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Geo Chromatic Number of a Graph

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Abstract— A set $S \subseteq V(G)$ is called a geodetic set if every vertex of G lies on a shortest $u-v$ path for some $u, v \in S$, the minimum cardinality among all geodetic sets is called geodetic number and is denoted by $g(G)$. A set $C \subseteq V(G)$ is called a chromatic set if C contains all p vertices of different colors in G,the minimum cardinality among all chromatic sets is called chromatic number and is denoted by $\chi(G)$. The combination of a geodetic set and a chromatic set gives a new concept, which is called a geo chromatic set (briefly $\chi_{gc} - set$) of G. A geo chromatic set $S_c \subseteq V(G)$ is both a geodetic set and a chromatic set. The geo chromatic number $\chi_{gc}(G)$ of G is the minimum cardinality among all geo chromatic sets of G. We determined the geo chromatic numbers of certain standard graphs and bounds of the geo chromatic number is proved. Also we illustrated that for positive integers x, y, z with $2 \leq x < y < z$, there is a connected graph G with $\chi(G) = x$, $g(G) = y$, and $\chi_{gc}(G) = z$.

Keywords— Geodetic number, chromatic number, Geo chromatic number

I. INTRODUCTION

We consider finite graphs without loops and multiple edges. For any graph G the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$. The order and size of G are denoted by k and m respectively. For basic graph theoretic terminology we refer to Bondy and Murthy [1].

The distance $d(u,v)$between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. An $u-v$ path of length $d(u,v)$ is called an $u-v$ geodesic. A vertex x is said to lie on an $u-v$ geodesic P if x is an internal vertex of P. The closed interval $I[u,v]$ consists of u, v and all vertices lying on some $u-v$ geodesic of G, and for a non-empty set $I[S] = \bigcup_{u,v \in S} I[u,v]$. If G is a connected graph, then a set S of vertices is a geodetic set if $I[S] = V(G)$. The geodetic number $g(G)$ of G is the minimum cardinality of a geodetic set of G. Geodetic number was introduced in [2, 5] and further studied in [3, 6].

A p-vertex coloring of G is an assignment of p colors, $1,2 \dots .p$ to the vertices of G; the coloring is proper if no two distinct adjacent vertices have the same color [1]. If $\chi(G) = p, G$ is said to be p- chromatic, where $p \leq k$. A set $C \subseteq V(G)$ is called chromatic set if C contains all p vertices of distinct colors in G. Chromatic number of G is the minimum cardinality among all the chromatic sets of G. That is $\chi(G) = \min\{|C_i| \,/\, C_i$ is a chromatic set of $G\}$. For references on chromatic sets see [8].

It is easily seen that a geodetic set not in general a chromatic set of G. In general, the converse is also not valid. This has motivated us to study the new geodetic conception of "Geo Coloring". We investigate those subsets of vertices of a graph G that are both a geodetic set and a chromatic set. We call these sets as geo chromatic sets. We call the minimum cardinality of a geo chromatic set of G, the geo chromatic number of G. The geo chromatic number of G is also called as geodetic chromatic number of G.

In section II, we introduce the definition and present the geo chromatic number of some standard graphs. In section III, we present bounds and general results. In section IV, we illustrate realization of the geo chromatic number of G.

The following theorems are used in sequel.

Theorem 1.1.[**5**] Every geodetic set of a graph contains its extreme vertices.

Theorem 1.2. [**5**] If G is a non trivial connected graph of order n and diameter d, then $g(G) \leq n - d + 1$.

Theorem 1.3. [1] If every chromatic sets of a graph G contains k vertices, then G has k vertices of degree at least $k - 1$.

II. GEO CHROMATIC NUMBER (GCN)

Definition 2.1. A set S_c of vertices in G is said to be geo chromatic set if S_c is both a geodetic set and a chromatic set. The minimum cardinality of a geo chromatic set of G is its geo chromatic number (GCN) and is denoted by $\chi_{gc}(G)$. A geo chromatic set of size $\chi_{gc}(G)$ is said to be χ_{gc}- set.

Example 2.2. For the graph G in Figure 1, G is 3 – colorable. $C = \{u_1, u_2, u_3\}$ is a minimum chromatic set, it is not a minimum geodetic set. The set $S = \{u_4, u_7\}$ is a minimum geodetic set of G but is not a chromatic set. The set $S_c = \{u_2, u_3, u_4, u_7\}$ is a geo chromatic set. But it is not minimum. The set $S_c = \{u_3, u_4, u_7\}$ is a minimum geo chromatic set of G. Thus $\chi_{gc}(G) = 3$.

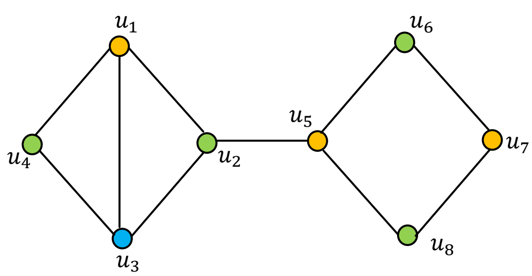

Figure1: The Graph G with $\chi_{gc}(G) = 3$.

Remark 2.3. For the graph G in figure 2, $S_c = \{u_1, u_4\}$, $S_c' = \{u_2, u_5\}$ are the two minimum geo chromatic sets. Thus, there can be more than one minimum geo chromatic sets for G.

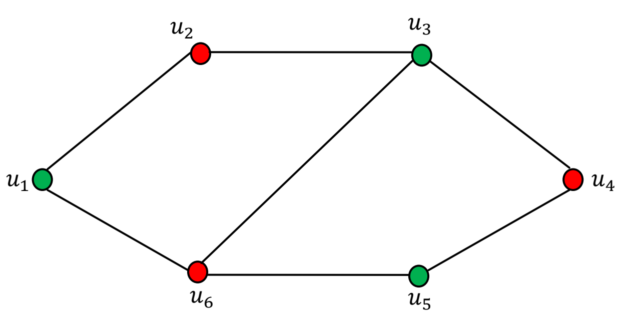

Figure 2: The Graph G with $\chi_{gc}(G) = 2$.

Remark 2.4. Consider the graph G in Figure 3

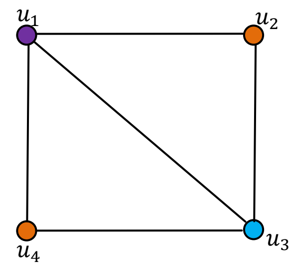

Figure 3: The graph G with $\chi(G) > g(G)$.

Here $\chi(G) = 3, S = \{u_2, u_4\}$ is a minimum geodetic set, $g(G) = 2$ and so $\chi(G) > g(G)$. For the graph G in Figure 4, $\chi(G) = 3, g(G) = 4$ and so $\chi(G) < g(G)$

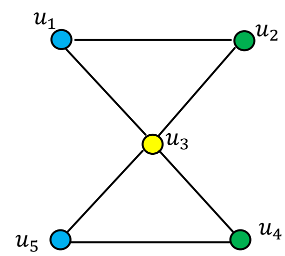

Figure 4: The graph G with $\chi(G) < g(G)$.

Observation 2.5. Every geo chromatic set of a connected graph G is a geodetic set of G.

Observation 2.6. Let G be a connected graph of order k, then $2 \leq \max\{\chi(G), g(G)\} \leq \chi_{gc}(G) \leq k$.

Theorem 2.7. For any path with k vertices,

$$\chi_{gc}(P_k) = \begin{cases} 3 \text{ if } k \text{ is odd} \\ 2 \text{ if } k \text{ is even} \end{cases}$$

Proof. Let P_k be a path with vertex set $\{v_1, v_2, \ldots \ldots v_k\}$. Let us consider two cases.

Case 1. For P_{2k}, since the pendent vertices say u_1 and u_k is a geodetic set, which is minimum. Also, the set S is a minimum chromatic set C of P_{2k}. That is $g(G) = |S|$ and $\chi(G) = |C| = |S|$. Hence $|S| = \chi_{gc}(P_{2k}) = |S_c|$

Case 2. For $P_{2k+1}, S = \{v_1, v_k\}$ is a minimum geodetic set of P_{2k+1}, but not a chromatic set and so $\chi_{gc}(P_{2k+1}) > |S| = 2$. If $N(v_1), N(v_k) \subseteq S$ has vertices with distinct color, then $S_c = S \cup \{N(v_1)\} \cup \{N(v_k)\}$ is a geo chromatic set of G, not a minimum, and so $\chi_{gc}(P_{2k+1}) < 4 = |S|$. Obtaining a minimum geo chromatic set either $N(v_1)$ or $N(v_k)$ is in S. Hence $\chi_{gc}(P_{2k+1}) = 3$. □

Theorem 2.8. For a cycle C_k of order $k \geq 3$, $\chi_{gc}(C_k) = \begin{cases} 2\; if\; k \equiv 2\;(mod\; 4) \\ 3\; otherwise \end{cases}$

Proof. Let $G = C_k$. Let G be the cycle with k vertices, $k \geq 3$. Since the set $S = \left\{v_1, v_{\frac{k+2}{2}}\right\}$ or $S = \left\{v_1, v_{\frac{k+1}{2}}, v_{\frac{k+3}{2}}\right\}$ is a minimum geodetic set of G. Let us consider two cases.

Case 1. Suppose $S = \left\{v_1, v_{\frac{k+2}{2}}\right\}$ is a geodetic set of G. By assigning the proper color of G, the vertices in S belongs to the same color class or distinct color class, then we consider two subcases.

Subcase 1.1 . Suppose the vertices in S belongs to the different color classes in G, then it is also a chromatic set of G. Since both geodetic set S and a chromatic set C of G are minimum.It shows that $S_c = \left\{v_1, v_{\frac{k+2}{2}}\right\}$ is a minimum geo chromatic set of G. Hence in $G = C_k$, $\chi_{gc}(G) = 2$ if $k \equiv 2(mod\; 4)$.

Subcase 1.2. Suppose the vertices in S belongs to the same color class, say C_1, then S is not a chromatic set of G. Since $\chi(C_{2k}) = 2$. Choose another vertex from G which belongs to the distinct color class. Let $v_i \in C_j, j \neq 1$. If v_i is in S, then the set becomes $S_c = \left\{v_1, v_{\frac{k+2}{2}},\right\} \cup \{v_i\}$ is a geodetic set as well as chromatic set of G . Hence $\chi_{gc}(G) \leq 3$. Since $\chi_{gc}(G) < 3$ is not possible. Therefore $\chi_{gc}(G) = 3$.

Case 2. Suppose $S_c = \left\{v_1, v_{\frac{k+1}{2}}, v_{\frac{k+3}{2}},\right\}$ is a minimum geodetic set of G, Obviously S is a chromatic set which is minimum. Since $\chi(C_{2k+1}) = 3$. It shows that the set $S_c = \left\{v_1, v_{\frac{k+1}{2}}, v_{\frac{k+3}{2}},\right\}$ is a minimum geo chromatic set of G. Hence $\chi_{gc}(G) = 3$. □

Theorem 2.9. For a complete graph K_k of order $k \geq 3, \chi_{gc}(K_k) = k$.

Proof. Let K_k be a complete graph with k vertices. Every vertex in K_k is the minimum geodetic set $S, g(G) = k$. Since each vertex in K_k was assigned by distinct colors, $\chi(G) = k$ Hence it is clear that $\chi_{gc}(K_k) = k$. □

Theorem 2.10. For a star $K_{1,k-1}$ of order $k > 2, \chi_{gc}(K_{1,k-1}) = k$.

Proof. Let $G = K_{1,k-1}$with $k > 2$. Since the set of all leaves is the minimum geodetic set S for G. But S is not a chromatic set and so $\chi_{gc}(G) \geq k$. If the neighborhood of leaves of G is contained in S, then S is a geodetic set as well as chromatic set of G. The set $S_c = \{u_1, u_2 \ldots\ldots u_{k-1}\} \cup \{u_o\}$ is the geo chromatic set of G, $\chi_{gc}(G) \leq k$. Hence it follows that $\chi_{gc}(G) = k$. □

Theorem 2.11. For a double star $S_{a,b,}$ $a + b \geq 4$ and $a, b < k$, $\chi_{gc}(S_{a,b}) = a + b$.

Proof. Let $G = S_{a,b}$ be the double star with bridge $x_o\, y_o$. Let $degx_o = a \geq 2$ and $degy_o = b \geq 2$, the set of all leaves which is adjacent to the vertex x_o be $h_1, h_2, \ldots, h_a$such that $|h_i| = a$,where $i = 1,2, \ldots, a$and the set of all leaves which are adjacent to the vertices $l_1, l_2, \ldots, l_b$ such that $|l_j| = b$,where $j = 1,2, \ldots, b$. Since, set at all leaves of $S_{a,b}$ is the minimum geodetic set $S, g(G) = a + b$. Let C be the proper coloring of G such that the vertices $h_1, h_2, \ldots, h_a$ receive a color say $c_1$and the vertices $l_1, l_2, \ldots, l_b$ receive a color say c_2 and so $\chi(G) = 2$. The vertices in S has proper colors which is the minimum cardinality of G. Hence $\chi_{gc}(G) \leq a + b$. By the observation 2.6, $\chi_{gc}(G) \geq g(G) = a + b$. Therefore $\chi_{gc}(G) = a + b$. □

Theorem 2.12. For a complete bipartite $K_{m,n}$, where $m, n < k$,

(i) $\chi_{gc}(K_{m,n}) = min\{m, n\} + 1, m = 2, n \geq 2$

(ii) $\chi_{gc}(K_{m,n}) = min\{m, n, 4\}, m, n > 2$.

Proof. Let $G = k_{m,n}$. Let $X = \{u_1, u_2 \ldots\ldots u_m\}$ and $Y = \{v_1, v_2 \ldots\ldots v_n\}$ be the set of all vertices of G, where $m, n \geq 2$.

(i) First assume that $m = n = 2$, the vertices in X is the minimum geodetic set S but not a minimum chromatic set. Hence $\chi_{gc}(G) > 2 = |S|$. Adding one of the vertex $v_i \in Y, 1 \leq i \leq 2$ with the minimum geodetic set S, to obtain the minimum geo chromatic set of G, $\chi_{gc}(G) < 4$. Hence $\chi_{gc}(K_{m,n}) = min\{m, n\} + 1$. Suppose that $m = 2, n > 2$, now the vertices in X forms a geodetic set S with same color and so $\chi_{gc}(G) \geq 3$. Add any one of $v_i, 1 \leq i \leq k$ with S is the geo chromatic set. $\chi_{gc}(G) > 3$ is not possible. Hence the only possible is $\chi_{gc}(G) = 3, \chi_{gc}(K_{m,n}) = \min\{m, n\} + 1$.

(ii) For $m, n > 2$, the set $X = \{u_1, u_2 \ldots\ldots u_m\}$ is a geodetic set of G, but not a chromatic set. By the proof of (i), adding one of $v_i, 1 \leq i \leq k$ with S, then S is a geo chromatic set which is not minimum. Take any two vertices $u_i \in X, 1 \leq i \leq m$ and any three vertices $v_i \in Y, 1 \leq i \leq n$ in G. This forms a geodetic chromatic set for G. But it is not minimum and so $\chi_{gc}(G) < \min\{m, n, 5\}$. Choosing any two vertices $u_i \in X, 1 \leq i \leq m$ and any one vertex in Y, does not form a geodetic set S, and so $\chi_{gc}(G) > \min\{m, n, 3\}$. Therefore choosing any two vertices from each $X\; and\; Y$ partite sets separately then clearly it forms a geo chromatic set of G, which is minimum, $\chi_{gc}(G) = \min\{m, n, 4\}$. □

Theorem 2.13. For a wheel $W_k, k > 4, \chi_{gc}(W_k) = \left\lceil \frac{k}{2} \right\rceil + 1$

Proof: Let $G = W_k$. We consider two cases.

Case 1. If k is odd, the set $S = \{v_i \,/\, i \equiv 1\ (mod 2)\}$ is the geodetic set of G. And so $|S| = \left\lceil \frac{k}{2} \right\rceil$. Let C be the proper coloring of G such that $S = \{v_i \,/\, i \equiv 1\ (mod 2)\}$ was assigned by the color say c_1 and $S_1 = \{v_j \,/\, j \equiv 0\ (mod 2)\}$ was assigned by the color say c_2. The centre vertex of W_k was assigned by the new color say c_3. It is clear that the minimum chromatic set C contains three vertices of distinct colors, $|C| = 3$. So that S is not a chromatic set of G, $|S_c| > \left\lceil \frac{k}{2} \right\rceil$. Since $N(v_i)$ receive distinct color other than S. Let one of the vertex of $N(v_i)$ be v_{i+1}, which is not the centre vertex of W_k. Now $S' = \{v_i \,/\, i \equiv 1\ (mod 2)\} \cup \{v_{i+1}\}$ is the geodetic set of G. Since $\chi(G) = 3, S'$ is not a chromatic set of G. If the centre vertex of W_k, say $u_o \in S'$, then the set $S_c = \{v_i \,/\, i \equiv 1\ (mod 2)\} \cup \{v_{i+1}\} \cup \{u_o\}$ is a geo chromatic set of G, $\chi_{gc}(G) \leq \left\lceil \frac{k}{2} \right\rceil + 1$. Since $\chi_{gc}(G) < \left\lceil \frac{k}{2} \right\rceil + 1$ is not possible. Hence $\chi_{gc}(G) = \left\lceil \frac{k}{2} \right\rceil + 1$.

Case 2. If k is even. The set $S = \{v_i \,/\, i \equiv 1\ (mod 2)\}$ is a geodetic set of G. Let C be the proper coloring of G such that the centre vertex of W_k say u_o was assigned by new color which is not assigned in C_{k-1}. It is clear that $|C| = 4$. The set $S = \{v_i \,/\, i \equiv 1\ (mod 2)\}$ is the geodetic set of G, not a chromatic set. And so $\chi_{gc}(G) > \left\lceil \frac{k}{2} \right\rceil$. If $u_o \in S$, the set $S_c = \{v_i \,/\, i \equiv 1\ (mod 2)\} \cup \{u_o\}$ is a geo chromatic set of G, $\chi_{gc}(G) \leq \left\lceil \frac{k}{2} \right\rceil + 1$. Since $\chi_{gc}(G) < \left\lceil \frac{k}{2} \right\rceil + 1$ is impossible .Therefore $\chi_{gc}(G) = \left\lceil \frac{k}{2} \right\rceil + 1$. □

III. BOUNDS FOR THE GEO CHROMATIC NUMBER

Theorem 3.1. For a connected graph G of order $k \geq 2$, $2 \leq \chi_{gc}(G) \leq k$ and these bounds are sharp.

Proof: Sharpness is clear from $\chi_{gc}(P_k) = \chi_{gc}(C_k) = 2$, where k is even and $\chi_{gc}(K_k) = \chi_{gc}(K_{1,k-1}) = k$. Note also that there is a connected graph G of order $k \geq 3$ with geo chromatic number $l \geq 2$ and $k \geq l$. Construct a graph G with geo chromatic number l and order $k \geq 3$. We obtain G from the complete graph $K_l: v_1, v_2 \ldots. v_l$ by adding an even path $P_k: u_1, u_2 \ldots. u_b$ with one of v_i, $1 \leq i \leq l$. Observe that $S_c = \{u_b, v_2, v_3, \ldots. v_l\}$ is a geo chromatic set. Thus $\chi_{gc}(G) = l \leq k$. □

Theorem 3.2. For a connected graph G, $2 \leq g(G) \leq \chi_{gc}(G) \leq k$.

Proof: At least two vertices are needed to form a geodetic set in G. So that $2 \leq g(G)$. Since every geo chromatic set of G is a geodetic set of G. Then $g(G) \leq \chi_{gc}(G)$. The vertex set $V(G)$ induces a geo chromatic set of G. And so $\chi_{gc}(G) \leq k$. It is clear that $2 \leq g(G) \leq \chi_{gc}(G) \leq k$. □

Remark 3.3. By theorem 3.1 the upper bound and the lower bound of theorem 3.2 are sharp. For P_{2k} and K_k, $\chi_{gc}(G) = g(G)$. Also, all the inequalities in theorem 3.2 are strict. For the graph G in figure 5, $g(G) = 3$, $\chi_{gc}(G) = 4$ and $k = 5$. Thus $2 < g(G) < \chi_{gc}(G) < k$.

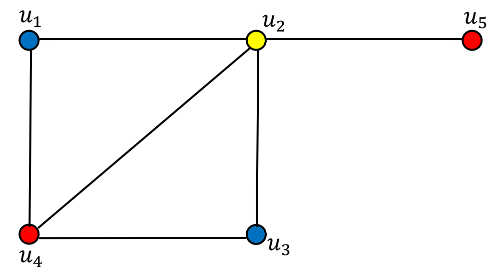

Figure 5: The Graph G with $2 < g(G) < \chi_{gc}(G) < k$

Theorem 3.4. If G is a disconnected graph with components $G_1, G_2, \ldots, G_k, (k \geq 2)$ with $|G_i| \geq 2$ then max $\chi_{gc}(G_i) < \chi_{gc}(G)$, Where $i = \{1,2 \ldots k\}$.

Proof: Let $G_1, G_2, \ldots, G_k, k \geq 2$ be the components of G. Let C be the proper coloring of G such that the color assigned by one component can be repeated to each of the other components. Assume that G_j, $j = 1, 2, \ldots, k$ be the component of G which has the geo chromatic set with maximum cardinality. That is $\chi_{gc}(G_1) + \chi_{gc}(G_2) + \cdots + \chi_{gc}(G_j) = r$ where r is the cardinality of the geo chromatic set of G_j. Since $G = G_1 \cup G_2 \cup \ldots \cup G_k$, so that $\chi_{gc}(G) = \chi_{gc}(G_1) + \chi_{gc}(G_2) + \cdots + \chi_{gc}(G_k)$ It follows that max $\chi_{gc}(G_i) < \chi_{gc}(G)$. □

Theorem 3.5. For a connected graph G, there is a graph G for which $\chi_{gc}(K) > \chi_{gc}(G)$ and there is a graph G for which $\chi_{gc}(K) < \chi_{gc}(G)$, where K is a sub graph of G.

Proof: Let G_1 be the complete graph with one vertex, let G_2 be the star $K_{1,k-1}$ and G_3 be the wheel W_k with k vertices. Clearly $G_1 \subseteq G_2 \subseteq G_3$. We have $\chi_{gc}(G_1) < \chi_{gc}(G_3)$ but $\chi_{gc}(G_3) < \chi_{gc}(G_2)$. □

Theorem 3.6. If G is a non- trivial connected graph of order k and $\chi_{gc}(G) = k$ then G has atleast one vertex of full degree.

Proof: Let G be a connected graph of order k with $\chi_{gc}(G) = k$. To prove that G has at least one vertex of full degree. Assume to the contrary that G does not have any vertex of full degree. We consider two cases:

Case 1: Suppose G contains the cycle with $\Delta(G) = k - 2$, then the vertex which has maximum degree does not belongs to a geodetic set S. Let v be a vertex such that $\deg v = k-2$, since $v \notin S$. There exists a vertex which is adjacent to v is not belongs to a geodetic set S of G. It is clear that $g(G) < k$. Consider the proper coloring of G such that the vertices which are adjacent to v receive the colors which was different from the assigned color of $v$such that the vertices of G receive the repeated colors. So that $\chi(G) \neq k$. The geodetic number and the chromatic number of G does not exceed k and so the geo chromatic number definitely not exceed k. It follows that $\chi_{gc}(G) < k$, which contradicts to our assumption.

Case 2: Suppose G does not contain a cycle with $\Delta(G) < k-1$, then the pendent vertices is the geodetic set S of G. Clearly $g(G) < k$. Let C be the proper coloring of G such that either the vertices of S receive the distinct colors or the vertices of S receive the same color. If the vertices of S receive distinct colors then $g(G) = \chi_{gc}(G)$ and so $\chi_{gc}(G) = k$ is not possible. If the vertices of S does not receive the distinct colors then $g(G) < \chi_{gc}(G)$. Also the set of vertices of G not form a minimum geo chromatic set S_c. So that $\chi_{gc}(G) \neq k$. This contradicts to our assumption that $\chi_{gc}(G) = k$. Hence G has at least one vertex of full degree. □

IV. REALIZATION RESULTS

Theorem 4.1. For any integer$k \geq 3$, there is a connected graph G with $\chi(G) = a$, $g(G) = a+1, \chi_{gc}(G) = a+2$.

Proof: Consider the complete graph K_k with the vertex set $\{v_1, v_2, \ldots . v_k\}$. Join one copy of $K_{1,2}$ with one of v_i, $1 \leq i \leq k$. There are distinct colors in K_k. Let us take the colors as $c_1, c_2 \ldots c_k$. Assign the color for $K_{1,2}$ which has been used in K_k. Therefore $\chi(G) = a$. Now $u_1 v_i\, v_{i+1}$ or $u_2 v_i\, v_{i+2}$, $1 \leq i \leq k$ is the geodesic. The vertex which is inbetween the path $u_1 - v_{i+1}$ *or* $u_2 - v_{i+1}$ does not occur in the geodetic set S. So that $g(G) = a+1$. By adding an internal vertex with S, we obtain the geo chromatic set S_c. The set $S_c = \{v_1, v_2, \ldots . v_k\} \cup \{u_1, u_2\} = V(G)$ is the minimum geo chromatic set of G. Hence $\chi_{gc}(G) = a+2$. □

Theorem 4.2. For $3 < q < r < k$, there is a connected graph G with $\chi(G) = q$ and $g(G) = \chi_{gc}(G) = r$.

Proof: Consider the graph H obtained as follows. Take one copy of an odd cycle C_m with the vertex set $\{v_1, v_2, \ldots . v_m\}$. Add one copy of an even path with any one vertex $v_3 \in C_m (1 < m < k)$. Let $\{v_3, w_1, w_2, \ldots, w_{r-q}\}$ be the vertex set of an even path. Let G be the graph obtained from H by adding a copy of $K_{1,j}$ $(2 \leq j \leq m-1)$ with the leaf of an even path. This is shown in figure 6.

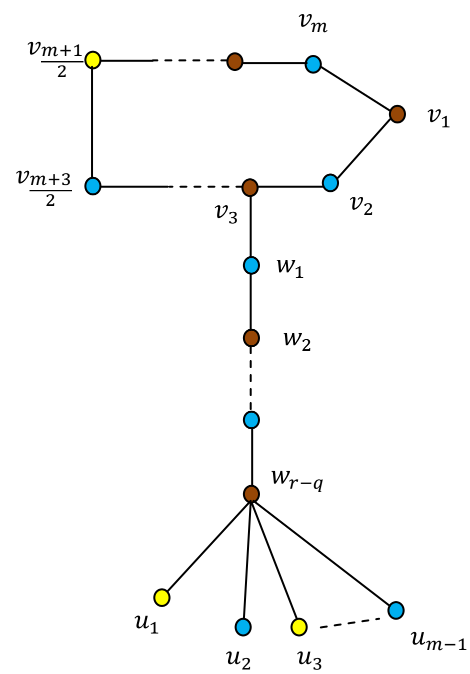

Figure 6: The Graph G with $\chi(G) = q$ and $g(G) = \chi_{gc}(G) = r$.

Since the chromatic number of an odd cycle is 3. Repeat the color to P_{r-q} and $K_{1,j}$ by the assigned color of C_m. Therefore $\chi(G) = q$ .The set $\left\{v_1, v_{\frac{m+3}{2}}\right\} \cup \{u_1, u_2 \ldots . u_{m-1}\}$ is a geodetic set of G, which is minimum. So that $g(G) = r$. For obtaining a geo chromatic set S_c of G, define the colors c_1, c_2, c_3 to the vertices of S. The vertices other than S is also colored by using these colors. So that S is aminimum geo chromatic set S_c of G. Hence $\chi_{gc}(G) = r$. □

Theorem 4.3. For any three positive integers x, y, z with $2 \leq x < y < z$, there is a connected graph G with $\chi(G) = x$, $g(G) = y$ and $\chi_{gc}(G) = z$.

Proof: Let $P_k : x_{10}\ x_{11}\ x_{12}\ \ldots . x_{1j}$ be an even path of G. Take $z-x$ copies of P_k. Let us take the copy as $\{x_{10}, x_{11}, x_{12}, \ldots\ x_{1j}\}$; $\{x_{20}, x_{21}, x_{22}, \ldots\ x_{2j}\}$; …;$\{x_{(z-x)o}, x_{(z-x)1}, \ldots\ x_{(z-x)j}\}$ respectively. Let H be the graph obtained from the copies of P_k by identifying the corresponding leaves into x_o and y_o. Now, each copy of P_k becomes an odd path with the vertex set $\{x_{11}, x_{12}, \ldots\ x_{1j}\}$; $\{x_{21}, x_{22}, \ldots\ x_{2j}\} \ldots \{x_{(z-x)1}, x_{(z-x)2}, \ldots\ x_{(z-x)j}\}$. Also take a vertex v_o, which is adjacent to y_o. Set of all leaves $\{v_1\ v_2\ \ldots . v_{z-x}\}$ are adjacent to v_o. Now we obtain the graph G from H by adding an odd path P_r with the vertex y_o.

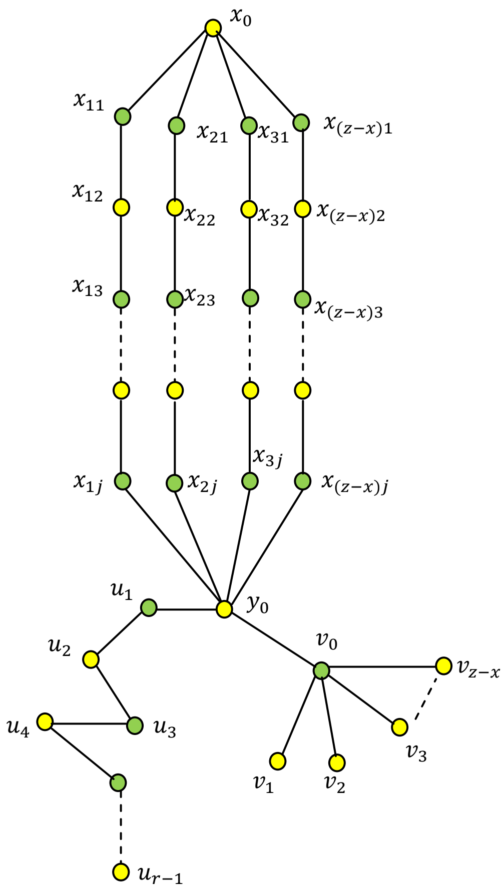

Figure7: The Graph G with $\chi(G) = x, g(G) = y$ and $\chi_{gc}(G) = z$.

Let the path be $P_r : y_o, u_1, u_2, \dots u_{r-1}$. Let C be the proper coloring of G such that the vertex x_o is assigned by the color c_1. Remaining vertices of G are colored by c_1 and c_2 in proper way. The graph G is shown in figure 7. It shows that minimum two colors are needed for the graph G. Thus $\chi(G) = x$. The set $S = \{x_o\} \cup \{v_1, v_2 \dots \dots v_{z-x}\} \cup \{u_{r-1}\}$ is the minimum geodetic set of G. So that $g(G) = y$. The set S has same color, say c_1, and so S is not a chromatic set of G. Take one vertex from G which has the color, say c_2. Let the vertex be x_{11}. By adding the vertex x_{11} with the minimum geodetic set S of G, then the set becomes $S_c = \{x_o, x_{11}\} \cup \{v_1, v_2 \dots \dots v_{z-x}\} \cup \{u_{r-1}\}$ is a minimum geo chromatic set of G. Therefore $\chi_{gc}(G) = z$. □

V. CONCLUSION AND FUTURE SCOPE

In this paper, the geo chromatic number $\chi_{gc}(G)$ have been discussed. We have presented bounds and general results of the geo chromatic number. Also, the realization results involving the chromatic number, geodetic number and the geo chromatic number of a connected graph were discussed. The concept, "geo coloring" can be extended to find edge geo chromatic number, total geo chromatic number, open geo chromatic number, connected geo chromatic number, detour chromatic number and so on.

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