.21 (0.08)	$\hat{\alpha}$ =0.12 (0.13) $\hat{\theta}$ =0.721 (0.001)	68.803
AIC	1722.452	1789.256				
AICC	1722.501	1789.28				
BIC	1735.096	1797.685				

IX. APPLICATIONS OF WEIGHTED QUASI LINDLEY DISTRIBUTIONS

Here we analyze the strength data, reported by Badar and Priest (1982), using the Weighted Quasi Lindley Distribution (WQLD) in comparison with Quasi Lindley Distribution (QLD). Estimates of the unknown parameters is carried out in R software along with calculation of model comparison criterion values like AIC, AICC and BIC values. It may be noted that Raqab et al. (2008) fitted the 3-parameter generalized exponential distribution to the same data set. Badar and Priest (1982) reported strength data measured in GPA for single carbon fibre and impregnated 1000 carbon fibre tows. Single fibres were tested at gauge lengths of 10, 20 and 50 mm. Impregnated tows of 1000 fibres were tested at gauge lengths of 20, 50, 150 and 300 mm. The transformed data sets that were considered by Raqab and Kundu (2005) are used here. Table 4, data set 1 (of size 69) and table 5, data set 2 (of size 63) correspond to single fibre with 20 mm and 10 mm of gauge length, respectively.

Table 4: Data set 1

0.031	0.314	0.479	0.552	0.700	0.803	0.861	0.865	0.944	0.958
0.966	0.977	1.006	1.021	1.027	1.055	1.063	1.098	1.140	1.179
1.224	1.240	1.253	1.270	1.272	1.274	1.301	1.301	1.359	1.382
1.382	1.426	1.434	1.435	1.478	1.490	1.511	1.514	1.535	1.554
1.566	1.570	1.586	1.629	1.633	1.642	1.648	1.684	1.697	1.726
1.770	1.773	1.800	1.809	1.818	1.821	1.848	1.880	1.954	2.012
2.067	2.084	2.090	2.096	2.128	2.233	2.433	2.585	2.585	

Table 5: Data set 2

0.101	0.332	0.403	0.428	0.457	0.55	0.561	0.596	0.597	0.645
0.954	0.674	0.718	0.722	0.725	0.732	0.775	0.814	0.816	0.818
0.824	0.859	0.875	0.938	0.94	1.056	1.117	1.128	1.137	1.137
1.177	1.196	1.23	1.325	1.339	1.345	1.42	1.423	1.435	1.443
1.464	1.472	1.494	1.532	1.546	1.577	1.608	1.635	1.693	1.701
1.737	1.754	1.762	1.828	2.052	2.071	2.086	2.171	2.224	2.227
2.425	2.595	3.22							

Fig. 4 and fig. 5, provides a graphical overview of the fitted distributions to a data given in table 4 and table 5. It is evident graphically, that WQLD is providing a better and close fit to the data sets. In order to compare the two models using the AIC (Akaike information criterion) given by Akaike (1976), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion) given by Schwarz (1987). The better distribution corresponds to lesser AIC, AICC and BIC values.

$$\text{AIC} = 2k\text{-}2\log L \qquad \text{AICC} = \text{AIC}+\frac{2k(k+1)}{n-k-1} \text{ and BIC} = k \log n\text{-}2\log L$$

where k is the number of parameters in the statistical model, n is the sample size and -log$L$ is the maximized value of the log-likelihood function under the considered model.

From table 6 and table 7, it is observed that Weighted Quasi Lindley distribution have the lesser AIC, AICC, -log$L$ and BIC values as compared to Quasi Lindley distribution, which witness that WQLD fits better than QLD for data given in table 4 and table 5. In case of data set 1, Kolmogorov Smirnov p-value is greater than 0.05 for both WQLD and QLD but favors WQLD as it has greater p-value as compared to QLD. In case of data set 2, QLD has non-significant p-value, hence does not fit statistically to the data set II but WQLD has Kolmogorov Smirnov p-value greater than 0.05. Hence we can conclude that the weighted Quasi Lindley distribution leads to a better fit than the Quasi Lindley distribution.

Table 6: ML estimates, -logL, AIC, AICC, BIC, KS-distance, KS p-values for fitted WQAD and QAD for data set 1.

Distribution	**Weighted Quasi Lindley**	**Quasi Lindley**
-logL	56.37293	62.239
AIC	118.74585	128.478
AICC	119.15263	128.8848
BIC	125.17526	132.7643
KS-Distance	0.082484	0.13975
P-value	0.7847	0.1706
ML Estimates	$\hat{c} = 2.18$ $\hat{\alpha} = 0.93$ $\hat{\theta} = 3.13$	$\hat{\alpha} = 0.001$ $\hat{\theta} = 1.582$

Fig.4: Weighted Quasi Lindley Distribution Fitting in comparison with Quasi Lindley Distribution

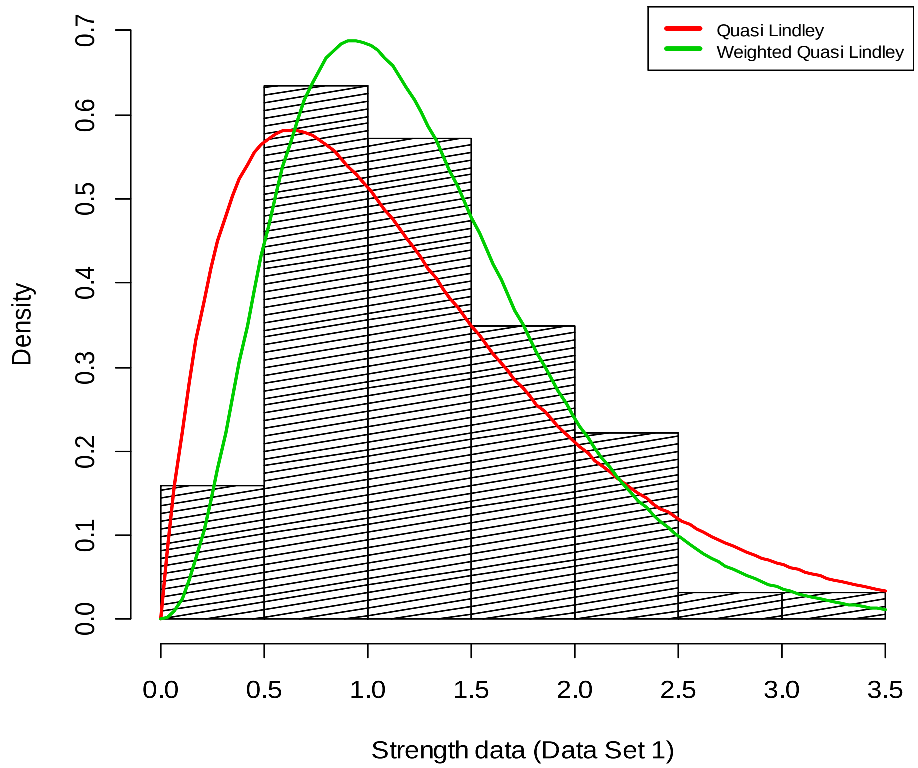
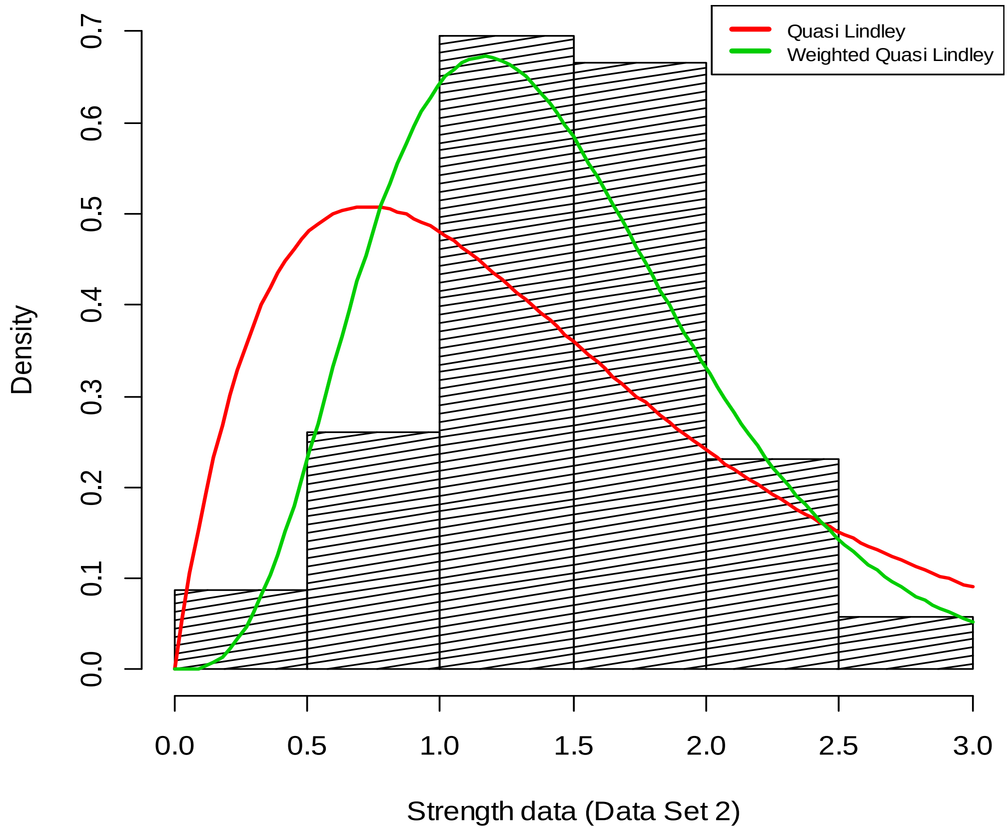


Table 7: ML estimates, -logL, AIC, AICC, BIC, KS-distance, KS p-values for fitted WQAD and QAD for data set 2.

Distribution	Weighted Quasi Lindley	Quasi Lindley
-logL	62.1195	75.0097
AIC	130.2390	154.0195
AICC	130.6083	154.3887
BIC	136.9413	158.4877
KS-Distance	0.12476	0.25908
P-value	0.233	0.00018
ML Estimates	$\hat{c} = 3.32$ $\hat{\alpha} = 0.87$ $\hat{\theta} = 3.55$	$\hat{\alpha} = 0.001$ $\hat{\theta} = 1.381$

Fig.5: Weighted Quasi Lindley Distribution Fitting in comparison with Quasi Lindley Distribution



X. CONCLUSION

A new generalization of the Quasi Lindley distribution called the Weighted Quasi Lindley distribution has been introduced. The subject distribution is generated by using the weighting technique and taking the two parameter Quasi Lindley distribution as the base distribution. Some statistical properties along with reliability measures are discussed. Model is examined with two real life data sets for significance purpose.

REFERENCES

[1]. Akaike, H. (1976). A new look at the statistical model identification. *IEEE Trans. Autom.Control* , 19 , 716–723.

[2]. Bader, M. G., & Priest, A. M. (1982). Statistical aspects of fiber and bundle strength in hybrid composites, In; hayashi T, Kawata K. Umekawa S (Eds.), *Progress in Science in Engineering Composites, ICCM-IV Tokyo*, 1129 -1136.

[3]. Fisher, R.A. (1934). The effects of methods of ascertainment upon the estimation of frequencies. *Ann. Eugenics*, 6, 13-25.

[4]. Gove, J. H. (2003). Environmental and Ecological Statistics, 10(4), 455-467. doi:10.1023/a:1026000505636

[5]. Gupta, R.C., & Tripathi, R.C. (1996). Weighted Bivariate Logarithmic Series Distribution. Communications in Statistics-Theory and Methods, 25(5), 1099-1117.

[6]. Para, B.A., & Jan, T. R. (2018). On three Parameter Weighted Pareto Type II Distribution: Properties and Applications in Medical Sciences. *Applied Mathematics and Information Sciences Letters*, 6 (1), 13-26.

[7]. Rao, C.R (1965). On discrete distributions arising out of method of ascertainment, in classical and Contagious Discrete, G.P. Patil .ed ;*Pergamon Press and Statistical publishing Society, Calcutta*, pp-320-332.

[8]. Raqab, M. Z., and Kundu, D. (2005). Comparison of different estimators of P(Y < X) for a scaled Burr Type X distribution. Communications in Statistcs – Simulation and Computation, 34, 465-483.

[9]. Schwarz, G. (1987). Estimating the dimension of a model. *Ann. Stat.*, 5, 461–464.

[10]. Shanker, R., & Mishra, A. (2013). A quasi Lindley distribution. *African Journal of Mathematics and Computer Science Research, 6*(4), 64-71, DOI 10.5897/AJMCSR 12.067

[11]. Warren, W. G. (1975). Statistical Distributions in Forestry and Forest Products Research. *A Modern Course on Statistical Distributions in Scientific Work*, 369-384. doi:10.1007/978-94-010-1845-6_27

