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Oscillation Criteria of Second Order Nonlinear Impulsive Neutral Differential Equations

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Abstract- In this paper, we study the oscillatory behavior of second order non-linear impulsive neutral differential equations. By using the generalized Ricatti transformation and the integral averaging technique, we obtained some new oscillation criteria. Examples are given to show how impulse perturbations greatly affect the oscillation behavior of the solutions.

AMS Subject Classification: 34C10, 34A37

Keywords and Phrases: Oscillation, Second Order, Nonlinear, Impulsive, Neutral Differential Equations.

I. Introduction

We are concerned with the oscillation of second order nonlinear impulsive neutral differential equation of the form

$$\begin{cases} (r(t)[z'(t)]^{\alpha})' + q(t)f\left(t, x\left(\tau(t)\right)\right) = 0, & t \neq \theta_k, \\ \Delta r(t)[z'(t)]^{\alpha}|_{t=\theta_k} + b_k q(\theta_k) f\left(\theta_k, x\left(\tau(\theta_k)\right)\right) = 0, & t \in [t_0, \infty), \quad k \in \mathbb{N}, \end{cases} \qquad (1)$$

where $z(t) = x(t) - p(t)x\left(\sigma(t)\right)$, and $\Delta[u(t)]|_{t=\theta} = u(\theta^+) - u(\theta^-)$ in which $u\left(\theta^{\mp}\right) = \lim_{t\to\theta^{\mp}} u(t)$. For convenience we define $u(\theta) = u(\theta^-)$.

Throughout this paper we assume the following conditions to hold:

(H_1) $r(t) \in C'\left([t_0, \infty), (0, \infty)\right), r'(t) > 0, p \in C\left([t_0, \infty)\right), \int_{t_0}^{\infty} [r(s)]^{-1/\alpha} ds = \infty$;

(H_2) $p, q \in C\left([t_0, \infty)\right), q(t) \geq 0, 0 \leq p(t) \leq 1$;

(H_3) $\sigma(t) \in C([t_0, \infty), \mathbb{R}), \tau(t) \in C'([t_0, \infty), \mathbb{R}), \sigma(t) \leq t, \tau(t) \leq t, \tau(t) \to \infty$ as $t \to \infty$, $\sigma(t) \to \infty$ as $t \to \infty, \sigma \circ \tau = \tau \circ \sigma$;

(H_4) α is a quotient of odd positive integers;

(H_5) $\{\theta_k\}$ is a fixed strictly increasing unbounded sequence of positive real numbers and $\{b_k\}$ is a sequence of positive real numbers;

(H_6) $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R}), uf(u) > 0$ for all $u \neq 0$ and there exists a positive constant c such that $\frac{f(u)}{u^{\alpha}} \geq c$ for all $u \neq 0$.

By a solution of equation (1) we mean a function $x(t)$ which is defined on $[T_x, \infty)$ with $T_x \geq t_0$ such that $x, x', x'' \in PLC(J, \mathbb{R})$ and $x(t)$ satisfies the equation (1), where $PLC(J, \mathbb{R})$ denotes the set of all real-valued function $g(t)$ defined on $J \subset [t_0, \infty)$ such that $g(t)$ is continuous for all $t \in J$ except possibly at $t = \theta_k$, where $v\left(\theta_k^{\mp}\right)$ exist and $v(\theta_k) = v(\theta_k^-)$.

Oscillation theory is one of the directions which initiated the investigation of the qualitative properties of differential equations. This theory started with the classical works of Sturm and Kneser, and still attracts the attention of many mathematicians as much for the interesting results obtained as for their various applications.

The attractiveness of the oscillation theory links rather strongly the occurrence of new objects to be investigated. Such fast development can be observed in studying, the oscillatory properties of the impulsive differential equations. The paper of K. Gopalsamy and B. G. Zhang [4] is the first investigation on oscillatory properties of impulsive differential equations. For further applications and questions concerning existence and uniqueness of solutions of impulsive differential equation, see for example [8] and the references cited there in.

Compared to equations without impulses, little has been known about the oscillation problem for impulsive differential equations due to difficulties caused by impulsive perturbation, see for example [1-3, 6, 7 ,13, 17-19] and the references cited therein.

When $p(t) = 0$ and $\tau(t) = t$ and $q(t) = 1$ equation (1) reduces to the following second order nonlinear impulsive differential equation

$$\begin{cases}(r(t)[x'(t)]^\alpha)' + f\left(t, x(\tau(t))\right) = 0, & t \neq \theta_k, \\ \Delta(r(t)[x'(t)]^\alpha)|_{t=\theta_k} + b_k f(\theta_k, x(\theta_k)) = 0, & t \in [t_0, \infty), \quad k \in \mathbb{N},\end{cases}$$

which received a lot of attention in the literature. The main objective of this paper is to establish oscillation for the second order nonlinear impulsive neutral differential equation (1). By introducing the auxiliary function $\rho \in C'[t_0, \infty)$ and a function $H(t, s)$ defined below, we establish some new oscillation criteria for equation (1) which complement the oscillation theory of impulsive differential equations. Examples are provided to illustrate the main results.

This paper is organized as follows. In Section 2 we prove our main Theorems. To illustrate our results, examples are provided in Section 3.

II. MAIN RESULTS

In this section, we obtain the oscillation criteria for the solutions of equation (1).

Lemma 1 (16). Let $g(u) = Bu - Au^{\frac{\alpha+1}{\alpha}}$ where $A > 0$ and B are constants, α is a quotient of odd positive integers. Then g attains its maximum value on ℝ, at

$$u^* = \frac{\alpha^\alpha B^\alpha}{A^\alpha(\alpha+1)^\alpha}$$

and

$$max(g) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$$

Theorem 1. Assume that the conditions (H_1) to (H_6) hold. If there exists a differentiable function $\rho(t)$ such that $\rho'(t) > 0, \tau'(t) > 0, \alpha \geq 1$ and

$$\limsup_{t\to\infty} \left[\int_{t_1}^{t} \left(cq(s)\rho(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{[\rho'(s)]^{\alpha+1} r(\tau(s))}{\rho^\alpha(s)[\tau'(s)]^\alpha}\right) ds + \sum_{t_1 \leq \theta_k < t} c_k b_k \rho(\theta_k) q(\theta_k)\right] = \infty, \qquad (\mathbf{2})$$

then the impulsive differential equation (1) is oscillatory.

Proof. Suppose to the contrary that equation (1) has a non-oscillatory solution $x(t)$. Without loss of generality we may assume that $x(t)$ is positive, then there exist $t_1 \geq t_0$ sufficiently large such that $x(\sigma(t)) > 0$ and $x(\tau(t)) > 0$.
In view of equation (1), we obtain

$$(r(t)[z'(t)]^\alpha)' \leq -cq(t)x^\alpha(\tau(t)) \leq 0, t \geq t_1, t \neq \theta_k,$$

which implies that $(r(t)[z'(t)]^\alpha)'$ is non increasing on each interval (θ_k, θ_{k+1}). If $t = \theta_k$, then

$$r(\theta_k^+)[z'(\theta_k^+)]^\alpha - r(\theta_k^-)[z'(\theta_k^-)]^\alpha = -b_k c_k q(\theta_k) x^\alpha(\tau(\theta_k)) \leq 0$$

which means that $\Delta r(t)[z'(t)]^\alpha|_{t=\theta_k} \leq 0$. Thus $r(t)[z'(t)]^\alpha$ is non increasing on $[t_1, \infty)$. We may claim that $z'(t)$ is eventually non negative. In fact, if $z'(t^*) < 0$ for some $t^* \geq t_1$, then

$$r(t)[z'(t)]^\alpha \leq r(t^*)[z'(t^*)]^\alpha \text{ for all } t \geq t^*.$$

Integrating from t^* to t, we have

$$z(t) \leq z(t^*) + [r(t^*)]^{\frac{1}{\alpha}} z'(t^*) \int_{t^*}^{t} [r(s)]^{\frac{-1}{\alpha}} ds. \qquad (\mathbf{3})$$

Taking limit as t→ ∞ and using the hypothesis (H_1) in (3) we see that $z(t)$ must be eventually negative, a contradiction. Therefore, our claim is true.
From (1) and using (H_5), we have

$$(r(t)[z'(t)]^\alpha)' \leq -cq(t)x^\alpha(\tau(t)) \leq 0, \quad t \neq \theta_k, \qquad (\mathbf{4})$$

since $z(t) = x(t) - p(t)x(\sigma(t))$, we have

$$x(t) = z(t) + p(t)x(\sigma(t)) \geq z(t) \qquad (\mathbf{5})$$

From (4) and (5), we have

$$(r(t)[z'(t)]^{\alpha})' \le -cq(t)z^{\alpha}(\tau(t)), \qquad t \ne \theta_k. \tag{6}$$

Define

$$w(t) = \rho(t)\frac{r(t)[z'(t)]^{\alpha}}{z^{\alpha}(\tau(t))}, \qquad t \ne \theta_k. \tag{7}$$

Then $w(t) > 0$. Differentiating w(t) and using (6), we get

$$w'(t) \le \frac{\rho'(t)}{\rho(t)}w(t) - cq(t)\rho(t) - \alpha\frac{w(t)z'(\tau(t))\tau'(t)}{z(\tau(t))}. \tag{8}$$

Since

$$r(t)[z'(t)]^{\alpha} \le r(\tau(t))[z'(\tau(t))]^{\alpha}, \tag{9}$$

we have from (8) and (9),

$$w'(t) \le \frac{\rho'(t)}{\rho(t)}w(t) - cq(t)\rho(t) - \alpha\frac{w(t)z'(t)[r(t)]^{\frac{1}{\alpha}}\tau'(t)}{z(\tau(t))[r(\tau(t))]^{\frac{1}{\alpha}}}, \le \frac{\rho'(t)}{\rho(t)}w(t) - cq(t)\rho(t) - \alpha w^{\frac{\alpha+1}{\alpha}}(t)\frac{\tau'(t)}{[\rho(t)]^{\frac{1}{\alpha}}[r(\tau(t))]^{\frac{1}{\alpha}}},$$

$$\le -cq(t)\rho(t) + Bw(t) - Aw^{\frac{\alpha+1}{\alpha}}(t),$$

where $A = \alpha\frac{\tau'(t)}{[\rho(t)]^{\frac{1}{\alpha}}[r(\tau(t))]^{\frac{1}{\alpha}}}$ and $B = \frac{\rho'(t)}{\rho(t)}$. By using, Lemma 2.2, we have

$$w'(t) \le -cq(t)\rho(t) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}},$$

$$w'(t) \le -cq(t)\rho(t) + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho'(t))^{\alpha+1}}{\rho^{\alpha}(t)}\frac{r(\tau(t))}{(\tau'(t))^{\alpha}}. \tag{10}$$

$$\Delta w(t)|_{t=\theta_k} = \rho(\theta_k)\frac{r(\theta_k)[z'(\theta_k)]^{\alpha}}{[z(\tau(\theta_k))]^{\alpha}}, = -\rho(\theta_k)b_k q(\theta_k)\frac{x^{\alpha}(\tau(\theta_k))}{[z(\tau(\theta_k))]^{\alpha}},$$

$$\Delta w(t)|_{t=\theta_k} \le -c_k b_k \rho(\theta_k)q(\theta_k). \tag{11}$$

In view of (11) and

$$\int_{t_1}^{t} w'(s)ds = w(t) - w(t_1) - \sum_{t_1 \le \theta_k < t}\Delta w(\theta_k), \tag{12}$$

if we integrating (10) from t_1 to t, then

$$w(t) \le w(t_1) - \left[\int_{t_1}^{t}\left(c\rho(s)q(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho'(s))^{\alpha+1}}{\rho^{\alpha}(s)}\frac{r(\tau(s))}{(\tau'(s))^{\alpha}}\right)ds + \sum_{t_1 \le \theta_k < t} c_k b_k \rho(\theta_k)q(\theta_k)\right].$$

Taking $\lim_{t\to\infty} sup$, we obtain $w(t) \to -\infty$ which contradicts with $w(t)$ is positive, then equation (1) is oscillatory.

Next, let us introduce the class of functions P defined as in [14, 15] which will be extensively used in the sequel. Let $D = \{(t,s): t \ge s \ge 0\}$. The function $H \in C'(D, \mathbb{R}_+)$ is said to belong to the class P denoted by $H \in P$, if

(i) $H(t,t) = 0, \quad t \ge 0$

(ii) $H(t,s) > 0$ on D,

(iii) $\frac{\partial H(t,s)}{\partial s} \ge 0$ for all $(t,s) \in D$.

Theorem 2. Assume conditions H_1 to H_6 hold. If there exists a positive differentiable function $\rho(t)$ and a function $H \in P$ such that

$$\limsup_{t\to\infty}\left[\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(cH(t,s)\rho(s)q(s) - \frac{\left[\frac{\partial H}{\partial s} + H(t,s)\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1} r(\tau(s))\rho(s)}{[H(t,s)\tau'(s)]^{\alpha}}ds\right) + \sum_{t_1 \le \theta_k < t} c_k b_k H(t,\theta_k)\rho(\theta_k)q(\theta_k)\right] = \infty, \tag{13}$$

then the impulsive differential equation (1) is oscillatory.

Proof. Let $x(t)$ be a non oscillatory solution of equation (1). Proceeding as in the proof of Theorem 1, we have the following

$$w'(t) \le \frac{\rho'(t)}{\rho(t)} w(t) - cq(t)\rho(t) - \alpha w^{\frac{\alpha+1}{\alpha}}(t) \frac{\tau'(t)}{[\rho(t)]^{\frac{1}{\alpha}}\left[r\left(\tau(t)\right)\right]^{\frac{1}{\alpha}}}, \qquad (\mathbf{14})$$

$$\Delta w(t)|_{t=\theta_k} \le -b_k c_k \rho(\theta_k) q(\theta_k). \qquad (\mathbf{15})$$

Multiplying (14) by $H(t,s)$, we obtain

$$cH(t,s)\rho(t)q(t) \le H(t,s)\frac{\rho'(t)}{\rho(t)} w(t) - H(t,s)w'(t) - \alpha H(t,s) w^{\frac{\alpha+1}{\alpha}}(t) \frac{\tau'(t)}{[\rho(t)]^{\frac{1}{\alpha}}\left[r\left(\tau(t)\right)\right]^{\frac{1}{\alpha}}}, \qquad (\mathbf{16})$$

Integrating the last inequality from t_1 to t, using the equation (12) and inequality (15), we obtain

$$\int_{t_1}^{t} cH(t,s)\rho(s)q(s)ds \le \int_{t_1}^{t} H(t,s)\frac{\rho'(s)}{\rho(s)} w(s)ds - \int_{t_1}^{t} H(t,s)w'(s)ds - \alpha \int_{t_1}^{t} H(t,s) w^{\frac{\alpha+1}{\alpha}}(s) \frac{\tau'(s)}{[\rho(s)]^{\frac{1}{\alpha}}\left[r\left(\tau(s)\right)\right]^{\frac{1}{\alpha}}} ds - \sum_{t_1 \le \theta_k < t} b_k c_k H(t,\theta_k)\rho(\theta_k)q(\theta_k). \qquad (\mathbf{17})$$

Since

$$\int_{t_1}^{t} H(t,s)w'(s)ds = -H(t,t_1)w(t_1) - \int_{t_1}^{t} w(s)\frac{\partial H}{\partial s} ds \qquad (\mathbf{18})$$

From inequality (17) and the equation (18), we have

$$\int_{t_1}^{t} cH(t,s)\rho(s)q(s)ds \le H(t,t_1)w(t_1) + \int_{t_1}^{t} \left(\frac{\partial H}{\partial s} + H(t,s)\frac{\rho'(s)}{\rho(s)}\right) w(s)ds - \alpha \int_{t_1}^{t} H(t,s) w^{\frac{\alpha+1}{\alpha}}(s) \frac{\tau'(t)}{[\rho(s)]^{\frac{1}{\alpha}}\left[r\left(\tau(s)\right)\right]^{\frac{1}{\alpha}}} ds - \sum_{t_1 \le \theta_k < t} b_k c_k H(t,\theta_k)\rho(\theta_k)q(\theta_k). \qquad (\mathbf{19})$$

If we choose $A = \alpha H(t,s) \frac{\tau'(t)}{[\rho(s)]^{\frac{1}{\alpha}}\left[r\left(\tau(s)\right)\right]^{\frac{1}{\alpha}}}$ and $B = \frac{\partial H}{\partial s} + H(t,s)\frac{\rho'(s)}{\rho(s)}$, we have from (18)

$$\int_{t_1}^{t} cH(t,s)\rho(s)q(s)ds \le H(t,t_1)w(t_1) + \int_{t_1}^{t} \frac{\alpha^{\alpha}}{(\alpha+1)^{(\alpha+1)}} \frac{B^{\alpha+1}}{A^{\alpha}} ds - \sum_{t_1 \le \theta_k < t} b_k c_k H(t,\theta_k)\rho(\theta_k)q(\theta_k).$$

Therefore

$$\frac{1}{H(t,t_1)}\left[\int_{t_1}^{t} cH(t,s)\rho(s)q(s)ds - \int_{t_1}^{t} \frac{1}{(\alpha+1)^{(\alpha+1)}} \frac{\left[\frac{\partial H}{\partial s} + H(t,s)\frac{\rho'(s)}{\rho(s)}\right]^{\alpha+1} r\left(\tau(s)\right)\rho(s)}{[H(t,s)\tau'(s)]^{\alpha}} ds + \sum_{t_1 \le \theta_k < t} b_k c_k H(t,\theta_k)\rho(\theta_k)q(\theta_k)\right] \le w(t_1). \qquad (\mathbf{20})$$

This limsup in (20), we obtain a contradiction with (13). This completes the proof.

III. EXAMPLE

Example 1 Consider the following second order impulsive type neutral differential equation

$$\begin{cases} \left[\left(\left(x(t)+\left(1-\frac{1}{t^{\frac{1}{2}}}\right)x(t-2)\right)'\right]' + \frac{6}{5t^{\frac{1}{2}}}x\left(\frac{t}{2}\right)=0, t\geq 1, t\neq k \\ \Delta\left(x(t)+\left(1-\frac{1}{t^{\frac{1}{2}}}\right)x(t-2)\right)|_{t=k} + k^3 x\left(\frac{k}{2}\right)=0, t=k. \end{cases} \quad (21)$$

Here, we have $r(t)=1, \alpha=1, p(t)=1-\frac{1}{t^{\frac{1}{2}}}, q(t)=\frac{6}{5t^{\frac{1}{2}}}, \tau(t)=\frac{t}{2}, \sigma(t)=t-2, \theta_k=k, t_0=1$ and $b_k=k^3$. Choose $\rho(t)=t$.

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{1}{r^{\frac{-1}{\alpha}}(s)}ds=\infty.$$

Now

$$\limsup_{t\to\infty}\left[\int_{t_1}^{t}\left(cq(s)\rho(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{[\rho'(s)]^{\alpha+1}\,r(\tau(s))}{\rho^{\alpha}(s)[\tau'(s)]^{\alpha}}\right)ds+\sum_{t_1\leq\theta_k<t}c_k b_k\rho(\theta_k)q(\theta_k)\right]$$
$$=\limsup_{t\to\infty}\left[\int_{1}^{t}\left(c\frac{6s}{5s^{\frac{1}{2}}}-\frac{1}{2s}\right)ds+\frac{6}{5}\sum_{1\leq k<t}c_k k^{\frac{5}{2}}\right]=\infty.$$

Since all conditions of Theorem 1 are satisfied, equation (21) is oscillatory.

Example 2 Consider the following second order impulsive type neutral differential equation

$$\begin{cases} \left[t\left(\left(x(t)+\frac{3}{4}x(t-3)\right)'\right)^3\right]' + \frac{1}{t^6}x^3(t-1)=0, t\geq 1, t\neq k \\ \Delta\left[\sqrt{t}\left(\left(x(t)+\frac{3}{4}x(t-3)\right)'\right)^3\right]_{t=k} + kx^3(k-1)=0, t=k. \end{cases} \quad (22)$$

Here, we have $r(t)=t, \alpha=3, p(t)=\frac{3}{4}, q(t)=\frac{9}{t^2}, \tau(t)=t-1, \sigma(t)=t-3, \theta_k=k, c_k=1$ and $b_k=k^5$. Choose $\rho(t)=t^3$.

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{1}{r^{\frac{-1}{\alpha}}(s)}ds=\infty.$$

Now

$$\limsup_{t\to\infty}\left[\int_{t_1}^{t}\left(cq(s)\rho(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{[\rho'(s)]^{\alpha+1}\,r(\tau(s))}{\rho^{\alpha}(s)[\tau'(s)]^{\alpha}}\right)ds+\sum_{t_1\leq\theta_k<t}c_k b_k\rho(\theta_k)q(\theta_k)\right]$$
$$=\limsup_{t\to\infty}\left[\int_{1}^{t}\left(\frac{c}{s^3}-\frac{3^4}{4^4}\frac{s-1}{s^4}\right)ds+\sum_{1\leq k<t}k^2\right]=\infty.$$

So, by Theorem 1, every solution of equation (22) is oscillatory.

The above example shows that the impulses play a very important role in the oscillatory behavior of equation under perturbing impulses.

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REFERENCES

[1]. L. Berenzansky and E. Braverman, On oscillation of a second order impulsive delay differential equation, J. Math. Anal. Appl., 39 (2000), 217-225.

[2]. E. M. Bonotto, L. P. Gimenes and M. Federson, Oscillation for a second order neutral differential equation with impulses, Cad. Mat., 9 (2008),169-190.
[3]. L. P. Gimenes and M. Federson, Oscillation by impulses for a second order delay differential equation, Cad. Mat., 06 (2005), 181-191.
[4]. K. Gopalsamy and B. G. Zhang, On delay differential equations with impulses, J. Math. Anal. Appl., 139 (1989), 110-122.
[5]. I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, (1991).
[6]. C. Guo and Z. Xu, On the oscillation of second order linear impulsive differential equations, Diff. Eqns. Apps., 3 (2010), 319-330.
[7]. Z. He and W. Ge, Oscillations of second order nonlinear impulsive ordinary differential equations, J. Copmut. Appl. Math., 158(2) (2003),397-406.
[8]. V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore 32 (1989).
[9]. W. Li, C. Yang and H. Ji, Oscillation of a class of second order nonlinear ODE with impulses, Appl. Math. Comput., 138(2-3) (2003), 181-188.
[10]. X. Li, Oscillation properties of second order delay differential – difference equations with impulses, Adv. Appl. Math. Anal., 3 (2008), 55-66.
[11]. M. Peng, Oscillation caused by impulses, J. Math. Anal. Appl., 255 (2001), 163-176.
[12]. M. Peng and W. Ge, Oscillation criteria for second order nonlinear differential equations with impulses, Compu. Math. Appl., 39 (2000), 217-255.
[13]. M. Peng and R. P. Agarwal, Oscillation theorems of second order non linear neutral delay differential equations under impulsive perturbations, Indian. J. Pure. Appl. Math., 33(7) (2002), 1017-1029.
[14]. CH. G. Philos, Oscillation theorems for linear differential equation of second order, Arch. Math., 53 (1989), 482-492.
[15]. Y V. Rogovchenko, Oscillation criteria for certain nonlinear differential equations, J. Math. Anal. Appl., 229 (1999), 399-416.
[16]. A. A. Soliman, R. A. Sallam, A. M. Hassan, Oscillation criteria of second order nonlinear neutral differential equations, Int. J. of App. Math. Research., 1 (3) (2012), 314 - 322.
[17]. J. Yan, Oscillation properties of a second order impulsive delay differential equation, Comp. Math. Appl., 47 (2004), 253-258.
[18]. Y. Zhang, A. Zhao and J. Yan, Oscillation criteria for impulsive delay differential equations, J. Math. Anal. Appl., 205 (1997), 461-470.
[19]. A. Zhao and J. Yan, Existence of positive solutions for delay differential equations with impulses, J. Math. Anal. Appl., 210 (1997), 667-678.

