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On Prime Labeling of Some Union Graphs and Circulant Graphs

S.K. Patel [1], J.B. Vasava[2*]

[1]Department of Mathematics, Government Engineering College, Gujarat Technological University, Bhuj, India
[2]General Department, Dr. S. and S. S. Ghandhy College of Engineering and Technology, Gujarat Technological University, Surat, India

*Corresponding author: jayeshvasava1910@gmail.com Tel: +91 9712694492

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Abstract—A graph G of order n is said to admit prime labeling if its vertices can be labeled with distinct positive integers not exceeding n such that each pair of adjacent vertices have relatively prime labels. A graph G that admits a prime labeling is called a prime graph. In this paper we investigate for prime labeling of some union graphs and later study some necessary and sufficient conditions for prime labeling of certain circulant graphs.

Keywords—Prime labeling, Prime graphs, Independence number, Union of graphs, Circulant graphs.

AMS Subject Classification (2010): 05C78

I. INTRODUCTION

We consider only finite, simple and undirected graphs. For a graph G, V(G) and E(G) denote its vertex set and edge set respectively. We shall denote the cardinality of these sets by $|V(G)|$ and $|E(G)|$ respectively. We refer to Gross and Yellen [1] for graph theoretic terminology and notations and Burton [2] for number theory results. We begin with the definition of prime labeling.

Definition 1.1: Let G be a graph of order n. A bijection $f : V(G) \rightarrow \{1, 2, ..., n\}$ is said to be a prime labeling of G, if for every pair of adjacent vertices u and v, $\gcd(f(u),f(v))=1$. A graph that admits a prime labeling is called a prime graph.

Prime labeling was originated by Entringer and was discussed in a paper by Taut et al.[3]. In the past thirty-five years, varieties of graphs have been studied for primality and in recent times, some of the variants of prime labeling like cordial prime labeling [4] and neighborhood prime labeling [5] are also studied extensively. A brief summary on prime labeling and its variants is available in the dynamic survey of graph labeling by Gallian[6]. In this paper, we find some new results related to prime labeling.

We now give the organization of our paper. Section I contains a brief introduction of prime labeling. Section II deals with main results and related examples. In Section III we briefly review the main results of the paper and lastly discuss about future scope along with conclusion in Section IV.

II. MAIN RESULTS

The independence number of a graph G is the maximum cardinality of an independent set of G. It is denoted by $\beta_0(G)$. For proving some of the results we use the following lemma [7].

Lemma 2.1: If $\beta_0(G) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor$, then G is not a prime graph (where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x).

It is proved in [3] that the wheel graph $W_n = C_n + K_1$ is prime if and only if n is even. Also it is easy to prove that the cycle C_n is prime for all n. Here we prove the result for union of wheel graph and cycle graph.

Theorem 2.2: $W_n \cup C_m$ is a prime graph if and only if n and m both are even.

Proof: First we show that $W_{2n+1} \cup C_{2m+1}$ is not a prime graph. Let G denote the graph $W_{2n+1} \cup C_{2m+1}$. It may be verified that $\beta_0(W_{2n+1}) = n$ and $\beta_0(C_{2m+1}) = m$. Therefore

$$\beta_0(G) = n + m. \quad (1)$$

Since $|V(G)| = 2n + 2m + 3$,

$$\left\lfloor \frac{|V(G)|}{2} \right\rfloor = n + m + 1. \quad (2)$$

So by (1) and (2) ,

$$\beta_0(G) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$$

Therefore in view of Lemma 2.1, G is not a prime graph.
Next we claim that if either $G' = W_{2n} \cup C_{2m+1}$ or $G' = W_{2n+1} \cup C_{2m}$ then G' is not a prime graph. It is easy to see that $\beta_0(G') = n + m$ and $|V(G')| = 2n + 2m + 2$. So

$$\left\lfloor \frac{|V(G')|}{2} \right\rfloor = n + m + 1.$$

Therefore

$$\beta_0(G') < \left\lfloor \frac{|V(G')|}{2} \right\rfloor.$$

Thus G' is not a prime graph.
Finally we prove that $W_{2n} \cup C_{2m}$ is a prime graph. Let G'' denote the graph $W_{2n} \cup C_{2m}$. Let the sets $\{v_1, v_2, ..., v_{2n+1}\}$ and $\{v_{2n+2}, v_{2n+3}, ..., v_{2n+2m+1}\}$ be the sets of vertices of W_{2n} and C_{2m} respectively, where v_1 is an apex vertex of W_{2n}.
Define $f : V(G'') \to \{1, 2, ..., 2n + 2m + 1\}$ as per the following two cases.

Case 1: $n \not\equiv 1 \pmod 3$

$f(v_1) = 1,$

$f(v_i) = i + 2, \qquad i = 2, 3, ..., 2n + 1,$

$f(v_{2n+2}) = 2,$

$f(v_{2n+3}) = 3,$

$f(v_i) = i, \qquad i = 2n + 4, 2n + 5, ..., 2n + 2m + 1,$

Case 2: $n \equiv 1 \pmod 3$

$f(v_1) = 1,$

$f(v_i) = i + 1, \qquad i = 2, 3, ..., 2n + 1,$

$f(v_{2n+2}) = 2,$

$f(v_i) = i, \qquad i = 2n + 3, 2n + 4, ..., 2n + 2m + 1.$

The definition of f given in Case 1 and Case 2 above is illustrated in Figure 1 and Figure 2 respectively. Under the given assumptions, it may be verified that f defines a prime labeling.■

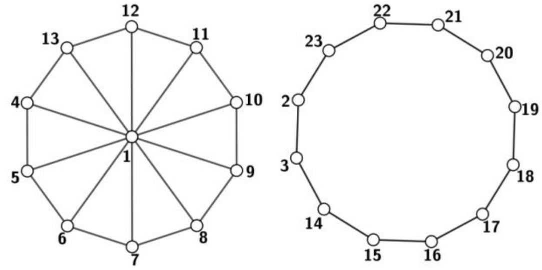

FIGURE 1. Prime labeling of $W_{10} \cup C_{12}$ (i.e. $W_{2\cdot5} \cup C_{2\cdot6}$)

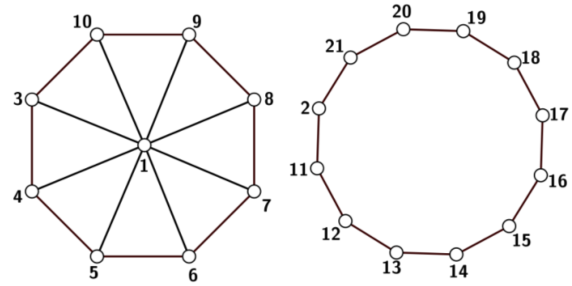

FIGURE 2. Prime labeling of $W_8 \cup C_{12}$ (i.e. $W_{2\cdot4} \cup C_{2\cdot6}$)

Note that $P_n + \bar{K}_2$ is prime if and only if either n is odd or n is 2[8]. The next result is about union of $P_n + \bar{K}_2$ and cycle graph.

Theorem 2.3: $(P_n + \bar{K}_2) \cup C_m$ is a prime graph if and only if either n=2, or n is odd and m is even.

Proof: First we show that $(P_2 + \bar{K}_2) \cup C_m$ is a prime graph. Let $\{v_1, v_2\}$, $\{u_1, u_2\}$ and $\{w_1, w_2, ..., w_m\}$ be the sets of consecutive vertices of P_2, $\bar{K}_2$ and C_m respectively. Define $f : V((P_2 + \bar{K}_2) \cup C_m) \to \{1, 2, ..., m + 4\}$ as

$f(v_1) = 3,$

$f(v_2) = 5,$

$f(u_1) = 2,$

$f(u_2) = 4,$

$f(w_1) = 1,$

$f(w_i) = i + 4, \qquad i = 2, 3, ..., m.$

It may be verified that f is a prime labeling on $(P_2 + \bar{K}_2) \cup C_m$.
To prove that for $n > 1$ neither $(P_{2n} + \bar{K}_2) \cup C_{2m+1}$ nor $(P_{2n} + \bar{K}_2) \cup C_{2m}$ is a prime graph, we let $G = (P_{2n} + \bar{K}_2) \cup C_{2m+1}$ and $G' = (P_{2n} + \bar{K}_2) \cup C_{2m}$.
Since for $n > 1$, $\beta_0(P_{2n} + \bar{K}_2) = n$ and $\beta_0(C_{2m+1}) = \beta_0(C_{2m}) = m$, we have

$$\beta_0(G) = \beta_0(G') = n + m. \qquad (3)$$

Also, $|V(G)| = 2n + 2m + 3$ and $|V(G)| = 2n + 2m + 2$. Therefore

$$\left\lfloor \frac{|V(G)|}{2} \right\rfloor = \left\lfloor \frac{|V(G')|}{2} \right\rfloor = n + m + 1. \qquad (4)$$

Then by (3) and (4),

$$\beta_0(G) = \beta_0(G') < \left\lfloor \frac{|V(G)|}{2} \right\rfloor = \left\lfloor \frac{|V(G')|}{2} \right\rfloor.$$

Therefore in view of Lemma 2.1, neither G nor G' is a prime graph.

Now we claim that $(P_{2n+1} + \bar{K}_2) \cup C_{2m+1}$ is not a prime graph. Let $G'' = (P_{2n+1} + \bar{K}_2) \cup C_{2m+1}$. Note that $\beta_0(G'') = n + m$ and $|V(G'')| = 2n + 2m + 4$, and therefore $\beta_0(G') < \left\lfloor \frac{|V(G')|}{2} \right\rfloor$. So G'' is not a prime graph.

Finally we prove that $G^* = (P_{2n+1} + \bar{K}_2) \cup C_{2m}$ is a prime graph for all n and m. Let $\{v_1, v_2, ..., v_{2n+1}\}$, $\{u_1, u_2\}$ and $\{w_1, w_2, ..., w_{2m}\}$ be the sets of consecutive vertices of P_{2n+1}, $\bar{K}_2$ and C_{2m+1} respectively. Now due to Bertrand's postulate, there exists a prime number p lying strictly between $\frac{2n+3}{2}$ and 2n+3.

Define $f : V(G^*) \to \{1, 2, ..., 2n + 2m + 3\}$ using this number p as per the following two cases.

Case 1: $n \not\equiv 0(\mathrm{mod}\, 3)$

$f(u_1) = 1,$

$f(u_2) = p,$

$f(v_i) = p - i, \qquad i = 1, 2, ..., p - 4,$

$f(v_i) = 2n + p - i + 2, \qquad i = p - 3, p - 2, ..., 2n + 1,$

$f(w_1) = 2,$

$f(w_2) = 3,$

$f(w_i) = i + 2n + 3, \qquad i = 3, 4, ..., 2m.$

Case 2: $n \equiv 0(\mathrm{mod}\, 3)$

$f(u_1) = 1,$

$f(u_2) = p,$

$f(v_i) = p - i, \qquad i = 1, 2, ..., p - 3,$

$f(v_i) = 2n + p - i + 2, \qquad i = p - 2, p - 1, ..., 2n + 1,$

$f(w_1) = 2,$

$f(w_i) = i + 2n + 3, \qquad i = 2, 3, ..., 2m.$

It may be verified that f is a prime labeling of G*.■

The graph $C_n^{(k)}$ (where $k > 1$) is the one point union of k copies of cycle C_n and it is obtained from the k copies of cycle C_n by identifying one vertex from each of these k copies of C_n. It is quite obvious that the graph $C_n^{(k)}$ is prime but there are some interesting results about prime labeling of union of such graphs which we studied in [9]. Here we derive a result about union of $C_n^{(2)}$ and the cycle graph C_m.

Theorem 2.4: $C_n^{(2)} \cup C_m$ is a prime graph if and only if at least one of n and m is even.

Proof: First we show that $C_{2n+1}^{(2)} \cup C_{2m+1}$ is not a prime graph. Let G denote the graph $C_{2n+1}^{(2)} \cup C_{2m+1}$. It may be verified that $\beta_0(C_{2n+1}^{(2)}) = 2n$ and $\beta_0(C_{2m+1}) = m$. Therefore

$$\beta_0(G) = 2n + m. \qquad (5)$$

Since $|V(G)| = 4n + 2m + 2$,

$$\left\lfloor \frac{|V(G)|}{2} \right\rfloor = 2n + m + 1. \qquad (6)$$

So by (5) and (8),

$$\beta_0(G) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$$

Therefore by Lemma 2.1, G is not a prime graph.

Now we prove that $C_{2n}^{(2)} \cup C_m$ is a prime graph. Let $\{v_1, v_2, ..., v_{2n}\}$ and $\{v_1, v_{2n+1}, v_{2n+2}, ..., v_{4n-1}\}$ be the sets of consecutive vertices of two cycles of $C_{2n}^{(2)}$ and, let $\{v_{4n}, v_{4n+1}, ..., v_{4n+m-1}\}$ be set of consecutive vertices of C_m.

Define $f : V(C_{2n}^{(2)} \cup C_m) \to \{1, 2, ..., 4n + m - 1\}$ as

$f(v_1) = 2n + 1,$

$f(v_i) = i, \qquad i = 2, 3, ..., 2n$ and $4n + 1, 4n + 2, ..., 4n + m - 1,$

$f(v_i) = i + 1, \qquad i = 2n + 1, 2n + 2, ..., 4n - 1,$

$f(v_{4n}) = 1.$

It is easy to verify that f is a prime labeling of $C_{2n}^{(2)} \cup C_m$.

Finally we prove that $C_{2n+1}^{(2)} \cup C_{2m}$ is a prime graph. Let $\{v_1, v_2, ..., v_{2n+1}\}$ and $\{v_1, v_{2n+2}, v_{2n+3}, ..., v_{4n+1}\}$ be the sets of consecutive vertices of two cycles of $C_{2n+1}^{(2)}$ and, let $\{v_{4n+2}, v_{4n+3}, ..., v_{4n+2m+1}\}$ be set of consecutive vertices of C_{2m}.

Define $f : V(C_{2n+1}^{(2)} \cup C_{2m}) \to \{1, 2, ..., 4n+2m+1\}$ as

$$f(v_1) = 1,$$

$$f(v_i) = i + 2m, \qquad i = 2, 3, ..., 4n+1,$$

$$f(v_i) = i - 4n, \qquad i = 4n+2, 4n+3, ..., 4n+2m+1.$$

It may be verified that f is a prime labeling of $C_{2n+1}^{(2)} \cup C_{2m}$.■

For $m > 2$, the (m,n)-gon star denoted by $S_n^{(m)}$, is the graph obtained from the cycle C_n and n copies of the path P_{m-2} by joining the two end vertices of a path P_{m-2} to each pair of consecutive vertices of the cycle such that each of the end vertices of the path is adjacent to exactly one vertex of the cycle. It has total n(m-1) vertices and nm edges as can be seen in the graph of $S_6^{(4)}$ in Figure 3.

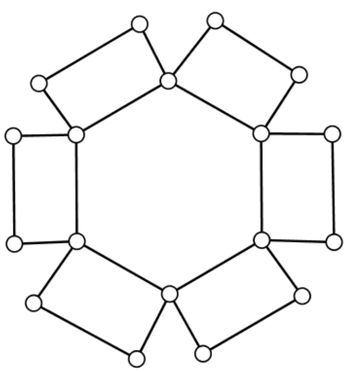

FIGURE 3. Graph $S_6^{(4)}$

In [8] it has been shown that $S_n^{(m)}$ is a prime graph for all n and m. Here we derive results for union of two (m, n)-gon stars.

Theorem 2.5: $S_n^{(m)} \cup S_k^{(j)}$ is not a prime graph if m, j are even and n, k are odd.

Proof: Let $G = S_n^{(m)} \cup S_k^{(j)}$. Since n is odd and m is even, the independence numbers of the cycle C_n and the path P_{m-2} are $\frac{n-1}{2}$ and $\frac{m-2}{2}$ respectively. So the number of elements in any independent set of $S_n^{(m)}$ is at most $\frac{n-1}{2} + n\left(\frac{m-2}{2}\right) = \frac{(m-1)n-1}{2}$. Similarly the cardinality of any independent set of $S_k^{(j)}$ is at most $\frac{(j-1)k-1}{2}$. Therefore

$$\beta_0(G) \le \frac{(m-1)n + (j-1)k - 2}{2}. \qquad (7)$$

Also, $|V(G)| = n(m-1) + k(j-1)$. Therefore

$$\left\lfloor \frac{|V(G)|}{2} \right\rfloor = \frac{n(m-1) + k(j-1)}{2}. \qquad (8)$$

By (7) and (8),

$$\beta_0(G) < \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$$

Thus G is not a prime graph.■

Theorem 2.6: $S_{2n}^{(2m)} \cup S_k^{(2m)}$ is a prime graph for all n, m and k.

Proof: Let G denote the graph $S_{2n}^{(2m)} \cup S_k^{(2m)}$. Let $\{u_1, u_{2,...,} u_{2n}\}$ and $\{v_1, v_{2,...,} v_k\}$ be the sets of consecutive vertices of the cycle C_{2n} and C_k respectively. Also for $1 \le i \le 2n$ and $1 \le j \le k$, let $\{u_q^i : 1 \le q \le 2m-2\}$ and $\{v_r^j : 1 \le r \le 2m-2\}$ be the sets of consecutive vertices of the paths P_{2m-2} in $S_{2n}^{(2m)}$ and $S_k^{(2m)}$ respectively, such that the vertices u_1^i, u_{2m-2}^i, v_1^j and v_{2m-2}^j are adjacent to the vertices u_i, u_{i+1}, v_j and v_{j+1} respectively. Define $f : V(G) \to \{1, 2, ..., (2m-1)(2n+k)\}$ as

$$f(u_i) = (i-1)(2m-1) + 2, \qquad i = 1, 2, ..., 2n,$$

$$f(u_q^i) = (i-1)(2m-1) + q + 2, \qquad i = 1, 2, ..., 2n, \quad q = 1, 2, ..., 2m-2,$$

$$f(v_1) = 1,$$

$$f(v_j) = (2n + j - 1)(2m-1) + 1, \qquad j = 2, 3, ..., k,$$

$$f(v_r^j) = (2n + j - 1)(2m-1) + r + 1, \qquad j = 1, 2, ..., k, \quad r = 1, 2, ..., 2m-2,$$

The definition of f is illustrated in Figure 4.
Observe that

$$\begin{aligned}\gcd(f(u_{i}), f(u_{i+1})) &= \gcd((i-1)(2m-1)+2, (i(2m-1)+2)\\ &= \gcd((i-1)(2m-1)+2, 2m-1)\\ &= \gcd(2, 2m-1)\\ &= 1.\end{aligned}$$

$$\begin{aligned}\gcd(f(v_j), f(v_{j+1})) &= \gcd((2n+j-1)(2m-1)+1, (2n+j)(2m-1)+1)\\ &= \gcd((2n+j-1)(2m-1)+1, 2m-1)\\ &= \gcd(1, 2m-1)\\ &= 1.\end{aligned}$$

$$\begin{aligned}\gcd(f(u_{1}), f(u_{2n})) &= \gcd(2, (2n-1)(2m-1)+2)\\ &= 1.\end{aligned}$$

$$\begin{aligned}\gcd(f(u_{1}), f(u^{2n}_{2m-2})) &= \gcd(2, (2n-1)(2m-1)+2m-2+2)\\ &= \gcd(2, (2n-1)(2m-1)+2m)\\ &= 1.\end{aligned}$$

Thus f is a prime labeling on G.■

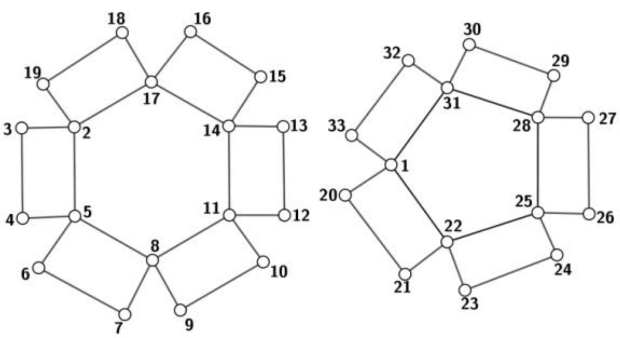

FIGURE 4. Prime labeling of $S_6^{(4)} \cup S_5^{(4)}$

The helm H_n is the graph obtained from a wheel by attaching a pendent edge at each vertex of the cycle C_n. The book graph B_n is the graph $S_n \times P_2$, where S_n is the star graph with n+1 vertices. Each of the graphs H_n and B_n is prime for all n, which is proved in [10] and [8] respectively. Our next result is about union of helm and book graph.

Theorem 2.7: $H_n \cup B_m$ is a prime graph for all n and m.

Proof: Let G denote the graph $H_n \cup B_m$. Let u_0 be a apex vertex of H_n. Let $\{u_1, u_2, ..., u_n\}$ be set of vertices of cycle C_n in H_n and let $\{u_1', u_2', ..., u_n'\}$ be set of pendant vertices of H_n such that u_i and u_i' are adjacent. Also let $\{(v_i, w_j): 0 \le i \le m, j = 1, 2\}$ be set of vertices of $B_m = S_m \times P_2$ where $\{v_0, v_1, ..., v_m\}$ and $\{w_1, w_2\}$ be sets of vertices of S_m and P_2 in which v_0 is a center vertex. Now there exists a prime number p lying strictly between $\frac{2n+3}{2}$ and 2n+3 which exist due to Bertrand's postulate.

We define $f: V(G) \to \{1, 2, ..., 2n+2m+3\}$ as

$$\begin{aligned}
&f(u_0) = p,\\
&f(u_i) = p-2i, && i = 1, 2, ..., \tfrac{p-5}{2},\\
&f(u_i') = p-2i+1, && i = 1, 2, ..., \tfrac{p-5}{2},\\
&f(u_i) = 4, && i = \tfrac{p-5}{2}+1,\\
&f(u_i') = 3, && i = \tfrac{p-5}{2}+1,\\
&f(u_i) = p+2(n-i)+2, && i = \tfrac{p-5}{2}+2, i = \tfrac{p-5}{2}+3, ..., n,\\
&f(u_i') = p+2(n-i)+1, && i = \tfrac{p-5}{2}+2, i = \tfrac{p-5}{2}+3, ..., n,\\
&f(v_0, w_1) = 1,\\
&f(v_0, w_2) = 2,\\
&f(v_i, w_1) = 2i+2n+2, && i = 1, 2, ..., m,\\
&f(v_i, w_2) = 2i+2n+3, && i = 1, 2, ..., m.
\end{aligned}$$

It may be verified that f is a prime labeling on G. The definition of f is illustrated in Figure 5.■

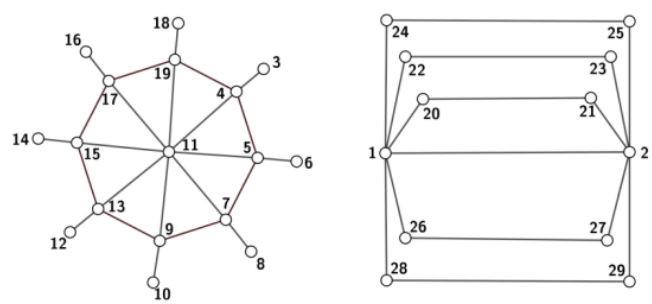

FIGURE 5. Prime labeling of $H_8 \cup B_5$

For a positive integer $n \ge 3$ and a subset $S \subseteq \{1, 2, ..., n\}$, the circulant graph Circ(n, S) is the graph with vertex set $\{v_1, v_2, ..., v_n\}$ and an edge between vertices v_i and v_j if and only if $|i-j| \in S \cup \{1, n-1\}$. Here we prove some results about circulant graph Circ(n, {k}), for $1 \le k \le \frac{n}{2}$. For simplicity we shall write Circ(n, {k}) as Circ(n, k).

Theorem 2.8: Circ(n, k) is not a prime graph in each of the following cases:

(i) n and k both are even.

(ii) n is odd

Proof: Case (i) n and k both are even.

In this case $\beta_0(C_n)=\frac{n}{2}$ for the cycle C_n of $Circ(n,k)$. Therefore since k is even, we have

$$\beta_0(Circ(n,k))<\beta_0(C_n)=\frac{n}{2}=\left\lfloor\frac{|V(Circ(n,k))|}{2}\right\rfloor.$$

So $Circ(n,k)$ is not a prime graph when n and k both are even.

Case (ii) n is odd.

Here $\beta_0(C_n)=\frac{n-1}{2}$ for the cycle C_n of $Circ(n,k)$. Suppose that $Circ(n,k)$ is a prime graph . Let f be a prime labeling of $Circ(n,k)$ and let $\{v_1,v_2,...,v_n\}$ be a set of consecutive vertices of $Circ(n,k)$. Without loss of generality suppose $f(v_{2i-1})$ is odd, for $i=1,2,...,\frac{n+1}{2}$ and $f(v_{2i})$ is even, for $i=1,2,...,\frac{n-1}{2}$. But if n is odd then v_2 is adjacent to at least one vertex with even label for any value of k. This is not possible. So $Circ(n,k)$ is not a prime graph when n is odd.■

There is a hope for positive results for $Circ(n,k)$ when n is even and k is odd. Our next result gives one of these positive results.

Theorem 2.9: Let p denote a prime number. Then $Circ(2p,p)$ is a prime graph if and only if $p\neq 2,3$.

Proof: We first show that $Circ(2p,p)$, where $p\neq 2,3$ is a prime graph. Let $G=Circ(2p,p)$ and let $\{v_1,v_2,...,v_{2p}\}$ be a set of consecutive vertices of $Circ(2p,p)$. We consider the following two cases.

Case 1: $p\equiv 1(\mathrm{mod}\,3)$.

Define $f:V(G)\to\{1,2,...,2p\}$ as

$$f(v_i)=i,\qquad i\neq p,p-2,$$
$$f(v_{p-2})=p,$$
$$f(v_p)=p-2.$$

We claim that $gcd(f(u),f(v))=1$ for any two adjacent vertices u and v.

If $i\neq p,2p,p-2$ then since p is a prime,

$gcd(f(v_{i.}),f(v_{i+p}))=gcd(i,i+p)=gcd(i,p)=1$. Also using $p\equiv 1(\mathrm{mod}\,3)$ we observe that

$gcd(f(v_{p-3.}),f(v_{p-2}))=gcd(p-3,p)=gcd(p,3)=1,$

$$\begin{aligned}gcd(f(v_{p+1.}),f(v_p))&=gcd(p+1,p-2)\\&=gcd(3,p-2)\\&=gcd(3,2)\\&=1.\end{aligned}$$

Using the fact that p is odd we get

$gcd(f(v_{2p.}),f(v_p))=gcd(2p,p-2)=gcd(4,p-2)=1$,

$gcd(f(v_{2p-2.}),f(v_{p-2}))=gcd(2p-2,p)=gcd(2,p)=1$.

Except these, the labels of any other pair of adjacent vertices are consecutive integers. Thus f is a prime labeling of G when $p\equiv 1(\mathrm{mod}\,3)$. Note that f is not a prime labeling when $p\equiv 2(\mathrm{mod}\,3)$. So we need to modify f for the resulting function g to be a prime labeling.

Case 2: $p\equiv 2(\mathrm{mod}\,3)$.

Define $g:V(G)\to\{1,2,...,2p\}$ as

$$g(v_i)=f(v_i),\qquad i\neq p-2,p,p+2,$$
$$g(v_{p-2})=p-2,$$
$$g(v_{p+2})=p,$$
$$g(v_p)=p+2.$$

The detailed verification that g is a prime labeling is almost similar to Case 1.

Now we show that $Circ(2p,p)$ is not prime when p=2, 3. Note that if p=2 then by Theorem 2.7, $Circ(2p,p)$ is not a prime graph. Also when p=3, $Circ(2p,p)$ is 3-regular graph and since 6 is relatively prime to only two numbers from 1 to 6, $Circ(6,3)$ cannot be a prime graph.■

In view of Theorem 2.8, we have complete information about the primality of $Circ(2n,n)$ when n is a prime number. However, if n is an odd integer which is not a prime number, then we do not have any general result about the primality of $Circ(2n,n)$. Along this line, so far we have been able to find prime labeling of $Circ(18,9)$ and $Circ(30,15)$ only, which are given in Figure 6 and Figure 7 respectively. At present it seems difficult to find a general formula for the prime labeling of $Circ(2n,n)$, where n is an arbitrary odd integer different from a prime number.

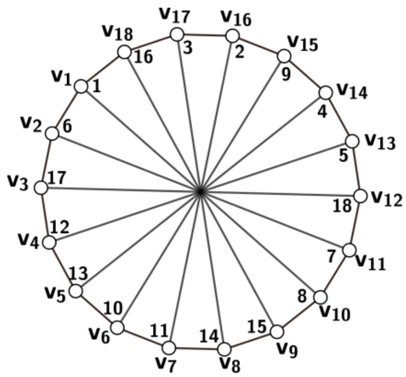

FIGURE 6. Prime labeling of $\mathrm{Circ}(18,9)$

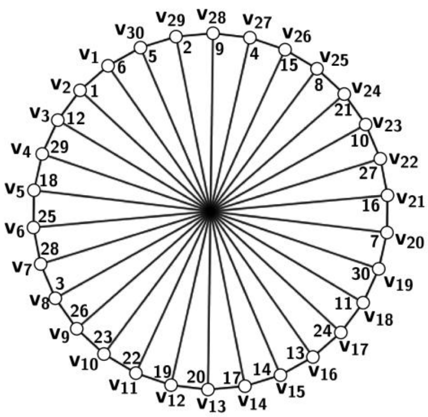

FIGURE 7. Prime labeling of $\mathrm{Circ}(30,15)$

In view of Theorem 2.7, Theorem 2.8, and the positive result of $\mathrm{Circ}(18,9)$ and $\mathrm{Circ}(30,15)$, we can make the following statement in the form of corollary.

Corollary 2.10: For $1 \le n \le 20$, $\mathrm{Circ}(2n,n)$ is a prime graph if and only if n is an odd integer.

III. RESULTS AND DISCUSSION

We have found the necessary and sufficient conditions for the graphs $W_n \cup C_m$, $(P_n + \bar{K}_2) \cup C_m$ and $C_n^{(2)} \cup C_m$ to be prime. We have also established some partial results on the prime labeling of $S_n^{(m)} \cup S_k^{(j)}$. We have considered a union of helm and book graph and shown it to be prime. Towards the end, we have considered circulant graphs for the study of prime labeling and in particular we have derived some interesting results about circulant graphs of the type $\mathrm{Circ}(2n,n)$.

IV. CONCLUSION AND FUTURE SCOPE

Although it seems difficult, some of the results derived in this paper can be made strong by further investigation. For instance, it will be interesting to get a full set of values of n, m, k and j for which $S_n^{(m)} \cup S_k^{(j)}$ is a prime graph. We also believe that by taking a more general set S, there is a lot more to explore about the prime labeling of the circulant graph $\mathrm{Circ}(n,S)$.

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AUTHORS PROFILE

S. K. Patel is a Professor of Mathematics at the Government Engineering College-Bhuj affiliated to the Gujarat Technological University. He has a teaching experience of more than 20 years and his research areas are Harmonic analysis and Graph theory. He is also a good problem solver especially in Real analysis. He has solved many problems posted in American Mathematical Monthly, Mathematics Magazine and College Mathematics Journal which are all published by the Mathematical Association of America and all his solutions are acknowledged in these journals. Recently, he was honored by GUDMAA (Gujarat University Department of Mathematics Alumni Association) for his achievements and contributions to the Mathematics community of Gujarat.

J. B. Vasava is working as a Lecturer in Mathematics at Dr. S. and S. S. Ghandhy College of Engineering- Surat which is affiliated to Gujarat Technological University. His area of research is Graph theory and he is currently pursuing his Ph.D under the guidance of Prof. (Dr.) S. K. Patel.

