International Journal of Scientific Research in Mathematical and Statistical Sciences

Research Paper

Volume-5, Issue-4, pp.299-305, August (2018)

E-ISSN: 2348-4519

Epidemic Model with Vital Dynamics and a Saturated Incidence rate

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Available online at: www.isroset.org

Accepted 18/Aug/2018, Online 30/Aug/2018

Abstract- In this paper, we proposes a new SEIRS model with vital dynamics and a saturated incidence rate. We find equilibrium point, basic reproduction number and stability of the system.

***Keywords*:** Epidemic Model, Saturated Incidence rate, Basic Reproduction number, Stability.

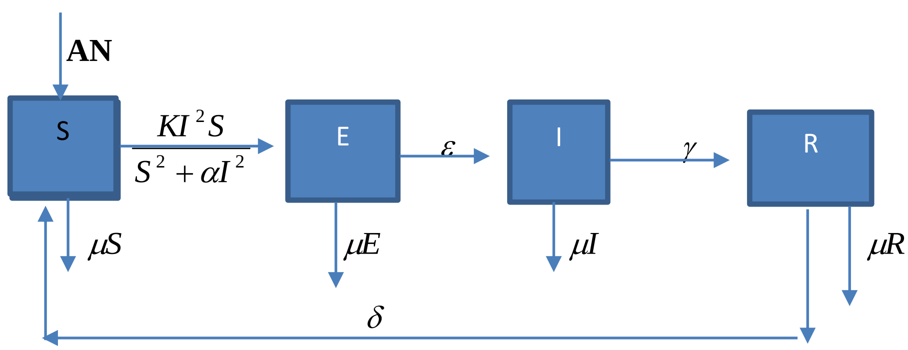
I. Introduction

To study the spread and control of infectious diseases we developed mathematical model. Mathematical model have been played a key role in policy making, including health economic aspects, emergency planning and risk assessment and optimizing various detections. Van den Driessche and Watmough [1] discussed reproduction number and sub-threshold endemic equilibria for compartmental models of disease transmission.

Several researchers proposed SEIR models with various incidence rates [2-6]. Xinli Wang [7] studies an SIRS epidemic model with Vital Dynamics and a Ratio Dependent Saturation incidence rate. Trawicki [8] studies deterministic SEIRS epidemic model for modeling Vital Dynamics, Vaccination, and Temporary Immunity.

In this paper, we use SEIRS epidemic model with vital dynamics and a saturated incidence rate $\frac{KI^2S}{S^2+\alpha I^2}$, where K is the proportionality constant and α is used to measure the effect of psychology. The existence of various types of equilibrium and stability results are discussed.

The Model:



Description of parameters

Parameter	**Name**
S	Susceptible individuals
E	Exposed individuals
I	Infectious individuals
R	Recovered individuals
A	Birth rate
μ	Death rate
δ	Transmission rate (recovered to susceptible)
ε	Transmission rate (exposed to infected)
γ	Recovered rate

Mathematically the SEIRS model is expressed as a system of ordinary differential equations given as:

$$\frac{dS}{dt} = AN - \frac{KI^2S}{S^2+\alpha I^2} - \mu S + \delta R$$
$$\frac{dE}{dt} = \frac{KI^2S}{S^2+\alpha I^2} - (\mu+\varepsilon)E \qquad (1)$$
$$\frac{dI}{dt} = \varepsilon E - (\mu+\gamma)I$$
$$\frac{dR}{dt} = \gamma I - (\delta+\mu)R$$

With population:

S(t)+E(t)+I(t)+R(t)=N(t) (2) Or

$$S'(t)+E'(t)+I'(t)+R'(t) = N'(t) \qquad (3)$$

where $\frac{KI^2S}{S^2+\alpha I^2}$ is the incidence rate at which the susceptible S becomes infected I by a disease.

The population N is governed by the ordinary differential equation:

$$\frac{dN}{dt} = AN - \mu N$$
$$\frac{dN}{dt} = (A-\mu)N$$
$$\int \frac{dN}{N} = \int (A-\mu)dt$$
$$\log N = (A-\mu)t$$
$$N = e^{(A-\mu)t} \qquad (4)$$

With time varying population N(t).

Instead of solving the ordinary differential equation system (1) with known population N from (4), the transformations:

$$s = \frac{S}{N}$$
$$e = \frac{E}{n}$$
$$i = \frac{I}{N}$$
$$r = \frac{R}{N}$$

Are applied to the system in (1), where s,e,I,r denote the fraction of the number of individuals in classes S,E,I,R with population N.

Now, the transformed system is formulated as:

$$\frac{dS}{dt} = A - \frac{ki^2 s}{s^2 + \alpha i^2} - As + \delta r$$
$$\frac{de}{dt} = \frac{ki^2 s}{s^2 + \alpha i^2} - (A + \varepsilon)e$$
$$\frac{di}{dt} = \varepsilon e - (A + \gamma)i$$
$$\frac{dr}{dt} = \gamma i - (\delta + A)r$$

Now, s+e+i+r =1

r =1-s-e-i

or

$$s' + e' + i' + r' = 0$$

Eliminate r and yield the simplified subsystem

$$\frac{ds}{dt} = A - \frac{ki^2 s}{s^2 + \alpha i^2} - AS + \delta((1 - s - e - i)$$
$$\frac{de}{dt} = \frac{ki^2 S}{s^2 + \alpha i^2} - (A + \varepsilon)e$$
$$\frac{di}{dt} = \varepsilon e - (A + \gamma)i$$
$$\frac{dr}{dt} = \gamma i - (A + \delta)(1 - s - e - i)$$

II. Local stability

From the transformed subsystem in (3), the local stability is analyzed to determined the DFE

DFE=(s,e,i)=(S^*,0,0) and endemic equilibrium $X_{EE} = (s,e,i) = (s^*, e^*, i^*)$

The equilibrium point are computed by setting $\frac{ds}{dt} = 0, \frac{de}{dt} = 0, \frac{di}{dt} = 0$ and solving for s,e and i in (3) to compute the two equilibrium points from the equilibrium points, the jacobian matrix J is calculated from:

$$\begin{bmatrix} G \\ H \\ K \end{bmatrix} = \begin{bmatrix} \frac{ds}{dt} \\ \frac{de}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} A - \frac{Ki^2 s}{s^2 + \alpha i^2} - As + \delta(1 - s - e - i) \\ \frac{Ki^2 s}{s^2 + \alpha i^2} - (A + \varepsilon)e \\ \varepsilon e - (A + \gamma)i \end{bmatrix}$$

$$J(s,e,i) = \begin{bmatrix} \frac{\partial G}{\partial s} & \frac{\partial G}{\partial e} & \frac{\partial G}{\partial i} \\ \frac{\partial H}{\partial s} & \frac{\partial H}{\partial e} & \frac{\partial H}{\partial i} \\ \frac{\partial K}{\partial s} & \frac{\partial K}{\partial e} & \frac{\partial K}{\partial i} \end{bmatrix}$$

$$= \begin{bmatrix} -(A + \frac{Ki^2 s}{s^2 + \alpha i^2} - \frac{2Ki^2 s^2}{(s^2 + \alpha i^2)^2} + \delta) & -\delta & \frac{2s^3 iK}{(s^2 + \alpha i^2)^2} - \delta \\ \frac{Ki^2}{s^2 + \alpha i^2} - \frac{2Ki^2 s^2}{(s^2 + \alpha i^2)^2} & -(A + \varepsilon) & \frac{2s^3 ik}{(s^2 + \alpha i^2)^2} \\ 0 & \varepsilon & -(A + \gamma) \end{bmatrix}$$

And evaluate at the equilibrium points to decide on the local stability which is directly determined from the eigen value of :

$$|J(X) - \lambda I| = 0 \qquad (5)$$

Based on the eigen values of (5) ,the linearized system will be stable or unstable for the transformed subsystem in (3)

III. Disease free equilibrium

$$(s,e,i) = (s^*,0,0) = (1,0,0)$$

At the DFE (1,0,0) the jacobian matrix J(x)is given as:

$$j(\bar{X}_{DFE}) = \begin{bmatrix} -(A+\delta) & -\delta & -\delta \\ 0 & -(A+\varepsilon) & 0 \\ 0 & \varepsilon & -(A+\gamma) \end{bmatrix}$$

With eigen values λ

$$\left| J(\bar{X}_{DFE}) - \lambda I \right| = 0$$

$$\begin{vmatrix} -(A+\delta+\lambda) & -\delta & -\delta \\ 0 & -(A+\varepsilon+\lambda) & 0 \\ 0 & \varepsilon & -(A+\gamma+\lambda) \end{vmatrix} = 0$$

$$\lambda_1 = -(A+\delta), \lambda_2 = -(A+\varepsilon), \lambda_3 = -(A+\gamma)$$

All the eigen values of the jacobian evaluated at the equilibrium point contain negative real part so the system is stable.

IV. Endemic equilibrium

From the transformed subsystem in (3), we have

$$s^* = -[\frac{(A+\varepsilon+\delta)(A+\gamma)+\varepsilon\delta}{\varepsilon}]i^* + 1$$

$$i^* = \frac{[\varepsilon K \pm \sqrt{(\varepsilon K)^2 - 4(A+\varepsilon)^2(A+\gamma)^2\alpha}]s^*}{2(A+\varepsilon)(A+\gamma)\alpha}$$

$$e^* = \frac{(A+\gamma)i^*}{\varepsilon}$$

Which only makes physical sense if

$$(\varepsilon K)^2 - 4(A+\varepsilon)^2(A+\gamma)^2\alpha \succ 0$$

$$(\varepsilon K)^2 > 4(A+\varepsilon)^2(A+\gamma)^2\alpha$$

$$\varepsilon K > 2(A+\varepsilon)(A+\gamma)\sqrt{\alpha}$$

$$\frac{\varepsilon K}{2\sqrt{\alpha}(A+\varepsilon)(A+\gamma)} > 1$$

The epidemic condition R_0 is given as:

$$R_0 = \frac{\varepsilon K}{2\sqrt{\alpha}(A+\varepsilon)(A+\gamma)}$$

At endemic equilibrium, the jacobian matrix is given by:

$$J(X)=\begin{bmatrix} -A+\frac{Ki^{*2}}{s^2+\alpha i^{*2}}-2\frac{Ki^{*2}s^{*2}}{(s^{*2}+\alpha i^{*2})^2}-\delta & -\delta & \frac{-\delta+2s^{*3}i^{*}K}{(s^{*2}+\alpha i^{*2})^2} \\ \frac{Ki^{*2}}{s^{*2}+\alpha l^{*2}}-\frac{2Ki^{*2}s^{*2}}{(s^{*2}+\alpha i^{*2})^2} & -(A+\varepsilon) & \frac{2s^{*3}i^{*}K}{(s^{*2}+\alpha i^{*2})^2} \\ 0 & \varepsilon & -(A+\gamma) \end{bmatrix}$$

$$|J(X_{EE})-\lambda I|=0$$

$$\begin{vmatrix} -(A+B+\delta+\lambda) & -\delta & C-\delta \\ B & -(A+\varepsilon+\lambda) & C \\ 0 & \varepsilon & -(A+\gamma+\lambda) \end{vmatrix}=0$$

Where,

$$B=\frac{Ki^{*2}}{s^{*2}+\alpha l^{*2}}-\frac{2Ki^{*2}s^{*2}}{(s^{*2}+\alpha i^{*2})^2}$$

$$C=\frac{2s^{*3}i^{*}K}{(s^{*2}+\alpha i^{*2})^2}$$

Which is expanded as:

$$-(A+B+\delta+\lambda)\begin{vmatrix} -(A+\varepsilon+\lambda) & C \\ \varepsilon & -(A+\gamma+\lambda) \end{vmatrix}+\delta\begin{vmatrix} B & C \\ 0 & -(A+\gamma+\lambda) \end{vmatrix}+C-\delta\begin{vmatrix} B & -(A+\varepsilon+\lambda) \\ 0 & \varepsilon \end{vmatrix}=0$$

$$-(A+B+\delta+\lambda)[(A+\varepsilon+\lambda)(A+\gamma+\lambda)-C\varepsilon]-\delta B(A+\gamma+\lambda)+(C-\delta)B\varepsilon=0$$

$$-(A+B+\delta+\lambda)[(A+\varepsilon)(A+\gamma)+(A+\varepsilon)\lambda+\lambda(A+\gamma)+\lambda^2-C\varepsilon]-\delta B(A+\gamma)-\delta B\lambda-\delta B\varepsilon+CB\varepsilon=0$$
$$-(A+B+\delta)(A+\varepsilon)(A+\gamma)-(A+B+\delta)(A+\varepsilon)\lambda-(A+B+\delta)(A+\gamma)\lambda-\lambda^2(A+B+\delta)+(A+B+\delta)C\varepsilon$$
$$-\delta B(A+\gamma)-\delta B\lambda-\delta B\varepsilon-(A+\varepsilon)(A+\gamma)\lambda-(A+\varepsilon)\lambda^2-\lambda^2(A+\gamma)-\lambda^3+C\varepsilon\lambda+CB\varepsilon=0$$
$$\lambda^3+\lambda^2(3A+B+\delta+\varepsilon+\gamma)+\lambda[(A+B+\delta)(2A+\varepsilon+\gamma)+(A+\varepsilon)(A+\gamma)+\delta B-C\varepsilon]$$
$$+(A+B+\delta)[(A+\varepsilon)(A+\gamma)-C\varepsilon]+\delta B(A+\gamma+\varepsilon)+CB\varepsilon=0$$

$$a_0\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0$$

Where

$a_0 = 1$

$a_1 = (3A + B + \delta + \varepsilon + \gamma)$

$a_2 = [(A + B + \delta)(2A + \varepsilon + \gamma) + (A + \varepsilon)(A + \gamma) + \delta B - C\varepsilon]$

$a_3 = (A + B + \delta)[(A + \varepsilon)(A + \gamma) - C\varepsilon] + \delta B(A + \gamma + \varepsilon) + CB\varepsilon$

Then by Routh Hurwitz criteria, if $if a_1 > 0, a_3 > 0 and\ a_1 a_2 > a_3$ then the polynomial is locally stable.

V. Conclusions

In this paper we developed and implemented a new SEIRS model. The studies shows that the system is stable for disease free equilibrium and we also proved that the system, for endemic equilibrium is locally stable by using Routh Hurwitz criteria.

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