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Centroidal, Logarithmic and Identric Mean Labeling of Graphs

Alagu S.[1*], R. Kala[2]

[1,2]Dept. of Mathematics, Manonmaniam Sundaranar University Tirunelveli – 627012, Tamilnadu, India

Correspondence Author: alagu391@gmail.com

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Abstract- Graph labeling was first introduced by Rosa in 1966 [1]. Labeling of graphs is an assignment of non-negative integers to vertices, edges or both according to some specified conditions. Mean labeling of graphs was introduced by Somasundaram.S and Ponraj.R in 2003[2]&[3]. Subsequently, labelings of graphs were done with geometric mean, harmonic mean etc.,. We have already introduced Centroidal mean labeling[4], Logarithmic and Identric mean labeling of graphs[5]. In this paper we acquire these mean labelings for some more standard graphs.
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Keywords: Centroidal Mean, Logarithmic Mean, Identric Mean, Labeling, Mean graphs, Centroidal mean labeling, Logarithmic mean labeling, Identric mean labeling.

I. INTRODUCTION

The study of a mean labeling is a two way process. It not only shows how different graphs behave with respect to the particular mean but also it shows how the mean behaves graphically. This is to present the academic point of view and not the application side of it, though it has some.

Notations : Let a, b be two positive integers. Then $C(a,b)$, $L(a,b)$, $I(a,b)$ denote the Centroidal mean, Logarithmic mean and Identric mean of a and b respectively.

Definition 1.1 [4]: Let $G=(V,E)$ be a graph with p vertices and q edges. G is said to be a centroidal mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1,2,\dots.q+1$ in such a way that on labeling each edge $e=xy$ with $\left\lfloor\frac{2(x^2+xy+y^2)}{3(x+y)}\right\rfloor$ or $\left\lceil\frac{2(x^2+xy+y^2)}{3(x+y)}\right\rceil$, the resulting edge labels are distinct and are from $1,2,\dots.q$. In this case, f is called a Centroidal mean labeling.

Definition 1.2 [5]: Let $G=(V,E)$ be a graph with p vertices and q edges. G is said to be a Logarithmic mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1,2,\dots.q+1$ in such a way that on labeling each edge $e=xy$ with $\left\lfloor\frac{y-x}{\log y-\log x}\right\rfloor$ or $\left\lceil\frac{y-x}{\log y-\log x}\right\rceil$, the resulting edge labels are distinct and are from $1,2,\dots.q$. In this case, f is called a Logarithmic mean labeling.

Definition 1.3 [5]: Let $G=(V,E)$ be a graph with p vertices and q edges. G is said to be a Identric mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1,2,\dots.q+1$ in such a way that on labeling each edge $e=xy$ with $\left\lfloor\frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}\right\rfloor$ or $\left\lceil\frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}\right\rceil$, the resulting edge labels are distinct and are from $1,2,\dots.q$. In this case, f is called a Identric mean labeling.

Definition 1.4 : A graph G is said to be complete, if every pair of distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.

Definition 1.5 [6]: An Alternate Triangular Snake $A(T_n)$ is obtained from a path $u_1u_2\dots u_n$ by joining u_i and u_{i+1} (alternately) to new vertex v_i. That is every alternate edge of a path is replaced by C_3.

Definition 1.6 [6]: An Alternate Quadrilateral Snake $A(Q_n)$ is obtained from a path $u_1u_2\dots u_n$ by joining $u_1, u_{3,\dots}u_{n-1}$ (alternately) to new vertices $v_1, v_2, \dots v_{\frac{n}{2}}$ respectively, $u_2, u_4, \dots u_n$ to new vertices $w_1, w_2, \dots w_{\frac{n}{2}}$ respectively and then joining v_i and w_i $(1 \le i \le \frac{n}{2})$ or by joining $u_2, u_4, \dots u_{n-2}$ (alternately) to new vertices $v_1, v_2, \dots v_{\frac{n}{2}-1}$ respectively, $u_{3,} u_5, \dots u_{n-1}$ to new vertices $w_1, w_2, \dots w_{\frac{n}{2}-1}$ respectively and then joining v_i and w_i $(1 \le i \le \frac{n}{2}-1)$.

That is every alternate edge of a path is replaced by C_4. The two types of Alternate Quadrilateral Snakes correspond to whether C_4 starts from u_1 or u_2 respectively.

Definition 1.7 [6]: The square G^2 of a graph G has $V(G^2) = V(G)$, with u, v adjacent in G^2 whenever $d(u,v) \leq 2$ in G.

II. MAIN RESULTS

Section I

Theorem 1 : Alternate Triangular Snakes are Centroidal, Logarithmic and Identric mean graphs.

Proof :
Let $A(T_n)$ be an Alternate Triangular snake. Two different cases arise depending on the structure of $A(T_n)$.

Case (i) : C_3 starts from u_1
Consider the function $f: V(A(T_n)) \to \{1,2, \dots q+1\}$ defined by $f(u_i) = 2i-1$ $(1 \leq i \leq n)$ and $f(v_i) = 2i$ $for\ i = 1,3,..n-1$. The edges u_iu_{i+1} $(1 \leq i \leq n)$ receive the label $2i$, u_iv_i $(i = 1,3,5..n-1)$ receive $2i-1$ as the label, v_iv_{i+1} $(i = 1,3,5..n-1)$ receive the label $2i+1$.

Case (ii) : C_3 starts from u_2
Let the function $f: V(A(T_n)) \to \{1,2, \dots q+1\}$ be defined by $f(u_1) = 1, f(u_2) = 2, f(u_i) = 2i-2$ $(3 \leq i \leq n)$, $f(v_i) = 2i-1$ $for\ i = 2,4,6 \dots n-2$. The edges $u_iu_{i+1}(1 \leq i \leq n-1)$, u_iv_i $(i = 2,4,6 \dots n-2)$, $v_iu_{i+1}(i = 2,4,6 \dots n-2)$ receive the label $2i-1, 2i-2, 2i$ respectively.

These functions work well as Centroidal, Logarithmic and Identric mean labeling.
Thus $A(T_n)$ is Centroidal, Logarithmic and Identric mean graph.

Theorem 2 : Alternate Quadrilateral Snakes are Centroidal, Logarithmic and Identric mean graphs.

Proof :
Let $A(Q_n)$ be an Alternate Quadrilateral snake. Two different cases arise depending on the structure of $A(Q_n)$.

Case (i) : C_4 starts from u_1
Define $f: V(A(T_n)) \to \{1,2, \dots q+1\}$ by $f(u_1) = 1$, $f(v_1) = 2, f(w_1) = 3, f(u_2) = 4$,
$f(u_i) = f(u_{i-2}) + 5$; $(3 \leq i \leq n)$,
$f(v_i) = f(v_{i-1}) + 5$; $2 \leq i \leq \frac{n}{2}$,
$f(w_i) = f(w_{i-1}) + 5$; $2 \leq i \leq \frac{n}{2}$.

Edges get labeled in the following manner.
$f(u_1v_1) = 1, f(v_1w_1) = 2, f(u_1u_2) = 3, f(u_2w_1) = 4, f(u_2u_3) = 5$,
$f(u_iu_{i+1}) = f(u_{i-2}u_{i-1}) + 5$; $3 \leq i \leq n-1$,
$f\left(u_iv_{\frac{i+1}{2}}\right) = f\left(u_{i-2}v_{\frac{i-1}{2}}\right) + 5$; $i = 3,5,7 \dots n-1$
$f(v_iw_i) = f(v_{i-1}w_{i-1}) + 5$; $2 \leq i \leq \frac{n}{2}$,
$f\left(u_iw_{\frac{i}{2}}\right) = f\left(u_{i-2}w_{\frac{i}{2}-1}\right) + 5$; $i = 4,6, \dots n$

Hence all the edge labels are distinct.

Case (ii) : C_4 starts from u_2

Define $f: V(A(T_n)) \to \{1,2, \dots q+1\}$ by
$f(u_1) = 1$, $f(u_2) = 2, f(v_1) = 3$,
$f(w_1) = 4, f(u_3) = 5$,
$f(u_i) = f(u_{i-2}) + 5$; $(4 \leq i \leq n)$,
$f(v_i) = f(v_{i-1}) + 5$; $2 \leq i \leq \frac{n}{2} - 1$,
$f(w_i) = f(w_{i-1}) + 5$; $2 \leq i \leq \frac{n}{2} - 1$.

Edges labels are as follows.
$f(u_1u_2) = 1, f(u_2v_1) = 2$,
$f(v_1w_1) = 3, f(u_2u_3) = 4, f(u_3w_1) = 5$,
$f(u_iu_{i+1}) = f(u_{i-2}u_{i-1}) + 5$; $4 \leq i \leq n-1$,
$f\left(u_iv_{\frac{i}{2}}\right) = f\left(u_{i-2}v_{\frac{i}{2}-1}\right) + 5$; $i = 4,6,8 \dots n-2$,
$f(v_iw_i) = f(v_{i-1}w_{i-1}) + 5$; $2 \leq i \leq \frac{n}{2} - 1$,
$f(u_{2i+1}w_i) = f(u_{i-2}w_{i-1}) + 5$; $2 \leq i \leq \frac{n}{2} - 1$.

The labels are all distinct.

Hence $A(Q_n)$ is Centroidal, Logarithmic and Identric mean graph.

In the above two theorems we see that the same function works for Centroidal, Logarithmic and Identric labeling. Now we have a theorem where Centroidal mean labeling and Logarithmic/Identric mean labeling have different functions.

Theorem 3 : P_n^2 is Centroidal, Logarithmic and Identric mean graph.

Proof :
The function $f: V(P_n^2) \to \{1,2,3, \dots .q+1\}$ defined by $f(u_1) = 1, f(u_i) = 2(i-1)$; $(2 \leq i \leq n)$ yields labels for the edges u_iu_{i+1} $(1 \leq i \leq n-1)$ as $2i-1$ and for the edges u_iu_{i+2} $(1 \leq i \leq n-2)$ as $2i$. Hence this function is a Centroidal mean labeling.

The function $f: V(P_n^2) \to \{1,2,3, \dots .q+1\}$ defined by $f(u_i) = 2(i-1)$; $(1 \leq i \leq n-1)$, $f(u_n) = 2n-2$ yields labels for the edges u_iu_{i+1} $(1 \leq i \leq n-1)$ as $2i-1$

and for the edges $u_i u_{i+2}$ $(1 \le i \le n-2)$ as $2i$. Hence this function is Logarithmic and Identric mean labeling.

Thus P_n^2 is a Centroidal, Logarithmic and Identric mean graph.

Section II

Centroidal mean labeling of K_n
Now we make some observations which will be useful in establishing the impossibility of Centroidal mean labeling of K_n , in general.

Observation 1 :
For $m \ge 2$,
$$\frac{2m}{3} - \left\lfloor \frac{2m}{3} \right\rfloor \begin{aligned} &= 0 && if\ \ m = 3k \\ &= 0.333 && if\ \ m = 3k-1 \\ &= 0.666 && if\ \ m = 3k-2 \end{aligned}$$

Observation 2 : $C(1,m) \simeq \frac{2m}{3}$ as m grows larger. That is, $C(1,m)$ approaches $\frac{2m}{3}$.

$$C(1,m) - \frac{2m}{3} = \frac{2}{3} - \frac{2m}{3(m+1)}$$

$$\frac{m}{m+1} \to 1 \text{ as } m \to \infty$$

Hence $\lim_{n\to\infty} \left[C(1,m) - \frac{2m}{3} \right] = 0$

Observation 3 : $\lfloor C(1,m) \rfloor = \left\lfloor \frac{2m}{3} \right\rfloor$

$$C(1,m) = \frac{2}{3}(m+1) - \frac{2m}{3(m+1)}$$
$$= \frac{2m}{3} + \frac{2}{3}\left(1 - \frac{m}{m+1}\right)$$
$$= \frac{2m}{3} + \frac{2}{3(m+1)}$$

Note that for $m = 1$, $\frac{2}{3(m+1)} = 0.333$

Case (i) : $\frac{2m}{3}$ is an integer $\Rightarrow \lfloor C(1,m) \rfloor = \left\lfloor \frac{2m}{3} \right\rfloor$

Case (ii) : $\frac{2m}{3} = \left\lfloor \frac{2m}{3} \right\rfloor + 0.333$

$$\frac{2m}{3} < C(1,m) < \frac{2m}{3} + 0.333 = \left\lfloor \frac{2m}{3} \right\rfloor + 0.666$$

Therefore $\lfloor C(1,m) \rfloor = \left\lfloor \frac{2m}{3} \right\rfloor$

Case (iii) : $\frac{2m}{3} = \left\lfloor \frac{2m}{3} \right\rfloor + 0.666$

$$\frac{2m}{3} < C(1,m) < \frac{2m}{3} + 0.333 = \left\lfloor \frac{2m}{3} \right\rfloor + 0.999$$

Therefore $\lfloor C(1,m) \rfloor = \left\lfloor \frac{2m}{3} \right\rfloor$

Observation 4 : For $m \ge 6$, $C(1,m) - \frac{2m}{3} < \frac{1}{10}$

$$C(1,m) = \frac{2(1+m+m^2)}{3(1+m)}$$
$$= \frac{2}{3}(1+m) - \frac{2m}{3(m+1)}$$

$$C(1,m) - \frac{2m}{3} = \frac{2}{3}\left[1 - \frac{m}{m+1}\right] = \frac{2}{3(m+1)}$$

$$m \ge 6 \Rightarrow m+1 \ge 7$$
$$\Rightarrow 3(m+1) \ge 21$$
$$\Rightarrow \frac{2}{3(m+1)} \le \frac{2}{21} < \frac{1}{10}$$
$$\Rightarrow C(1,m) - \frac{2m}{3} < \frac{1}{10}$$

Observation 5 : For $m \ge 6$,
$$C(2,m) - C(1,m) < \frac{1}{10}$$

$$C(2,m) - C(1,m) = \frac{2(4+2m+m^2)}{3(2+m)} - \frac{2(1+m+m^2)}{3(1+m)}$$
$$= \frac{2}{3}\left[1 - \frac{m^2}{(m+1)(m+2)}\right]$$

$$C(2,m) - C(1,m) < \frac{1}{10} \Leftrightarrow \frac{2}{3}\left[1 - \frac{m^2}{(m+1)(m+2)}\right] < \frac{1}{10}$$
$$\Leftrightarrow 1 - \frac{m^2}{(m+1)(m+2)} < \frac{3}{20}$$
$$\Leftrightarrow \frac{m^2}{(m+1)(m+2)} > \frac{17}{20}$$
$$\Leftrightarrow 3m^2 - 51m - 34 > 0$$
$$\Leftrightarrow m > 17.6 \ \&\ m < 0.64$$

Hence $C(2,m) - C(1,m) < \frac{1}{10} \ \ \forall\, m \ge 18$

Observation 6 : For $m \ge 30$, $C(3,m) - C(2,m) < \frac{1}{10}$

$$C(3,m) - C(2,m) = \frac{2(9+3m+m^2)}{3(3+m)} - \frac{2(4+2m+m^2)}{3(2+m)}$$

$= \frac{2}{3}(m+3) - \frac{2m}{m+3} - \frac{2}{3}(m+2) + \frac{4m}{3(m+2)}$

$$= \frac{2}{3}\left[\frac{5m+6}{(m+2)(m+3)}\right]$$

$$C(2,m) - C(1,m) < \frac{1}{10} \Leftrightarrow \frac{2}{3}\left[\frac{5m+6}{(m+2)(m+3)}\right] < \frac{1}{10}$$

$$\Leftrightarrow \frac{5m+6}{(m+2)(m+3)} < \frac{3}{20}$$

$$\Leftrightarrow 3m^2 - 85m - 102 > 0$$

$$\Leftrightarrow m < -1.15 \ \& \ m > 29.48$$

Therefore, $C(2,m) - C(1,m) < \frac{1}{10} \quad \forall m \geq 30$

Result : All the above observations lead to the result that, for $m \geq 30$,

$$\left\lfloor\frac{2m}{3}\right\rfloor < C(1,m) < C(2,m) < C(3,m) < \left\lfloor\frac{2m}{3}\right\rfloor + 1$$

Theorem 4 : For $n > 4$, K_n is not a Centroidal mean graph.

Proof :
Let $q = \frac{n(n-1)}{2}$ be the number of edges. To get 1 as an edge label, some pair of vertices receive (1,2). Therefore 1 and 2 are compulsory vertex labels. To get 2 as an edge label, some pair of vertices must receive one of the following : (1,2), (2,3), (1,3). Since (1,2) has already been used for the label 1, it is inevitable that we go for (2,3) or (1,3). Either of the choices demand the need for 3 as a vertex label. So it is mandatory that some triplet receives 1,2,3 as vertex labels. Now to get q as an edge label, one of the following must be given to some pair of vertices.

$(q-2,q), (q-1,q), (q-1,q+1), (q,q+1), \ (q-2,q+1), (q-3,q+1)$
Hence some vertex must receive either q or $q+1$.
But for $m \geq 30$, $\left\lfloor\frac{2m}{3}\right\rfloor < C(1,m) < C(2,m) < C(3,m) < \left\lfloor\frac{2m}{3}\right\rfloor + 1$.

It becomes impossible to give different labels to $(1,q),(2,q),(3,q)$ or $(1,q+1),(2,q+1),(3,q+1)$. Therefore, K_n $(n \geq 9)$ is not a Centroidal mean graph. It is easy to see that K_2, K_3, K_4 are Centroidal mean graphs. Now we give the argument for $5 \leq n \leq 8$. For $n = 6,7,8$, $q = 15, 21, 28$ respectively. For K_6 , one of the vertices must receive 15 or 16. But all of $C(1,15/16), C(2,15/16), C(3,15/16)$ lie between 10 and 11. Labeling (1,15/16), (2,15/16), (3,15/16) with different labels is impossible. As for K_7 , $C(1,7/8), C(2,7/8), C(3,7/8)$ all lie between 14 and 15 and hence different labels for those three edges are out of question.
Similarly for K_8,
$18 < C(1,28) < C(2,28) < C(3,28) < 19$ and
$19 < C(1,29) < C(2,29) < C(3,29) < 20$.
Now let us consider K_5. Since $q = 10$, some vertex must receive 10 or 11 as a label. If some vertex is given 11, then the inequality $7 < C(1,11) < C(2,11) < C(3,11) < 8$ makes it impossible in giving different labels to (1,11), (2,11), (3,11). Suppose that 10 is given for some vertex. Then the possibilities are (9,10) and (8,10).
But $6 < C(1,9) < C(2,9) < C(3,9) < 7$ and $5 < C(1,8) < C(2,8) < C(3,8) < 6$.
Therefore, K_5 is not a Centroidal mean graph.
Hence the theorem.

III. CONCLUSION

In [4] and [5], we have already investigated the Centroidal, Logarithmic and Identric mean labelings for the following types of graphs : Path, Cycle, Star, Triangular Snake, Quadrilateral Snake, Crown, Ladder, Comb, Caterpillar and Dragon.

In this paper, we have investigated these three mean labelings for a few more standard graphs like Alternate Triangular Snake, Alternate Quadrilateral Snake, product graph P_n^2 and Centroidal mean for the Complete graph K_n. As a scope for future work, further investigation with more graphs may be carried out; moreover, as in other mean labelings, general characterization results may be of interest.

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AUTHORS' PROFILE

Alagu. S, (M.Sc., M.Phil.) is a Research Scholar pursuing Ph.D degree in Mathematics at Manonmaniam Sundaranr University, Tirunelveli. She gave a brilliant academic performance securing first rank at the university level in both undergraduate and postgraduate studies. Presently, she is a DST INSPIRE fellow.

Dr.R.Kala is a Professor of Mathematics at Manonmaniam Sundaranar University, Tirunelveli. She has published 3 books, 45 papers in International journals and 9 in national journals. She has 19 years of teaching experience and 25 years of research experience.