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Cordial Labeling of an Arbitrary Supersubdivision of a Graph

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Abstract— In this paper, we have investigated a condition that if for a graph $\boldsymbol{G}$ there exist a binary vertex labeling $\boldsymbol{f}: \boldsymbol{V}(\boldsymbol{G}) \rightarrow \{\mathbf{0}, \mathbf{1}\}$ such that $\boldsymbol{e_f}(\mathbf{0}) = \mathbf{0}$ then an arbitrary supersubdivision of $\boldsymbol{G}$, $\boldsymbol{G} \times \boldsymbol{P_n}$, $\boldsymbol{G} \times \boldsymbol{C_{2n}}$, splitting graph of $\boldsymbol{G}$, disjoint union of graphs, path union of some graphs and a graph obtained by attaching $\boldsymbol{m}$ pendent edges to every vertex of the graph $\boldsymbol{G}$ are cordial.

Keywords— *Graph Labeling, Cordial Labeling, An Arbitrary Suppersubdivision.*

I. INTRODUCTION

Here, we have considered only simple, finite, undirected and non-trivial graph $G = (V, E)$ with the vertex set V and the edge set E. We follow Gross and Yellen [1] for various graph theoretic notations and terminology. We follow D. M. Burton [2] for number theory and J. A. Gallian [3] for survey on graph labeling. The definitions and other information which are useful for the this investigations are also given.

The aim of this work is to discuss and investigate some new results about cordial labeling.

The organization of the paper is as follows:

The paper opens with the introduction of Graph. In section II mentions Preliminaries. In the section III, we investigate the condition that if a graph G satisfies that condition then an arbitrary supersubdivision of that grapg is cordial. The last section concludes the paper.

II.PRELIMINARIES

Definition 1.1 *Let $G = (V, E)$ be a graph with vertex set V and edge set E. A binary vertex labeling $f: V(G) \rightarrow \{0,1\}$ induces an edge labeling $f^*: E(G) \rightarrow \{0,1\}$ defined by $f^*(e) = |f(u) - f(v)|$. For $i = 0$ and 1, let $v_f(i)$ be the number of vertices of G having label i under f and $e_f(i)$ be the number of edges of G having label i under f. Then f is called a cordial labeling if $|v_f(0) - v_f(1)| \leqslant 1$ and $|e_f(0) - e_f(1)| \leqslant 1$. If a graph G admits cordial labeling, then it is called cordial graph.*

Cahit[4] has introduced the concept of cordial labeling. In the same paper author proved that tree is cordial and K_n is cordial if and only if $n \leqslant 3$. Ho et al.[5] proved that unicyclic graph is cordial unless it is C_{4k+2}. Andar et al.[6] have discussed cordiality of multiple shells. Vaidya et al.[7, 8] have also discussed the cordiality of some graphs.

If we interchange the 0's and 1's in the binary vertex labeling f of G, the resulting labeling $\overline{f}$, called the dual labeling of f, will have the property:

$$v_{\overline{f}}(0) = v_f(1),\ v_{\overline{f}}(1) = v_f(0) \text{ and } e_{\overline{f}}(0) = e_f(0),\ e_{\overline{f}}(1) = e_f(1)$$

Definition 1.2 *Let $G(V, E)$ be a graph. A graph H is called a supersubdivision of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leqslant i \leqslant |E|$ in such a way that the ends of each e_i are merged with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G.*

Definition 1.3 *A supersubdivision H of a graph G is said to be an arbitrary supersubdivision of G if every edge of G is replaced by an arbitrary $K_{2,m}$ where m may vary for each edge arbitrarily.*

The concept of a supersubdivision and an arbitrary supersubdivision of a graph was introduced by G. Sethuraman and P. selvaraju [9]. Vaidya and Dani[10] have proved that the graphs obtained by arbitrary supersubdivision of tree, grid graph and complete bipartite graph are cordial. They have also discussed cordial labeling for the graph obtained by arbitrary supersubdivision of $C_n \odot P_m$. Vaidya and Kanani[11] have proved that the arbitrary supersubdivision of path and star admit cordial labeling. They have also proved that the arbitrary supersubdivision of cycle C_n is cordial except when n and all m_i are simultaneously odd numbers.

III CORDIAL LABELING OF AN ARBITRARY

SUPERSUBDIVISION OF GRAPH:

Theorem 2.1 *If for a graph G, there is a binary vertex labeling $f: V(G) \to \{0,1\}$ such that $e_f(0) = 0$ then the graph H obtained by an arbitrary supersubdivision of G is cordial.*

Proof: $G(V, E)$ is a graph with a binary vertex labeling $f: V(G) \to \{0,1\}$ such that $e_f(0) = 0$. Without loss of generality, for the graph G, assume that $v_f(0) = v_f(1) + k$, where $k \geqslant 0$. H is a graph obtained by an arbitrary supersubdivision of G where each edge e_i of the graph G is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer. Let $u_{i,j}$ be the vertices of m_i- vertices part, where $1 \leqslant i \leqslant |E|$, $1 \leqslant j \leqslant m_i$. The number of edges in H is $2m$, where $m = \sum_{i=1}^{|E|} m_i$. Define a binary vertex labeling $h: V(H) \to \{0,1\}$ as follows:

$h(v) = f(v)$ if $v \in G$.

Next, assign label 1 to first k vertices from $u_{i,j}$ and assign label 0 and 1 alternately to the remaining vertices $u_{i,j}$ starting with 0.

So for the graph H, we have $e_h(0) = e_h(1) = m$, $\therefore |e_h(0) - e_h(1)| = 0$.

Now,

$$v_h(0) = \begin{cases} v_f(0) + \frac{m-k}{2} & if \quad m-k \text{ is even;} \\ v_f(0) + \frac{m-k+1}{2} & if \quad m-k \text{ is odd.} \end{cases}$$

and

$$v_h(1) = \begin{cases} v_f(1) + k + \frac{m-k}{2} & if \quad m-k \text{ is even;} \\ v_f(1) + k + \frac{m-k-1}{2} & if \quad m-k \text{ is odd.} \end{cases}$$

$\therefore |v_h(0) - v_h(1)| \leqslant 1$.

Thus the graph H obtained by an arbitrary supersubdivision of G is cordial. ■

Corollary 2.2 *An arbitrary supersubdivision of the following graphs are cordial.*

1. Path P_n for all n.
2. Cycle C_{2n} for for all n.
3. Complete bipartite graph $K_{m,n}$.
4. Generalized Petersen graph $P(2n, 2k-1)$ for $1 \leqslant k < \left[\frac{n}{2}\right]$
5. t- ply graph $P_t(u, v)$, where the length of all the paths is odd.
6. Generalized Jahangir graph $J_{2m,n}$ for all $m \geqslant 1, n \geqslant 3$.
7. Tree T.

Proof: For (i) and (ii), label the vertices of path P_n and cycle C_{2n} by 0 and 1 alternately, then they satisfy the condition $e_f(0) = 0$. So from *Theorem 2.1* , an arbitrary supersubdivision of path P_n and cycle C_{2n} are cordial.

For (iii), label all the vertices of m-vertices part of $K_{m,n}$ by 0 and all the vertices of n-vertices part of $K_{m,n}$ by 1, then the number of edges with label 0 in $K_{m,n}$ is 0. Thus from *Theorem 2.1* , an arbitrary supersubdivision of Complete bipartite graph $K_{m,n}$ is cordial.

For (iv), let $u_1, u_2, u_3, \dots, u_{2n}$ be the outer consecutive vertices and $v_1, v_2, v_3, \dots, v_{2n}$ be the corresponding inner vertices of $P(2n, 2k-1)$. Now label the outer vertices from u_1 to u_{2n} by 0 and 1 alternately starting with 0 and label the inner vertices from v_1 to v_{2n} by 1 and 0 alternately starting with 1. So the edges $u_i u_{i+1}$ for $i = 1,2, \dots, 2n-1$, $u_{2n}u_1$ and $u_i v_i$ for $i = 1,2, \dots, 2n$ have label 1. Also the label of the edges $v_i v_{i+2k-1}$ for $i = 1,2, \dots, 2n$ (subscript modulo $2n$) have label 1 as i and $i + 2k - 1$ have different parity. Thus with this labeling, $P(2n, 2k-1)$ satisfies condition of *Theorem 2.1* . So an arbitrary supersubdivision of $P(2n, 2k-1)$ for $1 \leqslant k < \left[\frac{n}{2}\right]$ is cordial.

For (v), Suppose a t-ply $P_t(u, v)$ is obtained from t distinct paths P_i of length $2k_i + 1$ with the consecutive vertices $v_{i,0}, v_{i,1}, v_{i,2}, \dots, v_{i,2k_i+1}$ for each $i = 1,2, \dots, t$ by identifying all the vertices $v_{1,0}, v_{2,0}, v_{3,0}, \dots, v_{t,0}$ into a single vertex u and all the vertices $v_{1,2k_i+1}, v_{2,2k_i+1}, v_{3,2k_i+1}, \dots, v_{t,2k_i+1}$ into a single vertex v.

Now label u by 0 and v by 1. Next label the vertices $v_{i,1}, v_{i,2}, \dots, v_{i,2k_i}$ by 1 and 0 alternately starting with 1 for $1 \leqslant i \leqslant t$. Thus all the edges of $P_t(u, v)$ have label 1. Thus it follows from the *Theorem 2.1* that an arbitrary supersubdivision of t- ply graph $P_t(u, v)$, where the length of all the paths is odd, is cordial.

For (vi), let u_0 be an apex vertex and $u_1, u_2, \dots, u_{2mn}$ be the consecutive rim vertices of $J_{2m,n}$. The edges of $J_{2m,n}$ are $u_i u_{i+1}$ for $1 \leqslant i \leqslant 2mn - 1$, $u_{2mn}u_1$ and $u_0 u_{2m(i-1)+1}$ for $1 \leqslant i \leqslant n$. Now label u_0 by 0, u_i by 1 if i is odd and by 0 if i is even. Then all the edges have label 1. So from the *Theorem 2.1* that an arbitrary supersubdivision of generalized Jahangir graph $J_{2m,n}$ for $m \geqslant 1, n \geqslant 3$ is cordial.

For (vii), let v_1 be a pendant vertex of tree T. Define the binary vertex labeling $f: V(T) \to \{0,1\}$ as follows:

$$f(v_i) = \begin{cases} 0 & if \quad v_i = v_1; \\ 1 & if \quad d(v_1, v_i) \text{ in T is odd;} \\ 0 & if \quad d(v_1, v_i) \text{ in T is even.} \end{cases}$$

By the above labeling, every edge of T has label 1. Thus from *Theorem 2.1* , an arbitrary supersubdivision of tree T is cordial. ■

Theorem 2.3 *An arbitrary supersubdivision of a graph obtained by an arbitrary supersubdivision of any graph is cordial.*

Proof: Let $G(V,E)$ be any graph and H be the graph obtained by an arbitrary supersubdivision of G where each edge e_i of the graph G is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer. Let $u_{i,j}$ be the vertices of m_i- vertices part, where $1 \leqslant i \leqslant |E|$, $1 \leqslant j \leqslant m_i$. Define a binary vertex labeling $h: V(H) \to \{0,1\}$ as follows:

$$h(v) = \begin{cases} 0 & if \quad v \text{ is a vertex of } G; \\ 1 & if \quad v = u_{i,j} \text{ for all } i \text{ and } j. \end{cases}$$

We can easily verify that the above binary vertex labeling h satisfies the condition $e_h(0) = 0$. Thus from the *Theorem 2.1* , an arbitrary supersubdivision of the graph H is cordial. ■

Theorem 2.4 *If for a graph G, there is a binary vertex labeling $f: V(G) \to \{0,1\}$ such that $e_f(0) = 0$ then an arbitrary super subdivision of the graph $G \times P_n$ is cordial.*

Proof: We can consider the graph $G \times P_n$ as n layers graph, in which every layer is graph G and each vertex of the graph G of i^{th} layer is adjacent with the corresponding vertex of the graph G of $(i+1)^{th}$ layer, where $1 \leqslant i \leqslant n-1$. Define a binary vertex labeling $h: V(G \times P_n) \to \{0,1\}$ as follows:

$$h(v) = \begin{cases} f(v) & if \quad v \text{ lies in odd layer;} \\ \overline{f}(v) & if \quad v \text{ lies in even layer.} \end{cases}$$

We can easily verify that the above binary vertex labeling h satisfies the condition $e_h(0) = 0$. Thus from the *Theorem 2.1* , an arbitrary supersubdivision of the graph $G \times P_n$ is cordial. ■

Corollary 2.5 *For each graph G shown in corollary 3.4, an arbitrary supersubdivision of graphs $G \times P_n$ is cordial.*

Proof: The result is clear from the above *Corollary 2.2* and *Theorem 2.4* . ■

Theorem 2.6 *If for a graph G, there is a binary vertex labeling $f: V(G) \to \{0,1\}$ such that $e_f(0) = 0$ then an arbitrary super subdivision of the graph $G \times C_{2n}$ is cordial.*

Proof: We can consider the graph $G \times C_{2n}$ as $2n$ layers graph, in which every layer is graph G and each vertex of the graph G of i^{th} layer is adjacent with the corresponding vertex of the graph G of $(i+1)^{th}$ layer, where $1 \leqslant i \leqslant n-1$ and each vertex of the graph G of $2n^{th}$ layer is adjacent with the corresponding vertex of the graph G of 1^{st} layer. Define a binary vertex labeling $h: V(G \times C_{2n}) \to \{0,1\}$ as follows:

$$h(v) = \begin{cases} f(v) & if \quad v \text{ lies in odd layer;} \\ \overline{f}(v) & if \quad v \text{ lies in even layer.} \end{cases}$$

We can easily verify that the above binary vertex labeling h satisfies the condition $e_h(0) = 0$. Thus from the *Theorem 2.1* , an arbitrary supersubdivision of the graph $G \times C_{2n}$ is cordial. ■

Theorem 2.7 *If for a graph G, there is a binary vertex labeling $f: V(G) \to \{0,1\}$ such that $e_f(0) = 0$ then an arbitrary super subdivision of the splitting graph $S(G)$ is cordial.*

Proof: Let v_i be the vertices of the graph G and v'_i be the corresponding vertices of v_i in $S(G)$, where $i = 1,2,\dots,n$. Define a binary vertex labeling $h: V(S(G)) \to \{0,1\}$ as follows:

$$h(v) = \begin{cases} f(v_i) & if \quad v = v_i \text{ and } v = v'_i \text{ for } i = 1,2,\dots,n; \end{cases}$$

From the above labeling, for the graph $S(G)$, we have $e_h(0) = 0$ as $e_f(0) = 0$ and neighbourhood of v_i is same as the neighbourhood of v'_i. Thus from the *Theorem 2.1* , an arbitrary supersubdivision of $S(G)$ is cordial. ■

Theorem 2.8 *If for the graphs $G_1, G_2, \dots, G_n$, there is a binary vertex labeling $f_i: V(G) \to \{0,1\}$ for each G_i, such that $e_{f_i}(0) = 0$ for $1 \leqslant i \leqslant n$, then an arbitrary supersubdivision of disjoint union of G_i is cordial.*

Proof: Let H be the graph obtained by disjoint union of G_i. Label each G_i by f_i for $1 \leqslant i \leqslant n$. Thus the number of edges with label 0 of H is 0. So from *Theorem 2.1* , an arbitrary supersubdivision of H is cordial. ■

Theorem 2.9 *If for the graphs $G_1, G_2, \dots, G_n$, there is a binary vertex labeling $f_i: V(G) \to \{0,1\}$ for each G_i, such that $e_{f_i}(0) = 0$ for $1 \leqslant i \leqslant n$, then an arbitrary supersubdivision of the graph obtained by any path union of the graphs G_i is cordial.*

Proof: Let H be the graph obtained by path union of the graphs $G_1, G_2, \dots, G_n$. Let $e_i = v_i v_{i+1}$ be the edge, joining the graphs G_i and G_{i+1} in H, where v_i and v_{i+1} are the vertices of the graphs G_i and G_{i+1} respectively for $i = 1,2,\dots,n-1$. Define a binary vertex labeling $h: V(H) \to \{0,1\}$ as follows:

$$h(v) = \begin{cases} f_1(v) & if \quad v \text{ is a vertex of } G_1; \\ f_i(v) & if \quad v \text{ is a vertex of } G_i \text{ and } f_{i-1}(v_{i-1}) \neq f_i(v_i) \text{ for } i = 2,3,\dots,n; \\ \overline{f}_i(v) & if \quad v \text{ is a vertex of } G_i \text{ and } f_{i-1}(v_{i-1}) = f_i(v_i) \text{ for } i = 2,3,\dots,n. \end{cases}$$

It is routine to check that for the graph H, $e_h(0) = 0$. Thus from the *Theorem 2.1* , an arbitrary supersubdivision of the graph H is cordial. ■

Theorem 2.10 *If for a graph G, there is a binary vertex labeling $f: V(G) \rightarrow \{0,1\}$ such that $e_f(0) = 0$ then an arbitrary supersubdivision of a graph obtained by attaching m pendent edges to every vertex of the graph G is cordial, where m may vary for each vertex arbitrary.*

Proof: Here G is a graph with a binary vertex labeling $f: V(G) \rightarrow \{0,1\}$ such that $e_f(0) = 0$. Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of G and $v_{i,1}, v_{i,2}, v_{i,3}, \dots, v_{i,m_i}$ be the pendent vertices corresponding to each v_i for $i = 1,2,3, \dots, n$. Let H be a graph obtained by attaching m_i pendent vertices to every vertex v_i of the graph G for $i = 1,2,3, \dots, n$. Define a binary vertex labeling $h: V(H) \rightarrow \{0,1\}$ as follows:

$$h(v) = \begin{cases} f(v) & if \quad v = v_i \text{ for } i = 1,2,3, \dots, n; \\ \overline{f}(v_i) & if \quad v = v_{i,1}, v_{i,2}, \dots, v_{i,m_i} \text{ for } i = 1,2,3, \dots, n. \end{cases}$$

It is again routine to check that for the graph H, $e_h(0) = 0$. Thus from the *Theorem 2.1*, an arbitrary supersubdivision of the graph H is cordial. ■

IV. CONCLUDING REMARKS

In this paper, we have investigated a necessary condition for an arbitrary supersubdivision of a graph to be cordial graph. Here we have proved that if a graph G satisfy the condition given in the *Theorem 2.1* then $G \times P_n$, $G \times C_{2n}$, splitting graph of G, disjoint union of graphs, path union of some graphs and a graph obtained by attaching m pendent edges to every vertex of the graph G are cordial. To find such condition for other graph operations to be a cordial graph is again an open area for researchers.

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