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Some Fixed Point Theorems on Generalized Contractive mappings in cone metric space

R. Krishnakumar[1*], R. Livingston[2]

[1] PG & Research Department of Mathematics, Urumu Dhanalakshmi College, Tiruchirappalli, India
[2] PG & Research Department of Mathematics, Urumu Dhanalakshmi College, Tiruchirappalli, India

[*]Corresponding Author: srkudc7@gmail.com, Tel.: 94431 86332

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***Abstract*:** The purpose of this paper is to obtain sufficient conditions for the existence of unique fixed point of generalized contractive type mappings on complete metric spaces.

***Keywords*:** fixed point, generalized contractive mapping, complete cone metric space, sequently convergent.

I. INTRODUCTION

Guang and Xian generalized the notion of metric spaces, replacing the set of real numbers by an ordered Banach space, defining in this way, a cone metric space. These authors also described the convergence of sequences in cone metric spaces and introduced the corresponding notion of completeness. Afterwards, they proved some fixed point theorems of contractive mappings on complete cone metric spaces.

A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh introduced the classes of T-Contractive functions, extending the Banach Contraction principle and Edelstein's fixed point theorem.
In this paper, we generalized the notion of T-contractive mapping defined on a complete cone metric space (X, d), and exdend the results.

Definitions and preliminary

Definition 1: Let E be a real Banach space and P a subset of E . P is called cone if and only if :

1) P is closed, non-empty, and $P \neq \{0\}$,
2) $ax + by \in P \; for \; all \;\; x, y \in P$ and non-negative real numbers a, b;
3) $P \cap (-P) = \{0\}$.

Note also that the relations $\operatorname{int} P + \operatorname{int} P \subseteq \operatorname{int} P$ and $\lambda \operatorname{int} P \subseteq \operatorname{int} P (\lambda < 0)$ holds. For given cone $P \subseteq E$, we can define on E a partial ordering $\leq$ with respect to P by putting $x \leq y$ if and only if $y - x \in P$. Further, $x < y$ stands for $x \leq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P .

Definition 2: Let E be a real Banach space and $P \subset E$ be a cone. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$

The least positive number K satisfying the above inequality is called the normal constant of P.

In the following, we always suppose that E is a real Banach space, P is a cone in E with $\operatorname{int} P \neq 0$ and $\leq$ is the partial ordering with respect to P.

Definition 3: Let M be a non-empty set and $d : M \times M \to E$ a mapping such that:

(i) $0 \leq d(x,y)$ for all $x, y \in M$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in M$;

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in M$.

Then d is called a cone metric on M and (M,d) is called a cone metric space.

Notice that the notion of a cone metric space is more general than the corresponding of a metric space.

Definition 4: Let (M,d) be a cone metric space, $\{x_n\}$ a sequence in X . and $x \in X$.

(i) $\{x_n\}$ converges to x, if for every $c \in E \text{ with } 0 \ll c$, there is an n_0 such that for all $n \geq n_0$, $d(x_n, x) \ll c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x, (n \to \infty)$.

(ii) If for any $c \in E \text{ with } 0 \ll c$, there is an n_0 such thatfor all $n, m \geq n_0$, $d(x_n, x_m) \ll c$. Then $\{x_n\}$ is called a Cauchy sequence in M.

(iii) (M,d) is called a complete cone metric space, if every Cauchy sequence in M is convergent in M.

Lemma 1: Let (M,d) be a cone metric space, $P \in E$ a normal cone with normal constant K . let $\{x_n\}$, $\{y_n\}$ be a sequence in M and $x, y \in M$.

(i) $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

(ii) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$. That is the limit of $\{x_n\}$ is unique.

(iii) If $\{x_n\}$ converges to x, then $\{x_n\}$ is Cauchy sequence.

(iv) $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n\to\infty} d(x_n, x_m) = 0$

(v) If $x_n \to x$, and $x_n \to x$, $(n \to \infty)$ then $(dx_n, y_n) \to d(x,y)$.

Definition 5: Let (M,d) be a cone metric space, P be a normal cone with normal constant K and $T : M \to M$. Then

(i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$ for all $\{x_n\}$ in M.

(ii) T is said to be subsequentially convergent, if for every sequence $\{y_n\}$ that $\{Ty_n\}$ is convergent , implies $\{y_n\}$ has a convergent subsequences.

(iii) T is said to be sequentially convergent if for everysequence $\{y_n\}$,if $\{Ty_n\}$ is convergent, then of $\{y_n\}$ also is convergent.

Definition 6: Let (M,d) be a cone metric space and let $S : M \to M$ be a functions. S is said to be a generalized T – Contraction, if there exist non negative constants a, b, c such that $\alpha + 4\beta + 2\gamma < 1$ and

$$d(TSx, TSy) \leq \alpha d(Tx, Ty) + \beta\big(d(Tx, Ty) + d(Ty, TSx) + d(Ty, Tx) + d(Tx, TSy)\big) + \gamma\big(d(Tx, TSy) + d(Ty, TSx)\big)$$

Lemma 2: Let (M,d) be a complete cone metric space with normal cone P with normal constant K. Suppose $\lambda \in (0,1)$ and $\{x_n\}$ is a sequence in X such that $d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) for\ n = 0, 1, 2, 3, \ldots$ ...Then $\{x_n\}$ is a Cauchy sequence in X.

II. MAIN RESULTS

Theorem 1: Let (M,d) be a complete cone metric space with normal cone P with normal constant K. In addition let $T: M \to M$ be a continuous and $S: M \to M$ a generalized T – Contraction. Suppose S and T commute. Then,

(1) For every $x_0 \in M, \lim_{n\to\infty} d(TS^n x_0, TS^{n+1}x_0) = 0.$

(2) There is $v \in M$ such that $\lim_{n\to\infty} TS^n x_0 = v$

(3) If T is subsequentially convergent, then

   (i) $\{S^n x_0\}$ has a convergent subsequence;

   (ii) There is $u \in M$ such that $Su = u$;

(4) If T is sequentially convergent, then for each $x_0 \in M$ the iterate sequence $\{S^n x_0\}$ has a converges to u.

(5) T is constant on the fixed point set of S. If further T is one – one then S has unique fixed point.

Proof:

Let $x_0 \in M$. We define the iterate sequence$\{x_n\}$ by $x_{n+1} = Sx_n = S^{n+1}x_0$ Then

$d(Tx_n, Tx_{n+1}) = d(TSx_{n-1}, TSx_n)$

$d(Tx_n, Tx_{n+1}) \leq \alpha d(Tx_{n-1}, Tx_n) + \beta[d(Tx_{n-1}, Tx_n) + d(Tx_n, TSx_{n-1}) + d(Tx_n, Tx_{n-1}) + d(Tx_{n-1}, TSx_n)] + \gamma[d(Tx_{n-1}, TSx_n) + d(Tx_n, TSx_{n-1})]$

$d(Tx_n, Tx_{n+1}) \leq \alpha d(Tx_{n-1}, Tx_n) + \beta[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_n) + d(Tx_n, Tx_{n-1}) + d(Tx_{n+1}, Tx_{n-1})] + \gamma[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)]$

$d(Tx_n, Tx_{n+1}) \leq \alpha d(Tx_{n-1}, Tx_n) + \beta[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n-1}) + d(Tx_{n+1}, Tx_n) + d(Tx_n, Tx_{n-1})] + \gamma[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})]$

$d(Tx_n, Tx_{n+1}) \leq \alpha d(Tx_{n-1}, Tx_n) + 3\beta[d(Tx_{n-1}, Tx_n)] + b[d(Tx_{n+1}, Tx_n)] + \gamma[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})]$

$d(Tx_n, Tx_{n+1}) \leq (\alpha + 3\beta + \gamma)d(Tx_{n-1}, Tx_n) + (\beta + \gamma)\ [d(Tx_n, Tx_{n+1})]$

$d(Tx_{n+1}, Tx_n) \leq (\alpha + 3\beta + \gamma)d(Tx_{n-1}, Tx_n) + (\beta + \gamma)\ [d(Tx_n, Tx_{n+1})]$

$\big(1 - (\beta + \gamma)\big)d(Tx_{n+1}, Tx_n) \leq (\alpha + 3\beta + \gamma)d(Tx_{n-1}, Tx_n)$

$d(Tx_{n+1}, Tx_n) \leq \frac{(\alpha + 3\beta + \gamma)}{\big(1 - (\beta + \gamma)\big)} d(Tx_{n-1}, Tx_n) \ldots\ldots (1)$

Now

$\lambda = \frac{(\alpha + 3\beta + \gamma)}{\big(1 - (\beta + \gamma)\big)} < 1$

$\Rightarrow d(Tx_{n+1}, Tx_n) \leq \lambda d(Tx_{n-1}, Tx_n)$

Hence $\{Tx_n\}$ is a Cauchy sequence.

Since $x_n = s^n x_0$ consequently

$\lim_{n\to\infty} d(Ts^n x_0, Ts^{n+1}x_0) = \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0$

Thus (1) holds.

Since M is a complete there exists $v \in M$ such that

$\lim_{n\to\infty} Ts^n x_0 = \lim_{n\to\infty} Tx_n = v$ …… (2)

Thus (2) holds.

Now, suppose T is subsequentially convergent. Then from equation (2) $\{s^n x_0\}$ has a convergent subsequence.

Thus (3(i)) holds.

So, there are $u \in M$ and (x_{ni})such that $\lim_{n\to\infty} Ts^{ni}x_0 = u$ …… (3)

Since T is continuous

$\lim_{n\to\infty} Ts^n x_0 = Tu$ …… (4)

From (2) & (4)

$Tu = v$…… (5)

On the other hand

$d(TSu, Ts^{ni}x_0) = d\left(TSu, TS(S^{ni-1}x_0)\right)$

$= d(TSu, TSx_{ni})$

$d(TSu, Ts^{ni}x_0) \leq \alpha d(Tu, Tx_{ni-1}) + \beta[d(Tu, Tx_{ni-1}) + d(Tx_{ni-1}, TSu) + d(Tx_{ni-1}, Tu) + d(Tu, TSx_{ni-1})]$
$+ \gamma[d(Tu, TSx_{ni-1}) + d(Tx_{ni-1}, TSu)]$

On letting $n \to \infty$, we get

$d(TSu, Tu) \leq \alpha d(Tu, Tu) + \beta[d(Tu, Tu) + d(Tu, TSu) + d(Tu, Tu) + d(Tu, TSu)] + \gamma[d(Tu, Tu) + d(Tu, TSu)]$

$d(TSu, Tu) \leq 2\beta d(Tu, TSu) + \gamma d(Tu, TSu)$

$\leq (2\beta + \gamma) d(Tu, TSu)$

$TSu = Tu$ …… (6)

$d(STu, TS^{ni}x_0) = d\left(STu, TS(S^{ni-1}x_0)\right)$

$= d(STu, TSx_{ni})$

$d(STu, TS^{ni}x_0) \leq \alpha d(Tu, Tx_{ni-1}) + \beta[d(Tu, Tx_{ni-1}) + d(Tx_{ni-1}, STu) + d(Tx_{ni-1}, Tu) + d(Tu, STx_{ni-1})]$
$+ \gamma[d(Tu, STx_{ni-1}) + d(Tx_{ni-1}, STu)]$

On letting $n \to \infty$, we get

$d(STu, Tu) \leq \alpha d(Tu, Tu) + \beta[d(Tu, Tu) + d(Tu, STu) + d(Tu, Tu) + d(Tu, STu)] + \gamma[d(Tu, Tu) + d(Tu, STu)]$

$d(STu, Tu) \leq 2\beta d(Tu, STu) + \gamma d(Tu, STu)$

$\leq (2\beta + \gamma) d(Tu, STu)$

$STu = Tu$ …… (7)

From (7) & (6)

$STu = Tu = TSu$

Tu is a fixed point of S. Thus (3(ii)) holds,

Now, clearly (4) holds.

Suppose $x \& y$ are fixed point of S.

Then we show that $Tx = Ty$.

$d(TSx, TSy) \leq \alpha d(Tx, Ty) + \beta[d(Tx, Ty) + d(Ty, TSx) + d(Ty, Tx) + d(Tx, TSy)] + \gamma[d(Tx, TSy) + d(Ty, TSx)]$

$d(Tx, Ty) \leq \alpha d(Tx, Ty) + \beta[d(Tx, Ty) + d(Ty, Tx) + d(Ty, Tx) + d(Tx, Ty)] + \gamma[d(Tx, Ty) + d(Ty, Tx)]$

$d(Tx, Ty) \leq (\alpha + 4\beta + \gamma) d(Tx, Ty)$

$Tx = Ty$ …… (8)

$d(Tx, Ty) = 0$

Since T is constant on the fixed point of S.

Thus (5) holds.

If T is one – one, from (8) follows that S has unique fixed point if we assume that T is one – one instead of assuming that S and T commute.

Corollary 1: Let (M, d) be a complete cone metric space, P be a normal cone with normal constant K. In addition let $T: M \to M$ be a continuous and $S: M \to M$ a generalized T – Contraction. Suppose S and T commute. Then,

1. For every $x_0 \in M, \lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = 0$.
2. There is $v \in M$ such that $\lim_{n\to\infty} TS^n x_0 = v$
3. If T is subsequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence
4. There is $u \in M$ such that$Su = u$.

5. If T is sequentially convergent, then for each $x_0 \in M$ the iterate sequence $\{S^n x_0\}$ has a converges to u.

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AUTHOR PROFILE

Dr. R. Krishnakumar is the Associate Professor and Head of the PG & Research Department of Mathematics, Urumu Dhanalakshmi College. He has above 22 years experience in teaching. His field of research is Fixed point theory.

R. Livingston is pursuing his Ph.D in PG & Research Department of Mathematics, Urumu Dhanalakshmi College, Trichy. His field of research is Fixed point theory.

