International Journal of Scientific Research in Mathematical and Statistical Sciences
Review Paper
Vol.6, Issue.2, pp.184-192, April (2019)
E-ISSN: 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i2.184192

Coupled Common Fixed Point Theorems of *C*–Class Function on Ordered *S*–Metric Spaces

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Available online at: www.isroset.org

Received: 01/Apr/2019, Accepted: 18/Apr/2019, Online: 30/Apr/2019

Abstract– In this paper, we discuss some results on coupled common fixed point theorems of C – Class function on ordered *S* – metric spaces, which are study of generalisation of some existing results are given in form of corollaries.

Keywords–*S–metric space, ordered S–metric space, coupled fixed point, Common fixed point, C–Class function.*

I. INTRODUCTION

In 2012, S. Sedghi et al. introduced the concept of *S*–metric spaces[14]. In 2013, Animesh Gupta discussed the cyclic contraction on *S*–metric spaces[5]. S. Sedghi et al. developed the concept of generalization of fixed point theorems in *S*–metric spaces[15], [16]. In 1984, M.S. Khan, M. Swalech and S. Sessa expanded the research of the metric fixed point theory to a new category by introducing a control function which they called an altering distance function[11]. A.H. Ansari introduced the notion of *C* class function [2], [3] and many authors discussed in common fixed point and *S*–metric space [1], [7], [10], [12].
In this Paper, we proved some Coupled common fixed point theorems using C – Class function on ordered S – metric spaces, which are study of generalisation of some existing results.

The following definitions and properties will be needed in the sequel:

Definition 1.1. [14] Let *X* be a nonempty set. An *S*–metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

(S1) $S(x, y, z) \geq 0$ for all $x, y, z, a \in X$ with $x \neq y \neq z$,

(S2) $S(x, y, z) = 0$ if and only if $x = y = z$,

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an *S*–metric space.

Example 1.2. [5] Let *X* be a non-empty set, *d* is ordinary metric space on *X*, then $S(x, y, z) = d(x, z) + d(y, z)$ is an *S*–metric on *X*.

Lemma 1.3. [15] Let (X, S) be an *S*–metric space. Then we have $S(u,u,v) = S(v,v,u)$.

Definition 1.4. [16] Let (X, S) be an *S*–metric space.

(1) A sequence $\{u_n\}$ in *X* converges to *u* if and only if $S(u_n, u_n, u) \to 0$ as $n \to \infty$. That is, there exists $n_0 \in N$ such that for all $n \geq n_0, S(u_n, u_n, u) \leq \varepsilon$ for each $\varepsilon > 0$. We denote this by $\lim_{n\to\infty} u_n = u$ or $\lim_{n\to\infty} S(u_n, u_n, u) = 0$.

(2) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$. That is there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $S(u_n, u_n, u_m) \leq \varepsilon$ for each $\varepsilon > 0$.

(3) The S–metric space (X, S) is called complete if every Cauchy sequence is convergent.

Lemma 1.5. [16] Let (X, S) be an S–metric space. If there exists sequence $\{x_n\}, \{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma 1.6. [9] Let (X, S) be an S–metric space. Then

$$S(x,x,z) \leq 2S(x,x,y) + S(y,y,z),$$

and

$$S(x,x,z) \leq 2S(x,x,y) + S(z,z,y),$$

for all $x, y, z \in X$.

Definition 1.7. Let $(X, \preceq)$ be partially ordered set. Then $a, b \in X$ are called comparable if $(a \preceq b)$ or $(b \preceq a)$ holds.

Definition 1.8. Let X be a nonempty set. Then $(X, S, \preceq)$ is called an ordered S–metric space if:

(1) (X, S) is an S–metric space,

(2) (X, S) is a partially ordered set.

Definition 1.9. [6] Let $(X, \preceq)$ be partially ordered set and $H : X \times X \to X$. The mapping H is said to has the mixed monotone property if H is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, i.e., for any $a, b \in X$,

$$a_1, a_2 \in X, a_1 \preceq a_2 \Rightarrow H(a_1, b) \preceq H(a_2, b),$$
$$b_1, b_2 \in X, b_1 \preceq b_2 \Rightarrow H(a, b_1) \succeq H(a, b_2).$$

Definition 1.10. [8] Let $(X, \preceq)$ be partially ordered set and suppose $H : X \times X \to X$ and $g : X \to X$. The mapping H is said to has the mixed g–monotone property if $H$is monotone g–non-decreasing in its first argument and is monotone g–non-increasing in its second argument, i.e., for any $a, b \in X$,

$$a_1, a_2 \in X, g(a_1) \preceq g(a_2) \Rightarrow H(a_1, b) \preceq H(a_2, b),$$
$$b_1, b_2 \in X, g(b_1) \preceq g(b_2) \Rightarrow H(a, b_1) \succeq H(a, b_2).$$

Definition 1.11. [6] An element $(a, b) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(a,b) = ga$, $F(b,a) = gb$ and their common coupled fixed point if $F(a,b) = ga = a$ and $F(b,a) = gb = b$.

Definition 1.12. [13] Let X be a non-empty set. Then we say that the mappings $K : X \times X \to X$ and $g : X \to X$ are commutative if $gK(a,b) = K(ga, gb)$.

Definition 1.13. [13] An element $(a, b) \in X \times X$ is called a coupled fixed point of mapping $K : X \times X \to X$ if $K(a,b) = a$ and $K(b,a) = b$.

Definition 1.14. Let (X, S) and (X', S') be two S–metric spaces, and let $f : (X, S) \to (X', S')$ be a function. Then f is said to be continuous at a point $a \in X$ if and only if for every sequence x_n in X, $S(x_n, x_n, a) \to 0$ implies $S'(f(x_n), f(x_n), f(a)) \to 0$. A function f is continuous at X if and only if it is continuous at all $a \in X$.

In the following lemma we see the relationship between a metric and an S–metric.

Lemma 1.15. [4] Let (X, d) be a metric space. Then the following properties are satisfied:

(1) $S(u,v,z) = d(u,z) + d(v,z)$ for all $u, v, z \in X$ is an S–metric on X;

(2) $u_n \to u$ in $\{X, d\}$ if and only if $u_n \to u$ in (X, S_d);

(3) $\{u_n\}$ is Cauchy in $\{X, d\}$ if and only if $\{u_n\}$ is Cauchy in (X, S_d);

(4) $\{X, d\}$ is complete if and only if (X, S_d) is complete.

Definition 1.16. [2]A mapping $F:[0,\infty)^2 \to [0,\infty)$ is called *C*–Class function if it is continuous and satisfies following axioms:

(1) $F(s,t) \le s$;

(2) $F(s,t)=0$ implies that either $s=0$ or $t=0$; for all $s,t \in [0,\infty)$.

Note for some *F* we have that $F(0,0)=0$.

We denote *C*–Class functions as $\mathbb{C}$.

Example 1.17. [2]The following functions $F:[0,\infty)^2 \to R$ are elements of $\mathbb{C}$, for all $s,t \in [0,\infty)$:

(1) $F(s,t)=s-t, F(s,t)=s \Rightarrow t=0$;

(2) $F(s,t)=ms, 0<m<1, F(s,t)=s \Rightarrow s=0$;

(3) $F(s,t)=\dfrac{s}{(1+t)^r}; r\in(0,\infty), F(s,t)=s \Rightarrow s=0$ or $t=0$;

(4) $F(s,t)=\log(t+a^s)/(1+t), a>1, F(s,t)=s \Rightarrow s=0$ or $t=0$;

(5) $F(s,t)=\ln(1+a^s)/2, a>e, F(s,1)=s \Rightarrow s=0$;

(6) $F(s,t)=(s+l)^{\left(1/(1+t)^r\right)}-l, l>1, r\in(0,\infty), F(s,t)=s \Rightarrow t=0$;

(7) $F(s,t)=s\log_{t+a} a, a>1, F(s,t)=s \Rightarrow s=0$ or $t=0$;

(8) $F(s,t)=s-\left(\dfrac{1+s}{2+s}\right)\left(\dfrac{t}{1+t}\right), F(s,t)=s \Rightarrow t=0$;

(9) $F(s,t)=s\beta(s), \beta:[0,\infty)\to(0,1)$ and is continuous, $F(s,t)=s \Rightarrow s=0$;

(10) $F(s,t)=s-\dfrac{t}{k+t}, F(s,t)=s \Rightarrow t=0$;

Definition 1.18. [11] A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t)=0$ if and only if $t=0$.

Definition 1.19. [2]An ultra-altering distance function is a continuous, non-decreasing mapping $\varphi:[0,\infty)\to[0,\infty)$ such that $\varphi(t)>0, t\in[0,\infty)$ and $\varphi(0)\ge 0$.

We denote this set with Φ_u.

II. MAIN RESULT

Theorem 2.1. Let $(X,S,\preceq)$ be an ordered *S*–metric space. Let $\omega: X \to X$ and $H: X\times X \to X$ be mappings such that *H* has the mixed ω–monotone property on *X* and there exist two elements $\alpha_0,\beta_0 \in X$ with $\omega(\alpha_0)\preceq H(\alpha_0,\beta_0)$ and $\omega(\beta_0)\succeq H(\beta_0,\alpha_0)$ such that as follows:

$$\psi(S(H(\alpha,\beta),H(x,y),H(\gamma,\delta))) \le F\begin{pmatrix}\psi(k[S(\omega\alpha,\omega x,\omega\gamma)+S(\omega\beta,\omega y,\omega\delta)]),\\ \varphi(k[S(\omega\alpha,\omega x,\omega\gamma)+S(\omega\beta,\omega y,\omega\delta)])\end{pmatrix} \quad (2.1)$$

for $\alpha,\beta,\gamma,x,y,\delta \in X$ with $\omega\alpha\succeq\omega x\succeq\omega\gamma$ and $\omega\beta\preceq\omega y\preceq\omega\delta$ or $\omega\alpha\preceq\omega x\preceq\omega\gamma$ and $\omega\beta\succeq\omega y\succeq\omega\delta$ and $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function, $\varphi\in\Phi_u$ and $F\in\mathbb{C}$. Assume the following conditions:

(i) $H(X\times X)\subset\omega(X)$,

(ii) $\omega(X)$ is complete, continuous and commutes with *H*.

Then *H* and ω have a coupled coincidence point (α,β). If $\omega\alpha\succeq\omega\beta$ or $\omega\alpha\preceq\omega\beta$, then there $\omega(\alpha)=H(\alpha,\alpha)=\alpha$.

Proof. Let α_0, β_0 be two points such that $\omega(\alpha_0) \preceq H(\alpha_0, \beta_0)$ and $\omega(\beta_0) \succeq H(\beta_0, \alpha_0)$. As $H(X \times X) \subset \omega(X)$, we take α_1, β_1 in a way that $\omega(\alpha_1) = H(\alpha_0, \beta_0)$ and $\omega(\beta_1) = H(\beta_0, \alpha_0)$.

Again since $H(X \times X) \subset \omega(X)$, take $\alpha_2, \beta_2 \in X$ such that $\omega(\alpha_2) = H(\alpha_1, \beta_1)$ and $\omega(\beta_2) = H(\beta_1, \alpha_1)$.

Repeating in this way to construct two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in X such that,

$$\omega(\alpha_{n+1}) = H(\alpha_n, \beta_n) \text{ and } \omega(\beta_{n+1}) = H(\beta_n, \alpha_n), \text{ for all } n \geq 0 \tag{2.2}$$

Now, we claim that for all $n \geq 0$,

$$\omega(\alpha_n) \preceq \omega(\alpha_{n+1}), \tag{2.3}$$

and

$$\omega(\beta_n) \preceq \omega(\beta_{n+1}). \tag{2.4}$$

By induction principle, take $n = 0$. Since $\omega(\alpha_0) \preceq H(\alpha_0, \beta_0)$ and $\alpha(\beta_0) \succeq H(\beta_0, \alpha_0)$, we see that $\omega(\alpha_1) = H(\alpha_0, \beta_0)$ and $\omega(\beta_1) = H(\beta_0, \alpha_0)$, and so $\omega(\alpha_0) \preceq \omega(\alpha_1)$, and $\omega(\beta_0) \succeq \omega(\beta_1)$, i.e., (2.3) and(2.4) holds for $n = 0$. Suppose that (2.3) and (2.4) are valid for some $n > 0$. As H is a mixed ω –monotone property and also $\omega(\alpha_n) \preceq \omega(\alpha_{n+1})$, $\omega(\beta_n) \succeq \omega(\beta_{n+1})$, then from (2.2), we have

$$\omega(\alpha_{n+1}) = H(\alpha_n, \beta_n) \preceq H(\alpha_{n+1}, \beta_n)$$

and

$$H(\beta_{n+1}, \alpha_n) \preceq H(\beta_n, \alpha_n) = \omega(\beta_{n+1}).$$

similarly,

$$\omega(\alpha_{n+2}) = H(\alpha_{n+1}, \beta_{n+1}) \succeq H(\alpha_{n+1}, \beta_n)$$

and

$$H(\beta_{n+1}, \alpha_n) \succeq H(\beta_{n+1}, \alpha_{n+1}) = \omega(\beta_{n+2}).$$

Then from (2.2) and (2.3), we get

$$\omega(\alpha_{n+1}) \preceq \omega(\alpha_{n+2}) \text{ and } \omega(\beta_{n+1}) \succeq H(\beta_{n+2})$$

We conclude by induction principle that (2.3) and (2.4) holds for all $n \geq 0$.

Continuing this process, we see clearly that

$$\omega(\alpha_0) \preceq \omega(\alpha_1) \preceq \omega(\alpha_2) \preceq \cdots \preceq \omega(\alpha_{n+1}) \cdots$$

and

$$\omega(\beta_0) \preceq \omega(\beta_1) \preceq \omega(\beta_2) \preceq \cdots \preceq \omega(\beta_{n+1}) \cdots$$

If $(\alpha_{n+1}, \beta_{n+1}) = (\alpha_n, \beta_n)$,then H and ω have a coupled coincidence point. So, we suppose that $(\alpha_{n+1}, \beta_{n+1}) \neq (\alpha_n, \beta_n)$ for all $n \geq 0$, i.e., we suppose that either $\omega(\alpha_{n+1}) = H(\alpha_n, \beta_n) \neq \omega(\alpha_n)$ or $\omega(\beta_{n+1}) = H(\beta_n, \alpha_n) \neq \omega(\beta_n)$.

Next, we proves that, for all $n \geq 0$,

$$\psi(S(\omega\alpha_{n+1}, \omega\alpha_{n+1}, \omega\alpha_n)) \leq F\begin{pmatrix} \psi\left(\frac{1}{2}(2k)^n[S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]\right), \\ \varphi\left(\frac{1}{2}(2k)^n[S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]\right) \end{pmatrix}. \tag{2.5}$$

For n = 1, we have

$$\psi(S(\omega\alpha_2, \omega\alpha_2, \omega\alpha_1)) = F\begin{pmatrix} \psi(S(H(\alpha_1, \beta_1), H(\alpha_1, \beta_1), H(\alpha_0, \beta_0))), \\ \varphi(S(H(\alpha_1, \beta_1), H(\alpha_1, \beta_1), H(\alpha_0, \beta_0))) \end{pmatrix}$$

$$\leq F\begin{pmatrix} \psi(k[S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]), \\ \varphi(k[S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]) \end{pmatrix}$$

$$= F\begin{pmatrix} \psi\left(\frac{1}{2}(2k)[S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]\right), \\ \varphi(k[S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]) \end{pmatrix}.$$

And hence (2.5) holds for $n = 1$. Therefore, we assume that (2.5) holds for $n > 0$ .Since $\omega(\alpha_{n+1}) \succeq \omega(\alpha_n)$ and $\omega(\beta_{n+1}) \preceq g(\beta_n)$, by using (2.2) and (2.5), we have

$$\psi(S(\omega\alpha_{n+1}, \omega\alpha_{n+1}, \omega\alpha_n)) = F\begin{pmatrix} \psi(S(H(\alpha_n, \beta_n), H(\alpha_n, \beta_n), H(\alpha_{n-1}, \beta_{n-1}))), \\ \varphi(S(H(\alpha_n, \beta_n), H(\alpha_n, \beta_n), H(\alpha_{n-1}, \beta_{n-1}))) \end{pmatrix}$$

$$\leq F\begin{pmatrix} \psi(k[S(\omega\alpha_n, \omega\alpha_n, \omega\alpha_{n-1}) + S(\omega\beta_n, \omega\beta_n, \omega\beta_{n-1})]), \\ \varphi(k[S(\omega\alpha_n, \omega\alpha_n, \omega\alpha_{n-1}) + S(\omega\beta_n, \omega\beta_n, \omega\beta_{n-1})]) \end{pmatrix}. \qquad (2.6)$$

Now,

$$\psi(S(\omega\alpha_n, \omega\alpha_n, \omega\alpha_{n-1})) = F\begin{pmatrix} \psi(S(H(\alpha_{n-1}, \beta_{n-1}), H(\alpha_{n-1}, \beta_{n-1}), H(\alpha_{n-2}, \beta_{n-2}))), \\ \varphi(S(H(\alpha_{n-1}, \beta_{n-1}), H(\alpha_{n-1}, \beta_{n-1}), H(\alpha_{n-2}, \beta_{n-2}))) \end{pmatrix}$$

$$\leq F\begin{pmatrix} \psi(k[S(\omega\alpha_{n-1}, \omega\alpha_{n-1}, \omega\alpha_{n-2}) + S(\omega\beta_{n-1}, \omega\beta_{n-1}, \omega\beta_{n-2})]), \\ \varphi(k[S(\omega\alpha_{n-1}, \omega\alpha_{n-1}, \omega\alpha_{n-2}) + S(\omega\beta_{n-1}, \omega\beta_{n-1}, \omega\beta_{n-2})]) \end{pmatrix}. \qquad (2.7)$$

and

$$\psi(S(\omega\beta_n, \omega\beta_n, \omega\beta_{n-1})) = F\begin{pmatrix} \psi(S(H(\beta_{n-1}, \alpha_{n-1}), H(\beta_{n-1}, \alpha_{n-1}), H(\beta_{n-2}, \alpha_{n-2}))), \\ \varphi(S(H(\beta_{n-1}, \alpha_{n-1}), H(\beta_{n-1}, \alpha_{n-1}), H(\beta_{n-2}, \alpha_{n-2}))) \end{pmatrix}$$

$$\leq F\begin{pmatrix} \psi(k[S(\omega\beta_{n-1}, \omega\beta_{n-1}, \omega\beta_{n-2}) + S(\omega\alpha_{n-1}, \omega\alpha_{n-1}, \omega\alpha_{n-2})]), \\ \varphi(k[S(\omega\beta_{n-1}, \omega\beta_{n-1}, \omega\beta_{n-2}) + S(\omega\alpha_{n-1}, \omega\alpha_{n-1}, \omega\alpha_{n-2})]) \end{pmatrix}. \qquad (2.8)$$

From (2.7) and (2.8), we get that

$$\psi(S(\omega\alpha_n, \omega\alpha_n, \omega\alpha_{n-1})) + \psi(S(\omega\beta_n, \omega\beta_n, \omega\beta_{n-1})) \leq \\ \psi(2k[S(\omega\alpha_{n-1}, \omega\alpha_{n-1}, \omega\alpha_{n-2}) + S(\omega\beta_{n-1}, \omega\beta_{n-1}, \omega\beta_{n-2})]) \cdot$$

holds for $n \in N$. From (2.6), we have

$$\psi(S(\omega\alpha_{n+1}, \omega\alpha_{n+1}, \omega\alpha_n)) \leq \psi(k[S(\omega\alpha_n, \omega\alpha_n, \omega\alpha_{n-1}) + S(\omega\beta_n, \omega\beta_n, \omega\beta_{n-1})])$$
$$\leq \psi(2k^2[S(\omega\alpha_{n-1}, \omega\alpha_{n-1}, \omega\alpha_{n-2}) + S(\omega\beta_{n-1}, \omega\beta_{n-1}, \omega\beta_{n-2})])$$
$$\vdots$$
$$\leq \psi\left(\frac{1}{2}(2k)^n[S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]\right).$$

Hence for all $n \in N$, we have

$$\psi(S(\omega\alpha_{n+1}, \omega\alpha_{n+1}, \omega\alpha_n)) \leq \psi\left(\frac{1}{2}(2k)^n[S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]\right). \qquad (2.9)$$

Suppose $m, n \in N$, with $m > N$. First, let $m = 2p + 1,$ (2.9) we have

$$S(\omega\alpha_m, \omega\alpha_m, \omega\alpha_n) \leq 2(S(\omega\alpha_{n+1}, \omega\alpha_{n+1}, \omega\alpha_n) + \cdots + S(\omega\alpha_{m-1}, \omega\alpha_{m-1}, \omega\alpha_{m-2})) \\ + S(\omega\alpha_m, \omega\alpha_m, \omega\alpha_{m-1})$$

$$\leq \left(\sum_{i=n}^{m-2}(2k)^i + \frac{1}{2}(2k)^{m-1}\right) \times [S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]$$

$$\leq \left(\frac{(2k)^n}{1-2k} + \frac{1}{2}(2k)^{m-1} \right) \times [S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)].$$

Further, let $m = 2p$. Again, using (S3),(2.9) we obtain

$$S(\omega\alpha_m, \omega\alpha_m, \omega\alpha_n) \leq 2(S(\omega\alpha_{n+1}, \omega\alpha_{n+1}, \omega\alpha_n) + \cdots + S(\omega\alpha_m, \omega\alpha_m, \omega\alpha_{m-1}))$$

$$\leq \sum_{i=n}^{m-1} (2k)^i [S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)]$$

$$\leq \frac{(2k)^n}{1-2k} [S(\omega\alpha_1, \omega\alpha_1, \omega\alpha_0) + S(\omega\beta_1, \omega\beta_1, \omega\beta_0)].$$

Letting $n, m \to \infty$. Since $2k < 1$, using Lemma (1.2) we conclude that $\lim_{n,m\to\infty} S(\omega\alpha_m, \omega\alpha_m, \omega\alpha_n) = 0$.

Thus $\{\omega\alpha_n\}$ is Cauchy sequence in $\omega(X)$. Similarly $\{\omega\beta_n\}$ is Cauchy sequence in $\omega(X)$ .Since $\omega(X)$ is complete, we have $\{\omega\alpha_n\}$ and $\{\omega\beta_n\}$ are convergent to some $\alpha \in X$ and $\beta \in X$ respectively. Since ω is continuous, we have $\{\omega(\omega\alpha_n)\}$ is convergent to $\omega\alpha$ and $\{\omega(\omega\beta_n)\}$ is convergent to $\omega\beta$, that is,

$$\lim_{n\to\infty} (\omega(\omega\alpha_n)) = \omega(\alpha) \text{ and } \lim_{n\to\infty} (\omega(\omega\beta_n)) = \omega(\beta).$$

Since, *H* and ω are commutative, we have

$$H(\omega(\alpha_n), \omega(\beta_n)) = \omega(H(\alpha_n, \beta_n)) = \omega(\omega(\alpha_{n+1}))$$

and

$$H(\omega(\beta_n), \omega(\alpha_n)) = \omega(H(\beta_n, \alpha_n)) = \omega(\omega(\beta_{n+1})).$$

Next, we claim that (α, β) is coupled coincidence point of *H* and ω. From (2.1) we have

$$\psi(S(H(\alpha,\beta), H(\alpha,\beta), \omega\omega\alpha_{n+1})) = \psi(S(H(\alpha,\beta), H(\alpha,\beta), H(\omega\alpha_n, \omega\beta_n)))$$

$$\leq F\begin{pmatrix} \psi(k[S(\omega\alpha, \omega\alpha, \omega\omega\alpha_n) + S(\omega\beta, \omega\beta, \omega\beta_n)]), \\ \varphi(k[S(\omega\alpha, \omega\alpha, \omega\omega\alpha_n) + S(\omega\beta, \omega\beta, \omega\beta_n)]) \end{pmatrix}$$

Letting $n \to \infty$ and also ω is continuous, we get

$$\psi(S(H(\alpha,\beta), H(\alpha,\beta), \omega\alpha)) \leq F\begin{pmatrix} \psi(k[S(\omega\alpha, \omega\alpha, \omega\alpha) + S(\omega\beta, \omega\beta, \omega\beta)]), \\ \varphi(k[S(\omega\alpha, \omega\alpha, \omega\alpha) + S(\omega\beta, \omega\beta, \omega\beta)]) \end{pmatrix}$$

$$= F(0,0) = 0.$$

Hence $\omega\alpha = H(\alpha, \beta)$. Similarly, $\omega\beta = H(\beta, \alpha)$.

Next we claim that $H(\alpha, \alpha) = \omega(\alpha) = \alpha$. Since (α, β) is a coupled coincidence point of *H* and ω, we have $\omega\alpha = H(\alpha, \beta)$ and $\omega\beta = H(\beta, \alpha)$. Suppose that $\omega\alpha \neq \omega\beta$. Then from (2.1) we have

$$\psi(S(\omega\beta, \omega\beta, \omega\alpha)) = \psi(S(H(\beta,\alpha), H(\beta,\alpha), H(\alpha,\beta)))$$

$$\leq F\begin{pmatrix} \psi(k[S(\omega\beta, \omega\beta, \omega\alpha) + S(\omega\alpha, \omega\alpha, \omega\beta)]), \\ \varphi(k[S(\omega\beta, \omega\beta, \omega\alpha) + S(\omega\alpha, \omega\alpha, \omega\beta)]) \end{pmatrix}.$$

Also,

$$\psi(S(\omega\alpha, \omega\alpha, \omega\beta)) = \psi(S(H(\alpha,\beta), H(\alpha,\beta), H(\beta,\alpha)))$$

$$\leq F\begin{pmatrix} \psi(k[S(\omega\alpha, \omega\alpha, \omega\beta) + S(\omega\beta, \omega\beta, \omega\alpha)]), \\ \varphi(k[S(\omega\alpha, \omega\alpha, \omega\beta) + S(\omega\beta, \omega\beta, \omega\alpha)]) \end{pmatrix}.$$

Therefore,

$$\psi(S(\omega\beta, \omega\beta, \omega\alpha) + S(\omega\alpha, \omega\alpha, \omega\beta)) \leq F\begin{pmatrix} \psi(2k[S(\omega\beta, \omega\beta, \omega\alpha) + S(\omega\alpha, \omega\alpha, \omega\beta)]), \\ \varphi(2k[S(\omega\beta, \omega\beta, \omega\alpha) + S(\omega\alpha, \omega\alpha, \omega\beta)]) \end{pmatrix}.$$

Since 2k < 1, we get

$$\psi\big(S(\omega\beta,\omega\beta,\omega\alpha)+S(\omega\alpha,\omega\alpha,\omega\beta)\big)\le F\begin{pmatrix}\psi\big(S(\omega\beta,\omega\beta,\omega\alpha)+S(\omega\beta,\omega\beta,\omega\alpha)\big),\\ \varphi\big(S(\omega\beta,\omega\beta,\omega\alpha)+S(\omega\beta,\omega\beta,\omega\alpha)\big)\end{pmatrix},$$

which is contradiction. Hence $\omega\alpha=\omega\beta$ and $H(\alpha,\beta)=\omega\alpha=\omega\beta=H(\beta,\alpha)$.Since $\{\omega\alpha_{n+1}\}$ is a subsequence of $\{\omega\alpha_n\}$, we have $\{\omega\alpha_{n+1}\}$ is convergent to α.

Thus,

$$\psi\big(S(\omega\alpha,\omega\alpha,\omega\alpha_{n+1})\big)=F\begin{pmatrix}\psi\big(S(H(\alpha,\beta),H(\alpha,\beta),H(\alpha_n,\beta_n))\big),\\ \varphi\big(S(H(\alpha,\beta),H(\alpha,\beta),H(\alpha_n,\beta_n))\big)\end{pmatrix}$$

$$\le F\begin{pmatrix}\psi\big(k[S(\omega\alpha,\omega\alpha,\omega\alpha_n)+S(\omega\beta,\omega\beta,\omega\beta_n)]\big),\\ \varphi\big(k[S(\omega\alpha,\omega\alpha,\omega\alpha_n)+S(\omega\beta,\omega\beta,\omega\beta_n)]\big)\end{pmatrix}.$$

Letting $n\to\infty$ and also ω is continuous, we get

$$\psi\big(S(\omega\alpha,\omega\alpha,\alpha)\big)\le F\begin{pmatrix}\psi\big(k[S(\omega\alpha,\omega\alpha,\alpha)+S(\omega\beta,\omega\beta,\beta)]\big),\\ \varphi\big(k[S(\omega\alpha,\omega\alpha,\alpha)+S(\omega\beta,\omega\beta,\beta)]\big)\end{pmatrix}.$$

and also,

$$\psi\big(S(\omega\beta,\omega\beta,\beta)\big)\le F\begin{pmatrix}\psi\big(k[S(\omega\beta,\omega\beta,\beta)+S(\omega\alpha,\omega\alpha,\alpha)]\big),\\ \varphi\big(k[S(\omega\beta,\omega\beta,\beta)+S(\omega\alpha,\omega\alpha,\alpha)]\big)\end{pmatrix}.$$

Thus,

$$\psi\big(S(\omega\alpha,\omega\alpha,\alpha)+S(\omega\beta,\omega\beta,\beta)\big)\le F\begin{pmatrix}\psi\big(2k[S(\omega\alpha,\omega\alpha,\alpha)+S(\omega\beta,\omega\beta,\beta)]\big),\\ \varphi\big(2k[S(\omega\alpha,\omega\alpha,\alpha)+S(\omega\beta,\omega\beta,\beta)]\big)\end{pmatrix}.$$

Since $2k<1$, by last inequality, only if $S(\omega\alpha,\omega\alpha,\alpha)=0$ and $S(\omega\beta,\omega\beta,\beta)=0$. Hence $\alpha=\omega\alpha$ and $\beta=\omega\beta$. Thus we get $\omega\alpha=H(\alpha,\alpha)=\alpha$.

Corollary 2.2. Let $(X,S,\preceq)$ be an ordered *S*–metric space. Let $\omega:X\to X$ and $H:X\times X\to X$ be mappings such that *H* has the mixed ω–monotone property on *X* and there exist two elements $\alpha_0,\beta_0\in X$ with $\omega(\alpha_0)\preceq H(\alpha_0,\beta_0)$ and $\omega(\beta_0)\succeq H(\beta_0,\alpha_0)$ such that as follows:

$$\psi\big(S(H(x,y),H(x,y),H(\alpha,\beta))\big)\le F\begin{pmatrix}\psi\big(k[S(\omega x,\omega x,\omega\gamma)+S(\omega y,\omega y,\omega\delta)]\big),\\ \varphi\big(k[S(\omega x,\omega x,\omega\gamma)+S(\omega y,\omega y,\omega\delta)]\big)\end{pmatrix}$$

for $\alpha,\beta,x,y\in X$ with $\omega\alpha\succeq\omega x$ and $\omega\beta\preceq\omega y$ or $\omega\alpha\preceq\omega x$ and $\omega\beta\succeq\omega y$ and $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$. Assume the following conditions:

(i) $H(X\times X)\subset\omega(X)$,

(ii) ω is continuous and commutes with *H*.

(iii) $\omega(X)$ is complete.

Then there exist $\alpha\in X$ such that $\omega(\alpha)=H(\alpha,\alpha)=\alpha$.

Proof. From Theorem (2.1) by taking $\alpha=x$ and $\beta=y$.

Corollary 2.3. Let $(X,S,\preceq)$ be an ordered *S*–metric space. Let $H:X\times X\to X$ be mappings such that *H* has the mixed monotone property on *X* and there exist two elements $\alpha_0,\beta_0\in X$ with $\alpha_0\preceq H(\alpha_0,\beta_0)$ and $\beta_0\succeq H(\beta_0,\alpha_0)$. Let there exists a constant $k\in\left(0,\frac{1}{2}\right)$ such that the following holds:

$$\psi(S(H(x,y),H(x,y),H(\alpha,\beta)))\leq F\begin{pmatrix}\psi(k[S(x,x,\alpha)+S(y,y,\beta)]),\\ \varphi(k[S(x,x,\alpha)+S(y,y,\beta)])\end{pmatrix}$$

for $\alpha,\beta,x,y\in X$ with $\alpha\succeq x$ and $\beta\preceq y$ or $\alpha\preceq x$ and $\beta\succeq y$. If $(X,S,\preceq)$ regular then there exist $\alpha\in X$ such that $H(\alpha,\alpha)=\alpha$.

Proof. We defined $\omega: X\to X$ by $\omega\alpha=\alpha$. Then the mappings *H* and ω satisfies all the conditions of Corollary 2.2. Hence the result follows.

Corollary 2.4. Let $(X,S,\preceq)$ be an ordered *S*–metric space. Let $g: X\to X$ and $H: X\times X\to X$ be mappings has the mixed monotone property on *X* and there exist two elements $\alpha_0,\beta_0\in X$ with $\omega(\alpha_0)\preceq H(\alpha_0,\beta_0)$ and $\omega(\beta_0)\succeq H(\beta_0,\alpha_0)$. Let there exists a constant $k\in\left(0,\frac{1}{2}\right)$ such that the following holds:

$$\psi(S(H(x,y),H(x,y),H(\alpha,\beta)))\leq F\begin{pmatrix}\psi(k[S(\omega x,\omega x,\omega\alpha)+S(\omega y,\omega y,\omega\beta)]),\\ \varphi(k[S(\omega x,\omega x,\omega\alpha)+S(\omega y,\omega y,\omega\beta)])\end{pmatrix}$$

for $\alpha,\beta,x,y\in X$ with $\omega\alpha\succeq\omega x$ and $\omega\beta\preceq\omega y$ or $\omega\alpha\preceq\omega x$ and $\omega\beta\succeq\omega y$ and $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$. Assume the following conditions:

(i) $H(X\times X)\subset\omega(X)$,

(ii) $\omega(X)$ is complete,

(iii) ω is continuous and commutes with *H*.

Then there exist $\alpha\in X$ such that $H(\alpha,\alpha)=\omega(\alpha)=\alpha$.

Proof. Put $\varphi(t)=1$ for all $t\in[0,\infty)$ the result follows. Moreover, we get a generalization of theorem given in [16].

Corollary 2.5. Let $(X,S,\preceq)$ be complete ordered *S*–metric space. Let $H: X\times X\to X$ be mappings such that *H* has the mixed monotone property on *X* and there exist two elements $\alpha_0,\beta_0\in X$ with $\alpha_0\preceq H(\alpha_0,\beta_0)$ and $\beta_0\succeq H(\beta_0,\alpha_0)$. Let there exists a constant $k\in\left(0,\frac{1}{2}\right)$ such that the following holds:

$$\psi(S(H(x,y),H(x,y),H(\alpha,\beta)))\leq F\begin{pmatrix}\psi(k[S(x,x,\alpha)+S(y,y,\beta)]),\\ \varphi(k[S(x,x,\alpha)+S(y,y,\beta)])\end{pmatrix}$$

for $\alpha,\beta,x,y\in X$ with $\alpha\succeq x$ and $\beta\preceq y$ or $\alpha\preceq x$, $\beta\succeq y$ and $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$. If $(X,S,\preceq)$ regular then there exist $\alpha\in X$ such that $H(\alpha,\alpha)=\alpha$.

Proof. Let $\omega: X\to X$ be defined as $\omega(\alpha)=\alpha$. Then all conditions of Corollary 2.4 are satisfied.

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