International Journal of Scientific Research in Mathematical and Statistical Sciences **Research Paper**
Volume-5, Issue-6, pp.255-258, December (2018) **E-ISSN:** 2348-4519

On Quasi-Reduced Modules

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Available online at: www.isroset.org

Received 12/Oct/2018, Accepted 06/Dec/2018, Online: 31/Dec/2018

***Abstract*—**This paper generalizes the concept of a reduced modules. We introduced the concept of quasi-reduced-I i.e, QR-I and quasi-reduced-II i.e, QR-II in modules and studied some of their properties.

***Keywords*—**Reducedring,Armendarizring,Semi-primerinandRigidrings

I. INTRODUCTION

This paper deals with some quasi- condition on rings and modules. The term quasi- arises from Hirano's study of a number of concepts which arose from [1]. He defined quasi-Armendariz rings and established a number of interesting properties of these rings as follows:

Definition 1.1: A ring R is quasi-Armendariz if whenever polynomials $f(X) = \sum_{i=0}^{m} a_i X^i$ and $f(X) = \sum_{j=0}^{n} b_j X^j$ satisfy $f(X)R[X]g(X) = 0$ then $a_i R b_j = 0$ for all i, j.

Quasi-Armendariz modules can be defined analogously. They were studied by Baser [2] and other authors. The definition is as follows:

Definition 1.2:A left R-module M is a quasi-Armendariz if whenever two polynomials $f(X) = \sum_{i=0}^{m} a_i X^i \in R[X]$ and $m(X) = \sum_{j=0}^{n} m_j X^j \in M[X]$ satisfy $f(X)R[X]m(X) = 0$ then $a_i R m_j = 0$for i, j.

In view of these definitions it seems appropriate to call a vanishing condition in which an element a is replaced by the corresponding principal left ideal Ra as a quasi-condition. A few quasi-conditions are defined and studied in this paper. In section II, we have extended various quasi-compatibility condition from [6]. In section III we define and study two quasi-analogues of the reduced module concept.

II. QUASI-COMPATIBILITY

We begin with the following definitions

Definition:**2.1** A left R -module M is direct- α -quasicompatible(d-α-quasicompatible) whenever $a \in R$ and $m \in M$ satisfying $aRm = 0$, we have $\alpha(a)Rm = 0$, it is reverse- α -quasicompatible(r- α -quasicompitible) if whenever $a \in R$ and $m \in M$ satisfying $\alpha(a)Rm = 0$, we have $aRm = 0$.

Definition 2.2: A ring R is left d-α-quasicompatible(r- α-quasicompitible) if the left R-module R is left d- α -quasicompatible(r- α-quasicompitible).

Proposition 2.3: If a left R-module M is d-α-compatible then M is left d-α-quasicompatible.

Proof: Let $a \in R$ and $m \in M$ satisfy $aRm = 0$ that is $atm = 0$ for all $t \in R$. Since M is d-α-compatible then we have $\alpha(a)(tm) = 0$ for all $t \in R$. Hence it implies $\alpha(a)Rm = 0$.

Remark 2.4: If left R-module M is α-quasicompatible and semicommutative, then left R-module M is compatible.

Some results that hold in the compatibility case have straightforward analogues in the quasicompatibility case. We record a few of them.

Proposition 2.5: Let R be a ring and α be an onto endomorphism in R. Then ring R is left r- α-quasicompatible if and only if α is one-to-one and R is right d- α -quasicompatible.

Proof: Assume that R is a left r-α-quasicompatible. Then the condition $\alpha(a)Rb = 0$ implies $aRb = 0$.This yields, on letting b=1, α is one-to-one. Suppose next that elements a, b of Rsatisfy $aRb = 0$, that is $(at)b = 0$ for all $t \in R$. Then $\alpha(at)\alpha(b) = 0$ that is $\alpha(a)\alpha(t)\alpha(b) = 0$ for all $t \in R$.

Since α is onto, we have $\alpha(a)s\alpha(b) = 0$ for all $s \in R$. Therefore $\alpha(a)R\alpha(b) = 0$. By r-α-quasi-compatibility of R, we get $aR\alpha(b) = 0$. Next assume that R is right d-α-quasi-compatible and $a, b \in R$ satisfy $\alpha(a)Rb = 0$ implies $\alpha(a)(tb) = 0$ for all $t \in R$. Then $\alpha(a)\alpha(tb) = 0$ holds, implying $\alpha(atb) = 0$. Since α is one-to-one then $atb = 0$ that is $aRb = 0$.

Proposition 2.6: A flat module over a left d-α-quasi-compatible is d-α-quasi-compatible.

We remark that the analogue in the reverse α-compatibility case holds.

III. QUASI-REDUCEDNESS

For the definition of quasi-armendariz modules see **1.2.**

Definition 3.1: A left R-module M is linearly quasi-Armendariz if whenever two linear polynomials $f(X) = a + bX \in R[X]$ and $m(X) = m + nX \in M[X]$ satisfy $f(X)R[X]m(X) = 0$ then $aRn = 0$and $bRm = 0$.

Definition 3.2: A left R-module M is ps-Armendariz if whenever two polynomials $f(X) = \sum_{i=0}^{\infty} a_i X^i \in R[[X]]$ and $m(X) = \sum_{j=0}^{\infty} m_j X^j \in M[[X]]$ satisfy $f(X)R[[X]]m(X) = 0$ then $a_i R m_j = 0$ for all i, j.

It may be noted that ps-Armendariz modules are quasi-Armendariz and quasi-Armendariz modules are linearly quasi-Armendariz.

We next introduce two definitions which is to be called as quasi-reducedness conditions.

Definition 3.3: Let M be a left R-module. We say that M satisfies condition QR-I if whenever element $a \in R$ and $m \in M$ satisfy $aRam = 0$ we have $am = 0$.

Definition 3.4: Let M be a left R-module. We say that M satisfies condition QR-II if it satisfies the following equivalent conditions.

(i) For elements $a \in R$ and $m \in M$, the condition $aRaRm = 0$ implies $aRm = 0$.

(ii) For elements $a \in R$ and $m \in M$, the condition $aRaRm = 0$ implies $am = 0$.

We note that following easily verifiable fact: The classes of modules satisfying either of the conditions QR-I and QR-II over a ring are closed under direct products, submodules and direct sums.

Proposition 3.5: If a module M satisfies QR-I, then M satisfies QR-II.

Proof: Assume that QR-I holds for M and that for some element $a \in R$ and $m \in M$ we have $aRaRm = 0$. Then we certainly have $aRam = 0$ yielding $am = 0$. Hence QR-II holds for M.

The following result is a characterization of semiprime rings using the QR-I and QR-II conditions.

Proposition 3.6: The following conditions are equivalent.

(i) The left R-module R satisfies QR-II.

(ii) The condition $aRaRb = 0$ implies $ab = 0$.

(iii) The condition $aRaRb = 0$ implies $aRb = 0$.

(iv) The ring R is semiprime.

(v) The left R-module R satisfies QR-I.

Proof: (i) ⇒ (iv) Suppose that $aRa = 0$. Then $aRaR1 = 0$. By using the given condition we have $aR1 = 0$, showing that R is semiprime. (iv) ⇒ (i) let $aRaRb = 0$. Since $bRa \leq RaR$, we have $abRab \leq aRaRb = 0$, which implies since R is semiprime $ab = 0$. (iii) ⇒ (ii) Assume that for element $a, b \in R$ we have $aRaRb = 0$. Then $abRab \leq aRaRb = 0$. Therefore $1RabRab = 0$ yielding $Rab = 0$ and hence $ab = 0$. (ii) ⇒ (iii) Assume that $aRbRb = 0$ holds. Let $t \in R$. We have $atb \in aRb$ and $Rat \leq R$.Hence $atbRatb \leq aRbRb = 0$ yielding $atbRatbR.1. = 0$ Hence $atb = 0$. (iii) ⇒ (iv) let $a, b \in R$ satisfies $aRbRb = 0$. Now $(RaRb)^2 = RaRbRaRb \subseteq RaRbRb = 0$ implies that $(RaRb)^2 = 0$. Since R is semiprime we have $RaRb = 0$ which yields $aRaRb = 0$. Then by condition (i) we have $aRb = 0$.

(v) ⇒ (i) holds as a special case of proposition **3.5**.

(iv) ⇒ (v) Suppose that $aRab = 0$ holds for elements $a, b \in R$. Then $abRab \leq aRab = 0$ yields as the ring R is semiprime, $ab = 0$.

In the following results we relates the "QR' conditions with modules and rings satisfying other conditions.

Proposition 3.7: Rigid modules satisfy QR-I

Proof: Let left$R$-module Mbe a rigid module. Let $a \in R$ and $m \in M$ satisfy $aRam = 0$. This condition implies $a^2 m = 0$. By the rigidity of M we have $am = 0$. Hence M satisfies QR-I.

Corollary 3.8: Reduced modules satisfy QR-I.

Example 3.9: Let R be a non-reduced, semiprime ring, for example, the matrix ring $M_2(K)$ over a field K. Then, regarded as a left module over itself, R satisfies QR-I(by proposition **3.6**), but is non-reduced.

Proposition 3.10: Cyclically semiprime modules satisfy QR-I.

Proof: Suppose that for elements $a \in R$ and $m \in M$, a cyclically semiprime module, we have $aRam = 0$ and $am \neq 0$ .Now there exist $q \in (Rm)^*$ satisfying $[(am)q]am \neq 0$. Since $a(mq)am \in aRam = 0$.We arrive at a contradiction. Hence $am = 0$.

Definition 3.11: A ring R is left (right) weakly regular if every left(right) ideal is idempotent, equivalently, if for every $x \in R$ we have $x \in RxRx$ $(x \in xRxR)$. It is weakly regular if it is both left and right weakly regular.

Remark 3.12: Von Neumann regular rings as well as quasi-simple rings are left and right weakly regular.

Proposition 3.13: R is left weakly regular if and only if every left R-module satisfies QR-I.

Proof: $(\Rightarrow)$ The condition $aRam = 0$ certainly implies $RaRam = 0$. Since R is left weakly regular we have $= (Ra)^2$. So we have $Ram = 0$ and therefore $am = 0$.

$(\Leftarrow)$ For $a \in R$, consider the left R - module $M = R/RaRa$. As $aRa1 = 0$ in M, we have $a1 = 0$ yielding $a \in RaRa$. Therefore R is left weakly regular.

The following well-known result is a consequence of proposition **3.6** and **3.13**.

Corollary 3.14: If R is left weakly regular then it is semiprime.

Remark 3.15:If the ring R is right weakly regular we have $aR = aRaR$ for each $a \in R$. It follows from definition **3.4** that every left R-module satisfies QR-II.

Example 3.16: It was shown by Andruszkiewicz and Puczylowski (see the last paragraph of [8]) that the weakly regular condition is not left-right symmetric. Let R be a right weakly regular ring(with an identify element) which is not left weakly regular so that there exists an element $b \in R$ such that $RbRb \neq Rb$. Then by proof of the 'if' implication in proposition **3.13** the left R-module $M = R/RbRb$ does not satisfy QR-I. However, since R is right weakly regular, by remark **3.15**M satisfies QR-II. Thus the condition QR-I is strictly stronger than the condition QR-II.

Proposition 3.17: Modules satisfying QR-II are linearly quasi-Armendariz.

Proof: Suppose that $(a + bX)R(m + nX) = 0$.

Then $aRm = 0 = bRn$ and $atn + btm = 0$ for all $t \in R$.
We assert $bRm = 0 = aRn$, proving that M is linearly quasi-Armendariz. Suppose if possible,$bsm \neq 0$ for some $s \in R$. Now if $bRbRm \neq 0$, then we have $bsbum \neq 0$ for some $u \in R$. Now $aun + bum = 0$ implies $bum = -aun$, so $0 \neq bsaun \in bRn = 0$, which is a contradiction. Hence $bRbRm = 0$, yielding $bRm = 0$. Similarly, we can show $aRn = 0$.

Corollary 3.18: Modules which satisfy QR-I are linearly quasi-Armendariz.

Proposition 3.19: Cyclically semiprime modules are linearly quasi-Armendariz.

Corollary 3.20: Semiprime modules are linearly quasi-Armendariz.

Corollary 3.21: Semiprime rings are linearly quasi-Armendariz .

Corollary 3.22: Prime rings are linearly quasi-Armendariz .

Qusetion: Are all cyclically semiprime modules ps-quasi-Armendariz? Next we consider simple, semisimple and semiprimitive modules.

Proposition 3.23: Simple modules satisfy condition QR-I.

Proof: Suppose that for an element $a \in R$ and $m \in M$ we have $aRam = 0$ and $am \neq 0$. As $M = Ram$, yielding $0 = R(aRam) = RaRam = RaM = M$ as $0 \neq am \in aM$, a contradiction. So M satisfies condition QR-I.

Corollary 3.24: Semisimple Modules (more generally, semiprimitive modules) satisfy condition QR-I.

Proof: Semisimple modules(more generally, semiprimitive modules) are submodules of direct products of some families of simple modules.

It is easy to see that free modules over semiprime rings satisfy QR-I condition. In fact, more generally we have the following result.

Proposition 3.25: Flat modules over semiprime rings satisfy condition QR-I.

Proof: Let M be a flat module over the semiprime ring R. Let $m \in M$ satisfy $aRaRm = 0$. Now, let β be an R -epimorphism $\beta: F \longrightarrow M$, where F is free. We denote the kernel of β by K. Let $x \in F$ satisfy $\beta(x) = m$ and let $r, t \in R$ .Then $\beta(aratx) = arat\beta(x) = aratm = 0$. Therefore $aratx \in K$. Since M is flat, by well known proposition in ([15], Corollary 11.4 in Chapter 1) there exists an$R$-homomorphism $\gamma: F \longrightarrow K$ satisfying $\gamma(arat) = aratx$. Since F is free and$\gamma(x), x \in F$ then we write $x = (x_i)$ and $\gamma(x) = (p_i(\gamma(x)))$ where $p_i(\gamma(x)), x_i \in R$. Now $arat\gamma(x) = aratx$ implies $arat(\gamma(x) - x) = 0$ and therefore $arat\left(p_i(\gamma(x) - (x_i))\right) = 0$ in R. Hence $aRaR\left(p_i(\gamma(x) - (x_i))\right) = 0$ for all i. Since R is semi-simple we have $aR(p_i(x) - (x_i)) = 0$ that is $aR(\gamma(x) - x) = 0$ which implies $aR\gamma(x) = aRx$. Now $aRm = aR\,\beta(x) = \beta(aRx) = \beta(aR\gamma(x)) =$
$aR\left(\beta(\gamma(x))\right) = aR.0 = 0$

Therefore $aRm = 0$. Hence M satisfies QR-I

Proposition 3.26: Consider the following conditions for a ring R.

(i) R is left weakly regular.
(ii) Every left R-module satisfies condition QR-I
(iii) Every left R-module satisfies condition QR-II.

(iv) R is semiprime with regular centre.

Then (i) $\Leftrightarrow$ (ii) and (ii) $\Rightarrow$ (iii) $\Rightarrow$(iv) hold.

Proof: (i) $\Leftrightarrow$ (ii) holds by proposition **3.13**, (ii) $\Rightarrow$ (iii) follows from proposition **3.5** and it is easy to verify (iii) $\Rightarrow$ (iv).

The following example shows that there exist modules over semiprime rings with regular centres which do not satisfy the condition QR-II.

Example 3.27: Consider the domain $R = K\{X,Y\}$ of polynomials in noncommutative indeterminants X and Y over a field K. The ring R, being a domain, is semiprime and has the field K as its centre. Let M be a left R-module $R/RXRXR$. We have $XRXR1 = 0$ in M. However $XR1 \neq 0$. This shows that M does not satisfy QR-II.

V. CONCLUSION

By considering power series instead of polynomial, one can get the notion of ps-quasi-Armendariz modules. It may be noted that ps-quasi-Armendaiz modules are quasi-Armendariz. So a question which arises from our study and which we have not yet settled is the following.

Are all cyclically semiprime modules ps-quasi-Armendariz?

Acknowledgment: The authors would like to thank Prof. M.B Rege for the useful discussion that led to the outcome of this paper.

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