International Journal of Scientific Research in Mathematical and Statistical Sciences **Research Paper**
Vol.6, Issue.1, pp.237-240, February (2019) **E-ISSN:** 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i1.237240

Orthogonal Modular Stability of Radical Cubic Functional Equation

R. Murali [1*], P. Divyakumari [2]

[1, 2] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, TamilNadu, India

[*]*Corresponding Author: shcrmurali@yahoo.co.in, Tel.: +91-94422-84395*

Available online at: www.isroset.org

Received: 06/Jan/2019, Accepted: 09/Feb/2019, Online: 28/Feb/2019

Abstract—In this paper, authors newly introduce radical cubic type functional equation and obtain its general solution. Also, investigate the Hyers-Ulam-Rassias stability of introduced radical cubic type functional equation in modular space.

Keywords*—Modular space, orthogonality, cubic functional equation, generalized Hyers-Ulam-Rassias stability.*

I. INTRODUCTION

S.M. Ulam [10] is the pioneer for the stability problem in functional equations. In 1940, while he was delivering a talk before the Mathematics Club of University of Wisconsin, he dicussed a number of unsolved problems. Among those was the following question concerning the stability of homomorphisms:

Let G be a group G' be a metric group with metric $\rho(\cdot,\cdot)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to G'$ satisfies the inequality $d(f(xy), f(x, f(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $h : G \to G'$ exist with $d(f(x), h(x)) < \varepsilon$ for all x in G?

In 1941, D.H. Hyers [3] provided a partial solution to Ulam's question. Indeed, he proved the following celebrated theorem.

Theorem 1.1: [3] Assume that E_1 and E_2 be two Banach spaces. If a function $f : E_1 \to E_2$ satisfies the inequality

$$\| f(x+y) - f(x) - f(y) \| \leq \in$$

for some $\in \geq 0$ and for all $x, y \in E_1$, then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for each x in E_1 and $a : E_1 \to E_2$ is the unique additive mapping satisfying

$$\| f(x) - a(x) \| \leq \in$$

for any $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then a is linear.

From the above theorem, one can say that the additive functional equation

$$f(x+y) = f(x) + f(y)$$

is stable in the sense of Hyers and Ulam (or) it is called Hyers-Ulam stability. In 1978, Th.M. Rassias [8] gave a generalized solution to Ulam's problem for approximately linear mappings and he proved a new generalizations to the Hyers-Ulam stability theory where he used the controlled function as the sum of powers of norms. The phenomenon that was introduced and proved by Th.M. Rassias is called Hyers-Ulam-Rassias stability (or) generalized Hyers-Ulam stability. The definitions related to our main theorem can be referred in [4].

In this paper, authors newly introduce radical cubic type functional equation

$$f(2x+y) + f(2x-y) = 8f(x+y) + 8f(x-y) - 36f\left(\sqrt[3]{xy^2}\right) \quad (0.1)$$

for all $x, y \in \mathfrak{R}^+$. Using orthogonality, authors obtain its general solution and investigate the Hyers-Ulam-Rassias stability in modular space.

II. GENERAL SOLUTION OF (0.1)

In this section, we obtain the general solution of the functional equation (0.1). Throughout this section, let X and Y be real vector spaces.

Theorem 2.1. Let X and Y be real vector spaces. If a function $f : X \to Y$ satisfies the functional equation

$$f(2x+y) + f(2x-y) = 8f(x+y) + 8f(x-y) - 36f\left(\sqrt[3]{xy^2}\right) \quad (2.1)$$

for all $x, y \in X$, then $f : X \to Y$ is odd and cubic.

Proof: Suppose a function $f : X \to Y$ satisfies (2.1). Putting $x = y = 0$ in (2.1), we get $f(0) = 0$. Let $y = 0$ in (2.1), we obtain

$$f(2x) = 8f(x) \quad (2.2)$$

for all $x \in X$. Let $x = 0$ in (2.1), we obtain

$$f(-y) = -f(y) \tag{2.3}$$

for all $y \in X$. Hence, $f: X \to Y$ is odd. Setting $(x,y)=(x,x)$ and using (2.2), we obtain

$$f(3x) = 27f(x) \tag{2.4}$$

for all $x \in X$. From (2.2) and (2.4), we arrive

$$f(nx) = n^3 f(x) \tag{2.5}$$

for all $x \in X$. Hence, $f: X \to Y$ is cubic.

III. Orthogonal Stability of (0.1)

In this section we assume that the convex modular ρ has the Fatou property such that satisfies the Δ_8 – condition with $0 < \kappa \le 8$. In addition, we assume that $(E, \perp)$ denotes an orthogonality space and we define

$$Df(x,y) = f(2x+y) + f(2x-y) - 8f(x+y) - 8f(x-y) - 36f\left(\sqrt[3]{xy^2}\right)$$

for all $x, y \in E$ with $x \perp y$, on the other hand, we give the Hyers-Ulam-Rassias stability of the equation (0.1) in modular spaces.

Theorem 3.1: Let X_ρ is a ρ-complete modular space. Let $(E, \|\cdot\|)$ with $\dim E \ge 2$ be a real normed linear space and let $f: E \to X_\rho$ be a mapping fulfilling

$$\rho(Df(x,y)) \le \mu\left\{\|x\|^p + \|y\|^p\right\} \tag{3.1}$$

for all $x, y \in E$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $C: E \to X_\rho$ such that

$$\rho(C(x) - f(x)) \le \frac{\mu}{16 - \kappa 2^{p-2}}\|x\|^p \tag{3.2}$$

for all $x \in E$. The function $C(x)$ is defined by

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n} \qquad \forall x \in E. \tag{3.3}$$

Proof: Setting (x,y) by $(x,0)$ in (3.1), we obtain

$$\rho\left(\frac{f(2x)}{8} - f(x)\right) \le \frac{\mu}{16}\left(\|x\|^p\right) \tag{3.4}$$

for all $x \in E$. Now replacing x by $2x$ in (3.4), we arrive

$$\rho\left(\frac{f(2^2x)}{8} - f(2x)\right) \le \frac{\mu}{16}2^p\left(\|x\|^p\right) \tag{3.5}$$

for all $x \in E$. From (3.4) and (3.5), we obtain

$$\begin{aligned}
&\rho\left(\frac{f(2^2x)}{(2^3)^2} - f(x)\right) \\
&\le \rho\left(\frac{f(2^2x)}{(2^3)^2} - \frac{f(2x)}{(2^3)} + \frac{f(2x)}{(2^3)} - f(x)\right) \\
&\le \rho\left(\frac{f(2^2x)}{(2^3)^2} - \frac{f(2x)}{(2^3)}\right) + \rho\left(\frac{f(2x)}{(2^3)} - f(x)\right) \\
&\le \frac{\kappa}{(2^3)^2}\rho\left(\frac{f(2^2x)}{(2^3)^2} - \frac{f(2x)}{(2^3)}\right) + \frac{\kappa}{(2^3)}\rho\left(\frac{f(2x)}{(2^3)} - f(x)\right) \\
&\le \frac{\mu}{2\cdot 2^3}\left(1 + \kappa 2^{p-6}\right)\|x\|^p
\end{aligned} \tag{3.6}$$

In general, using induction on a positive integer n, we obtain that

$$\rho\left(\frac{f(2^n x)}{(2^3)^n} - f(x)\right) \le \frac{\mu}{2\cdot 2^3}\sum_{i=0}^{n-1}\left(\kappa^i 2^{i(p-6)}\right)\|x\|^p \tag{3.7}$$

for all $x \in E$. Indeed, for *n = 1* the relation (3.7) is true. Assume that the relation (3.7) is true for *n*, and we show this relation rest true for *n + 1*, thus we have

$$\begin{aligned}
&\rho\left(\frac{f(2^{n+1}x)}{(2^3)^{n+1}} - f(x)\right) \\
&\le \rho\left(\frac{f(2^{n+1}x)}{(2^3)^{n+1}} - \frac{f(2x)}{(2^3)} + \frac{f(2x)}{(2^3)} - f(x)\right) \\
&\le \rho\left(\frac{f(2^{n+1}x)}{(2^3)^{n+1}} - \frac{f(2x)}{(2^3)}\right) + \rho\left(\frac{f(2x)}{(2^3)} - f(x)\right) \\
&\le \frac{\mu}{2\cdot 2^3}\sum_{i=0}^{n}\left(\kappa^i 2^{i(p-6)}\right)\|x\|^p
\end{aligned}$$

for all $x \in E$. Hence the relation (3.7) is true for all $x \in E$, then (3.7) become

$$\rho\left(\frac{f(2^n x)}{(2^3)^n} - f(x)\right) \le \frac{\mu}{2\cdot 2^3}\frac{\left(1-\kappa 2^{p-6}\right)^n}{1-\kappa 2^{p-6}}\|x\|^p \tag{3.8}$$

for all $x \in E$. Replacing x by $2^m x$ in (3.8), we have

$$\rho\left(\frac{f(2^{n+m}x)}{(2^3)^n} - f(2^m x)\right) \le \frac{\mu}{2\cdot 2^3}\frac{\left(1-\kappa 2^{p-6}\right)^n}{1-\kappa 2^{p-6}}2^{mp}\|x\|^p \tag{3.9}$$

for all $x \in E$. Hence

$$\rho\left(\frac{f(2^{n+m}x)}{\left(2^3\right)^{n+m}}-\frac{f(2^m x)}{2^{3m}}\right)$$
$$\leq \frac{1}{\left(2^3\right)^m}\rho\left(\frac{f(2^{n+m}x)}{\left(2^3\right)^n}-f(2^m x)\right) \quad (3.10)$$
$$\leq \frac{\mu}{2\cdot 2^3}\frac{\left(1-\kappa 2^{p-6}\right)^n}{1-\kappa 2^{p-6}}2^{(p-3)m}\|x\|^p$$

for all $x\in E$. If $m,n\to\infty$ we get, the sequence $\left\{\frac{f(2^n x)}{8^n}\right\}$ is $\rho-$ Cauchy sequence in the $\rho-$ complete modular space X_ρ. Hence $\left\{\frac{f(2^n x)}{8^n}\right\}$ is $\rho-$convergent in X_ρ and we well define the mapping from E into X_ρ satisfying

$$\rho\left(C(x)-f(x)\right)\leq \frac{\mu}{16-\kappa 2^{p-2}}\|x\|^p.$$

To prove C satisfies (0.1), replace (x,y) by $(2^n x, 2^n y)$ in (3.1) and divide by 8^n then it follows that

$$\frac{1}{8^n}\rho\left(\begin{array}{l} f\left(2^n\left(2x+y\right)\right)+f\left(2^n\left(2x-y\right)\right) \\ -8f\left(2^n\left(x+y\right)\right)-8f\left(2^n\left(x-y\right)\right) \\ -36f\left(\sqrt[3]{2^n x\left(2^n y\right)^2}\right)\end{array}\right)$$
$$\leq \frac{\mu}{8^n}\left\{\left\|2^n x\right\|^p+\left\|2^n y\right\|^p\right\}.$$

Taking limit as $n\to\infty$ in the above inequality, we get

$$\rho\left(\begin{array}{c} C\left(2x+y\right)+C\left(2x-y\right)-8C\left(x+y\right) \\ -8C\left(x-y\right)-36C\left(\sqrt[3]{xy^2}\right)\end{array}\right)\leq 0,$$

which gives

$$C\left(2x+y\right)+C\left(2x-y\right)$$
$$=8C\left(x+y\right)+8C\left(x-y\right)-36C\left(\sqrt[3]{xy^2}\right)$$

for all $x,y\in E$ with $x\perp y$. Therefore $C:E\to X_\rho$ is an orthogonally cubic mapping which satisfies (0.1). To prove the uniqueness: Let C' be another orthogonally cubic mapping satisfying (0.1) and the inequality (3.2). Then

$$\rho\left(C(x)-C'(x)\right)$$
$$=\frac{1}{8^n}\rho\left(C(2^n x)-C'(2^n x)\right)$$
$$\leq \frac{1}{8^n}\left(\rho\left(C(2^n x)-f(2^n x)\right)+\rho\left(f(2^n x)-C'(2^n x)\right)\right)$$
$$\leq \frac{1}{2^{n(3-p)}}\left(\frac{2\mu}{16-\kappa 2^{p-2}}\|x\|^p\right)$$
$$\to 0 \; as\, n\to\infty$$

for all $x\in E$. Therefore C is unique. This completes the proof of the theorem.

IV. CONCLUSION

In this paper, authors introduced a new radical cubic functional equation and obtained its general solution, also investigated its Hyers-Ulam-Rassias stability in modular spaces using orthogonality.

REFERENCES

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc.Japan, Vol 2, pp 64-66, 1950.

[2] P.Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., Vol 184, pp 431-436, 1994.

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci., Vol 27, pp 222-224, 1941.

[4] Iz-iddine El-Fassi and Samir Kabbaj, *On the generalized orthogonal stability of mixed type additive-cubic functional equations in modular spaces*, Tbilisi Mathematical Journal, Vol 9, Issue 1, pp 231-243, 2016.

[5] Hark-Mahn Kim and Young Soon Hong, *Approximate Quadratic Mappings in Modular Spaces*, International Journal of Pure and Applied Mathematics, Vol 116, Issue 1, pp 31-43, 2017.

[6] J.M. Rassias, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, Vol 46, pp 126-130, 1982.

[7] K.Ravi, M. Arunkumar and J.M. Rassias, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, Vol 3, Issue 08, pp 36 – 47, Autumn 2008.

[8] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math.Soc.,Vol 72, pp 297-300, 1978.

[9] Wanchitra Towanlong and Paisan Nakmahachalasint, *A quadratic functional equation and its generalized Hyers-Ulam Rassias stability*, Thai Journal of Mathematics, special Issue (Annual Meeting in Mathematics, 2008), pp 85-91.

[10] S.M. Ulam, *A collection of the mathematical problems*, Interscience Publ., New York, 1960.

[11] Kittipong Wongkum, Poom Kumam, Yeol Je Cho, Phatiphat Thounthong, Parin Chaipunya, *On the generalized Ulam-Hyers-Rassias stability for quartic functional equation in modular spaces*, J. Nonlinear Sci. Appl., 10(2017), 1399-1406.

AUTHORS PROFILE

Dr. R. Murali is an Associate Professor & UG Head of PG and Research Department of Mathematics in Sacred Heart College (Autonomous), Tirupattur-635 601, Vellore Dt., Tamil Nadu, India. He has completed B.Sc., M.Sc., and M. Phil. Mathematics in

University of Madras, Chennai during 84-90. He pursued B.Ed., from Annamalai University in 1992 and Ph.D. from Thiruvalluvar University in 2013. He has 27 years of teaching experience. He has published more than 60 Research papers in various National/ International reputed Journals. His area of Research includes Stability of Functional equations, differential equations and difference equations.

Mrs.P.Divyakumari is a full-time research scholar in the broad field of Functional Equations under the guidance of Dr.R.Murali in the PG and Research Department of Mathematics, Sacred Heart College, Tirupattur. She qualified in the State Eligibility Test for Assistant Professor in 2017. She completed her B.Sc. from AAA Government Arts College, Walajapet, University of Madras in 2001 and M.Sc. from Voorhees College, Vellore, University of Madras in 2003. She pursued her M.Phil from Alagappa University in 2005. She has 2 years of teaching experience.

