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Fixed point Theorems of Multivalued Mappings in Cone Metric Spaces via Cone C-Class function

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Abstract— Let $\boldsymbol{P}$ be a subset of a Banach space $\boldsymbol{E}$ and $\boldsymbol{P}$ is normal and regular cone on $\boldsymbol{E}$, we prove the existence of the fixed point for multi valued maps and φ–ψ- contractive mappings in cone metric spaces via cone C class functions.

Keywords— *Cone metric space, Multivalued mappings, Fixed point, Cone C class function*

I. INTRODUCTION

In recent years, several authors (see [1-5]) have studied the strong convergence to a fixed point with contractive constant in cone metric spaces. Seong Hoon Cho and Mi sun Kim [5] have proved certain fixed point theorems by using Multivalued mapping in the setting of contractive constant in metric spaces. Note on $\varphi - \psi$ -cotractive type mappings and related fixed point are proved by Arslan Hojat Ansari [8]. Fixed point theorems of Multivalued mappings in Cone metric spaces proved by Dr.M.Marudai and Dr.R.Krishnakumar [1].

II. PRELIMINARIES

Definition 1.1: Let E be a Banach space and a subset, P of E is said to be a cone if it satisfies the following conditions
(i)$P \neq \emptyset$ and P is closed;

(ii) $ax + by \in P\ \forall\ x, y \in P$ and a, b are non-negative real numbers

(iii) $P \cap (-P) = \emptyset$

Given a cone $P \subset E$,we define a partial ordering $\leq$ with respect to the cone P by $x \leq y$ if and only if $y - x \in P$. If $y - x \in interior\ of\ P$, then it is denoted by $x \ll y$. The cone P is said to be Normal if a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The cone P is called regular if every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below convergent.

Definition 1.2 : Let X be a non-empty set, and suppose the mapping $d: X \times X \to E$ is said to be a cone metric space if it satisfies
(i) $0 \leq d(x, y) \forall\ x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$
(iii) $d(x, y) = d(x, z) + d(z, y)$ for all $x, y, z \in X$
Example 1.3: Let $E = R^2, P = \{(x, y) \in E; x, y \geq 0\}, X = R$ and $d: X \times X \to E$ defined by
$$d(x, y) = (|x - y|, \alpha|x - y|)$$
Where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.
Definition 1.4: Let (X, d) be cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then
(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

Definition 1.5. Let (X, d) is said to be a complete cone metric space, if every Cauchy sequence is convergent in X

Definition1.6: Let (X, d) be a metric space. We denote $CB(X)$ the family of nonempty closed bounded subset of X. Let $H(.,.)$ be the Hausdorff distance on $CB(X)$.
That is, for $A, B \in CB(X)$
$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$
Where $d(a,B) = \inf\{d(a,b); b \in B\}$ is the distance from the point a to the subset B. An element $x \in X$is said to be a fixed point of a multi-valued mapping $T: X \to 2^X$ if $x \in T(X)$

Definition 1.7: A function $\psi: P \to P$ is called an altering distance function if the following properties are satisfied:
(i) ψ is non-decreasing and continuous
(ii) $\psi(t) = 0$ if and only if $t = 0$

Definition 1.8. : An ultra altering distance function is a continuous, non decreasing mapping $\varphi: P \to P$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$
We denote this set with Φ_u

Definition 1.9.: A mapping $f: P^2 \to P$ is called cone C −class function if it is continuous and satisfies following axioms:
(1) $F(s,t) < s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in P$
We denote cone C −class functions as $\mathcal{C}$
Example 2.9 : The following functions $F: P^2 \to P$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:
(i) $F(s,t) = s - t$
(ii) $F(s,t) = ks$, where $0 < k < 1$,
(iii) $F(s,t) = s\beta(s)$, where $\beta: [0,\infty) \to [0,1)$,
(iv) $F(s,t) = \Psi(s)$, where $\Psi: \mathrm{P} \to \mathrm{P}, \Psi(0) = 0, \Psi(\mathrm{s}) > 0$ for all $s \in P$ with $s \neq 0$ and $\Psi(s) \leq s$ for all $s \in P$
(v) $F(s,t) = s - \varphi(s)$, where $\varphi: [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(vi) $F(s,t) = s - h(s,t)$, where $h: [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(s,t) = 0 \Leftrightarrow t = 0$ for all $t, s > 0$
(vii) $F(s,t) = \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi: [0,\infty) \to [0,\infty)$ is a upper semi continuous function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$

Lemma 1.10: Let ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ and $\{s_n\}$ a decreasing sequence in P such that

$$\psi(s_{n+1}) \leq F(\psi(s_n), \varphi(s_n))$$

For all $n \geq 1$. Then $\lim_{n\to\infty} s_n = 0$

III. MAIN RESULTS

Theorem 2.1: Let (X, d) be a complete cone metric space and the mapping $T: X \to CB(X)$ be multivalued map satisfying for each $x, y \in X$

$$\psi\big(H(Tx,Ty)\big) \leq F(\psi\big[a[d(x,Tx) + d(y,Ty)] + b[d(x,Ty) + d(Tx,y)]\big]$$

$$\phi(a[d(x,Tx) + d(y,Ty)] + b[d(x,Ty) + d(Tx,y)])$$

for all $x, y \in X$ and $a + b < \frac{1}{2}$, $a, b \in \left[0, \frac{1}{2}\right)$. ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s)$. Then T has a fixed point in X

Proof: for every $x_0 \in X$ and $n \geq 1, x_1 \in Tx_0$ *and* $x_{n+1} \in Tx_n$

$$\psi\big(d(x_{n+1}, x_n)\big) \leq \psi(H(Tx_n, Tx_{n-1}))$$

$$\leq F(\psi(a[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] + b[d(x_n, Tx_{n-1}) + d(Tx_n, x_{n-1})]),$$

$$\phi(a[d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1})]+b[d(x_n,Tx_{n-1})+d(Tx_n,x_{n-1})]\,))$$
$$\leq F(\psi(a[d(x_n,x_{n+1})+d(x_{n-1},x_n)]+b[d(x_n,x_n)+d(x_{n+1},x_{n-1})]),$$
$$\varphi(a[d(x_n,x_{n+1})+d(x_{n-1},x_n)]+b[d(x_n,x_n)+d(x_{n+1},x_{n-1})])\,)$$
$$\leq F(\psi(a[d(x_n,x_{n+1})+d(x_{n-1},x_n)]+b[d(x_{n+1},x_{n-1})]),$$
$$\varphi(a[d(x_n,x_{n+1})+d(x_{n-1},x_n)]+b[d(x_{n+1},x_{n-1})])\,)$$
$$\leq F(\psi(a[d(x_n,x_{n+1})+d(x_{n-1},x_n)]+b[d(x_{n+1},x_n)+d(x_n,x_{n-1})]),$$
$$\varphi(a[d(x_n,x_{n+1})+d(x_{n-1},x_n)]+b[d(x_{n+1},x_n)+d(x_n,x_{n-1})])\,)$$
$$\leq F(\psi\big((a+b)[d(x_n,x_{n+1})+d(x_{n-1},x_n)]\big),\varphi\big((a+b)[d(x_n,x_{n+1})+d(x_{n-1},x_n)]\big))$$
$$\leq \psi\big((a+b)[d(x_n,x_{n+1})+d(x_{n-1},x_n)]\big)$$

$d(x_{n+1},x_n)\leq Ld(x_{n-1},x_n)$ where $L=\frac{a+b}{1-(a+b)}$

$d(x_{n+1},x_n)\leq L^n d(x_1,x_0)$

For $n>m$ we have

$$d(x_n,x_m)\leq d(x_n,x_{n-1})+d(x_{n-1},x_{n-2})+\cdots+d(x_{m+1},x_m)$$
$$\leq [L^{n-1}+L^{n-2}+\cdots+L^m]d(x_1,x_0)$$
$$\leq \frac{L^m}{(1-L)}d(x_1,x_0)$$

Let $0\ll c$ be given, choose a natural number N_1 such that $\frac{L^m}{(1-L)}d(x_1,x_0)\ll c$ for all $m\geq N_1$ this implies $d(x_n,x_m)\ll c$. For $n>m$, $\{x_n\}$ is a Cauchy sequence in (X,d) is a complete cone metric space, there exists $p\in X$ such that $x_n\to p$. Choose a natural number N_2 such that $d(x_n,P)\ll\frac{c(1-L)}{3}$, for all $n\geq N_2$. Hence for $n\geq N_2$ we have $d(x_n,P)\ll\frac{c(1-K)}{3}$ where

$k=a+b$

$$\psi(d(Tp,P))\leq\psi(H(Tx_n,Tp)+d(Tx_n,p))$$
$$\leq F(\psi(a[d(x_n,Tx_n)+d(p,Tp)]+b[d(x_n,Tp)+d(Tx_n,p)]+d(x_{n+1},p)),$$
$$\varphi(a[d(x_n,Tx_n)+d(p,Tp)]+b[d(x_n,Tp)+d(Tx_n,p)]+d(x_{n+1},p))\,)$$
$$\leq F(\psi(a[d(x_n,x_{n+1})+d(p,Tp)]+b[d(x_n,Tp)+d(x_{n+1},p)]+d(x_{n+1},p))$$
$$,\varphi(a[d(x_n,x_{n+1})+d(p,Tp)]+b[d(x_n,Tp)+d(x_{n+1},p)]+d(x_{n+1},p))$$
$$\leq F(\psi(a[d(x_n,x_{n+1})+d(p,Tp)]+b[d(x_n,Tp)+d(p,Tp)+d(x_{n+1},p)]+d(x_{n+1},p)),$$
$$\varphi\big(a[d(x_n,x_{n+1})+d(p,Tp)]+b[d(x_n,Tp)+d(p,Tp)+d(x_{n+1},p)]+d(x_{n+1},p)\big)$$
$$\leq F(\psi(ad(x_n,x_{n+1})+ad(p,Tp)+bd(x_n,Tp)+bd(p,Tp)+bd(x_{n+1},p)+d(x_{n+1},p)),$$
$$\varphi(ad(x_n,x_{n+1})+ad(p,Tp)+bd(x_n,Tp)+bd(p,Tp)+bd(x_{n+1},p)+d(x_{n+1},p))\,)$$
$$\leq (ad(x_n,x_{n+1})+ad(p,Tp)+bd(x_n,Tp)+bd(p,Tp)+bd(x_{n+1},p)+d(x_{n+1},p)$$
$$(1-k)d(Tp,p)\leq kd(x_n,Tp)+kd(x_{n+1},p)+d(x_{n+1},p)$$
$$\leq d(x_n,Tp)+d(x_{n+1},p)+d(x_{n+1},p)$$
$$d(Tp,p)\leq\frac{[d(x_n,Tp)+d(x_{n+1},p)+d(x_{n+1},p)]}{(1-k)}$$
$$d(Tp,p)\ll\frac{c}{3}+\frac{c}{3}+\frac{c}{3}$$
$$d(Tp,p)\ll c$$

For all $n\geq N_2$, $d(Tp,p)\ll\frac{c}{m}$ for all $m\geq 1$, we get $\frac{c}{m}-d(Tp,p)\in P$ and $m\to\infty$ we get $\frac{c}{m}\to 0$ and P is closed $d(Tp,p)\in P$ bu $d(Tp,p)\in P$

$\therefore d(Tp,p)=0$ and so $p\in Tp$.

Corollary 2.1: Let (X,d) be a complete cone metric space and the mapping $T\colon X\to CB(X)$ be multivalued map satisfying for each $x,y\in X$

$$\psi\big(d(Tx,Ty)\big)\leq F(\psi(a[d(x,Tx)+d(y,Ty)]),\varphi(a[d(x,Tx)+d(y,Ty)])\,)$$

for all $x,y\in X$ and $a\in\left[0,\frac{1}{2}\right)$. ψ and φ are altering distance and ultra altering distance functions respectively, $F\in\mathcal{C}$ such that $\psi(t+s)\leq\psi(t)+\psi(s)$. Then T has a fixed point in X

Proof: The proof of the corollary immediately follows by putting $b=0$ in the previous theorem.

Theorem 2.2: Let (X,d) be a complete cone metric space and the mapping $T\colon X\to CB(X)$ be multivalued map satisfy the condition

$$\psi\big(H(Tx,Ty)\big)\leq F(\psi(r\max\{d(x,y),d(x,Tx),d(y,Ty)\}),\varphi(r\max\{d(x,y),d(x,Tx),d(y,Ty)\})\,)$$

For all $x, y \in X$ and $r \in [0,1)$. ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t+s) \le \psi(t) + \psi(s)$. Then T has a fixed point in X

Proof: for every $x_0 \in X$ and $n \ge 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$

$$\psi\big(d(x_{n+1}, x_n)\big) \le \psi\big(H(Tx_n, Tx_{n-1})\big)$$
$$\le F(\psi(r\max\{d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Tx_{n-1})\}), \varphi(r\max\{d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Tx_{n-1})\}))$$
$$\le F(\psi(r\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n)\}), \varphi(r\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n)\}))$$
$$\le F(\psi(rd(x_n,x_{n-1})), \varphi(rd(x_n,x_{n-1})))$$
$$\le r^n d(x_1, x_0)$$

For $n > m$ we have

$$d(x_n, x_m) \le d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)$$
$$\le [r^{n-1} + r^{n-2} + \cdots + r^m] d(x_1, x_0)$$
$$\le \frac{r^m}{(1-r)} d(x_1, x_0)$$

Let $0 \ll c$ be given, choose a natural number N_1 such that $\frac{r^m}{(1-r)} d(x_1,x_0) \ll c$ for all $m \ge N_1$ this implies $d(x_n, x_m) \ll c$. For $n > m$, $\{x_n\}$ is a Cauchy sequence in (X,d) is a complete cone metric space, there exists $p \in X$ such that $x_n \to p$. Choose a natural number N_2 such that $d(x_n, P) \ll \frac{c}{3}$, for all $n \ge N_2$. Hence for $n \ge N_2$ we have $d(x_n, P) \ll \frac{c}{3}$

$$\psi\big(d(Tp, P)\big) \le \psi\big(H(Tx_n, Tp) + d(Tx_n, p)\big)$$
$$\le F(\psi(r\max\{d(x_n,p), d(x_n,Tx_n), d(p,Tp)\} + d(x_{n+1},p)),$$
$$\varphi(r\max\{d(x_n,p), d(x_n,Tx_n), d(p,Tp)\} + d(x_{n+1},p)))$$
$$\le F(\psi(r\max\{d(x_n,p), d(x_n,x_{n+1}), d(p,Tp)\} + d(x_{n+1},p)),$$
$$\varphi(r\max\{d(x_n,p), d(x_n,x_{n+1}), d(p,Tp)\} + d(x_{n+1},p)))$$
$$\le F(\psi(r\max\{d(x_n,p), d(x_n,p) + d(p,x_{n+1}), d(p,Tp)\} + d(x_{n+1},p)),$$
$$\varphi(r\max\{d(x_n,p), d(x_n,p) + d(p,x_{n+1}), d(p,Tp)\} + d(x_{n+1},p))$$
$$d(Tp, P) \ll c$$

For all $n \ge N_2$, $d(Tp,p) \ll \frac{c}{m}$ for all $m \ge 1$, we get $\frac{c}{m} - d(Tp,p) \in P$ and $m \to \infty$ we get $\frac{c}{m} \to 0$ and P is closed $-d(Tp,p) \in P$ bu $d(Tp,p) \in P$

$\therefore d(Tp,p) = 0$ and so $p \in Tp$.

Corollary 2.2: Let (X,d) be a complete cone metric space and the mapping $T: X \to CB(X)$ be multivalued map satisfy the condition

$$H(Tx, Ty) \le kd(x,y)$$

For all $x, y \in X$ and $k \in [0,1)$. ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t+s) \le \psi(t) + \psi(s)$. Then T has a fixed point in X

Proof: The proof of the corollary immediately follows by taking $d(x,y)$as maximum value in previous theorem.

Note 2.3: We prove the above theorems in the setting of P is a normal cone with normal constant K

Theorem 2.4: Let (X,d) be a complete cone metric space and P a normal cone with normal constant K. Suppose the mapping $T: X \to CB(X)$ be multivalued map satisfy the condition

$$\psi\big(H(Tx,Ty)\big) \le F(\psi(r\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(Tx,y)\}),$$
$$\varphi(r\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(Tx,y)\}))$$

For all $x, y \in X$ and $r \in [0,1)$. ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t+s) \le \psi(t) + \psi(s)$. Then T has a fixed point in X

Proof: for every $x_0 \in X$ and $n \ge 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$

$$\psi\big(d(x_{n+1}, x_n)\big) \le \psi\big(H(Tx_n, Tx_{n-1})\big)$$
$$\le F(\psi(r\max\{d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_{n-1}), d(Tx_n,x_{n-1})\}),$$
$$\varphi(r\max\{d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_{n-1}), d(Tx_n,x_{n-1})\}))$$
$$\le F(\psi(r\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n), d(x_n,x_n), d(x_{n+1},x_{n-1})\}),$$
$$\varphi(r\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n), d(x_n,x_n), d(x_{n+1},x_{n-1})\}))$$
$$\le F(\psi(r\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n+1},x_{n-1})\}),$$
$$\varphi(r\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n+1},x_{n-1})\}))$$
$$\le F(\psi(r\max\{d(x_n,x_{n-1}), d(x_{n+1},x_{n-1})\}), \varphi(r\max\{d(x_n,x_{n-1}), d(x_{n+1},x_{n-1})\}))$$

$\leq \psi(r \max\{d(x_n, x_{n-1}), d(x_{n+1}, x_{n-1})\}$

Case (i) If $d(x_{n+1}, x_n) \leq rd(x_n, x_{n-1})$ then we get, $d(x_{n+1}, x_n) \leq r^n d(x_1, x_0)$ for $n > m$

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)$$
$$\leq [r^{n-1} + r^{n-2} + \cdots + r^m] d(x_1, x_0)$$
$$\leq \frac{r^m}{(1-r)} d(x_1, x_0)$$

We get $\|d(x_n, x_m)\| \leq K \frac{r^m}{(1-r)} \|d(x_1, x_0)$. $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $p \in X$. such that $x_n \to p$ as $n \to \infty$

$$\psi(d(Tp, P)) \leq \psi(H(Tx_n, Tp) + d(Tx_n, p))$$
$$\leq F(\psi(r \max\{d(x_n, p), d(x_n, Tx_n), d(p, Tp), d(x_n, Tp), d(Tx_n, p)\} + d(x_{n+1}, p)),$$
$$\varphi(r \max\{d(x_n, p), d(x_n, Tx_n), d(p, Tp), d(x_n, Tp), d(Tx_n, p)\} + d(x_{n+1}, p)))$$
$$\leq F(\psi(r \max\{d(x_n, p), d(x_n, x_{n+1}), d(p, Tp), d(x_n, Tp), d(x_{n+1}, p)\} + d(x_{n+1}, p)),$$
$$\varphi(r \max\{d(x_n, p), d(x_n, x_{n+1}), d(p, Tp), d(x_n, Tp), d(x_{n+1}, p)\} + d(x_{n+1}, p)))$$
$$\leq F(\psi(rd(p, Tp)), \varphi(rd(p, Tp)))$$

$d(Tp, P) = 0$. Hence $P \in Tp$

Case (ii) $d(x_{n+1}, x_n) \leq rd(x_{n+1}, x_{n-1})$ then we get

$$d(x_{n+1}, x_n) \leq r[d(x_{n+1}, x_n) + d(x_n, x_{n-1})]$$
$$\leq \frac{r}{1-r} [d(x_n, x_{n-1})]$$
$$\leq h[d(x_n, x_{n-1})] \qquad \text{where } h = \frac{r}{1-r} < 1$$

For $n > m$

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)$$
$$\leq [h^{n-1} + h^{n-2} + \cdots + h^m] d(x_1, x_0)$$
$$\leq \frac{h^m}{(1-h)} d(x_1, x_0)$$

We get $\|d(x_n, x_m)\| \leq K \frac{h^m}{(1-h)} \|d(x_1, x_0)$. $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $p \in X$. such that $x_n \to p$ as $n \to \infty$

$$\psi(d(Tp, P)) \leq \psi(H(Tx_n, Tp) + d(Tx_n, p))$$
$$\leq F(\psi(r \max\{d(x_n, p), d(x_n, Tx_n), d(p, Tp), d(x_n, Tp), d(Tx_n, p)\} + d(x_{n+1}, p)),$$
$$\varphi(r \max\{d(x_n, p), d(x_n, Tx_n), d(p, Tp), d(x_n, Tp), d(Tx_n, p)\} + d(x_{n+1}, p)))$$
$$\leq F(\psi(r \max\{d(x_n, p), d(x_n, x_{n+1}), d(p, Tp), d(x_n, Tp), d(x_{n+1}, p)\} + d(x_{n+1}, p)),$$
$$\varphi(r \max\{d(x_n, p), d(x_n, x_{n+1}), d(p, Tp), d(x_n, Tp), d(x_{n+1}, p)\} + d(x_{n+1}, p)))$$
$$\leq F(\psi(rd(p, Tp)), \varphi(rd(p, Tp)))$$

$d(Tp, P) = 0$. Hence $P \in Tp$

$$\psi(d(p, q)) = \psi(H(Tp, Tq))$$
$$\leq F(\psi(r \max\{d(x, y), d(p, Tp), d(q, Tq), d(p, Tq), d(Tp, q)\}),$$
$$\varphi(r \max\{d(x, y), d(p, Tp), d(q, Tq), d(p, Tq), d(Tp, q)\}))$$
$$\leq F(\psi(r \max\{d(p, q), d(p, p), d(q, q), d(p, q), d(p, q)\}),$$
$$\varphi(r \max\{d(p, q), d(p, p), d(q, q), d(p, q), d(p, q)\}))$$
$$\leq F(\psi(r[d(p, q)]), \varphi(r[d(p, q)]))$$
$$\leq \psi(r[d(p, q)])$$

This is contradiction and hence T has a unique fixed point in X

Corollary 2.3: Let (X, d) be a complete cone metric space and P a normal cone with normal constant K. Suppose the mapping $T: X \to CB(X)$ be multivalued map satisfy the condition

$$\psi(H(Tx, Ty)) \leq F(\psi(r \max\{d(x, y), d(x, Tx), d(y, Ty)\}), \varphi(r \max\{d(x, y), d(x, Tx), d(y, Ty)\}))$$

For all $x, y \in X$ and $r \in [0,1)$.). ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t + s) \leq \psi(t) + \psi(s)$. Then T has a fixed point in X

Proof: The proof of the corollary immediately follows since

$$\max\{d(x, y), d(x, Tx), d(y, Ty)\} \leq \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\}$$

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