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Limits in Sequence and Function Spaces

A.N. Kumar [1*], L.K. Sharan [2]

[1]Dept. of Mathematics , Deo Sharan Women's Evening College , Sohsarai, Nalanda,Patliputra University, Patna, India.
[2] Ex. Head , P.G. Dept. of Mathematics, Veer Kunwar Singh University, Ara , Bhojpur, India.

[*]Corresponding Author: dramarchaurasiya@gmail.com, Tel.: +91-94309-34292 , +91-97080-79073

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Abstract— —In this work we have developed the study of sequence spaces, by establishing some of the results. We have also extended the study by establishing a few results to the case of function space analogous to that for sequence spaces. We also construct some suitable sequence and function spaces. In Result and Discussion section , 1st part we have established some of the result on the sequence spaces and in 2nd part we have established a few results using the definitions of different limits in function spaces. These have done on account of previous work here we refer [Cooke, (1),chapter 10].

Keywords— Co-ordinate Convergent , Projective Convergent, Co-ordinate Limit ,Projective Limit , ,in case of sequence space and Dual Space, Perfect Space ,Parametric Convergent, Parametric Limit, Projective Convergent and Projective Limit in case of function space**.**

I. INTRODUCTION

Köthe-Toeplitz,(1) studied sequence and sequence spaces in detailed. They introduced the notions of different sequence spaces and the dual sequence spaces of the given sequence spaces. Later on the study of sequence and sequence spaces was made by Allen, (1) who considerably developed and generalized by establishing a good number of results specially for the dual space of different sequence spaces. A few results had also been established for the dual space of the dual space of a sequence space. An account of all these can be found in Cooke,(1) chapter 10. Later on Prasad, (1) developed the study for function spaces by introducing some of the definitions to function spaces analogous to that for sequence spaces. A good number of results had been established by him. Later on Kumar,(1), studied it and developed some of the results for function and function spaces. Further, to go ahead in order to advance our study in this paper we, in the next section, give in details the definitions of different convergences and limits but only for sequence and sequence spaces and thereafter we shall establish some of the results by making the use of these notions of different limits and convergences. After establishing some of the results for sequence and sequence spaces we shall give the notions of different limits and convergence for function and function spaces. Efforts will also be made to establish some of the results for function and function spaces. In this paper I try to make a comparative study of sequence and function spaces by establishing some of the results.

Section I gives the introduction of Limits in Sequence and Function Spaces, Section II introduces about previous work related to the topic by authors, Section III yields methodology of the research work. Section IV elaborates the definitions c-cgt , p-cgt, c-limit, p-limit, and something more .Section V discusses on results as a theorem. Section VI concludes the paper with the discussion on the work carried out in this paper.

II. RELATED WORK

The previous research works has been done by the title "Convergence in Dual Space" and "Limit in Dual Space" . Also a paper has accepted by Indian Science Congress Association for oral presentation in 2019 Conference at L.P.U. Punjab , with the title "Parametric convergence implies projective convergence in the dual space of a function space." .

III. METHODOLOGY

In this paper the main constituent of research methodology is adopted. i.e. Theoretical perspective or orientation that guide research and logic of inquiry only. Further more I can say that the research work is an example of Qualitative research because the research paper is comparative study of sequence and function spaces.

IV. DEFINITIONS

Def-4.1 Co-Ordinate Convergence (Or c-convergent) : Let α be a sequence space. We now consider a sequence of points in α denoted by $x^{(1)}, x^{(2)}, x^{(3)}, \ldots\ldots\ldots x^{(n)}, \ldots\ldots$ Where $x_k^{(n)}$ be the k^{th} coordinate of $x^{(n)}$. Then the sequence of points $x^{(n)}$ is said to be coordinate convergent (c-convergent or simply c-cgt) when $\lim_{n\to\infty} x_k^{(n)}$ exists for every k

Def-4.2 Projective Convergent (p-convergent Or Simply p-cgt) : If $\phi(s) \le \beta \le \alpha_*$ and if for sequence $x^{(n)}$ in α , the sequence $u_n = \sum_{k=1}^{\infty} x_k^{(n)} u_k$ converges for every u in β, we say that $x^{(n)}$ is projective convergent (p-convergent) relative to β or αβ- convergent. When $\beta = \alpha^*$ (the dual space of α) , we say that $x^{(n)}$ is projective convergent in α or α-convergent.

Def-4.3 Coordinate Limit (Or c-limit) : If the $\lim_{n\to\infty} x_k^{(n)}$ exists for every k and is x_k then the point $x=(x_k)$ is called the coordinate limit of $x^{(n)}$ and in this case we write C-lim $x^{(n)}=x$.

Def-4.4 Projective Limit (p-limit) : A sequence x in α or outside α is called the projective limit (p-limit) of $x^{(n)}$ in α relative to β or αβ-limit $x^{(n)}$, when (i) $\sum_{k=1}^{\infty} u_k x_k$ is absolutely convergent for every u_k in β. That is x is in β^* , and

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} x_k^{(n)} u_k = \sum_{k=1}^{\infty} x_k u_k$ for every u in β.When $\beta=\alpha_*$,x is called the projective limit of $x^{(n)}$ in α or α-lim $x^{(n)}$.

also we refer to Cook, (1) to show that if αβ-limit $x^{(n)}=x$ then c-limit $x^{(n)}=$ x also it follows from(i) that c-limits of αβ-convergent sequences are considered as possible αβ-limits only if they are in β^* also by Cooke ,(1) αβ-convergence implies coordinate convergence (c-convergence) but the converse is false.

Def-4.5 Normal Sequence Space : A sequence space α is said to be normal if, whenever x is in α and $|y_k| \le |x_k|$ for every k, then y is in α.We now give two results which will be useful in establishing results for the proof of which we refer to Cooke ,(1). A necessary and sufficient condition for the αβ-convergence of $x^{(n)}$ in α is that to every u inβ, and to every ϵ>0, there corresponds a positive number N(ϵ,u) such that for every p,$\mathcal{E}$≥N, $|\sum_{k=1}^{\infty} u_k \left(x_k^{(p)} - x_k^{(\mathcal{E})}\right)| \le \epsilon$.

When β is normal, the necessary and sufficient condition that $x^{(n)}$ in α should be αβ-convergent is that to every u in β, and to every ϵ>0, corresponds a number N(ϵ,u) such that for every p,$\mathcal{E}$ ≥N $\sum_{k=1}^{\infty} |u_k \left(x_k^{(p)} - x_k^{(\mathcal{E})}\right)| \le \epsilon$.

V. RESULTS AND DISCUSSION

In this section we establish some of the results firstly for sequence spaces and then for function spaces. Also before establishing some results to the case of function spaces we will use to give some related definitions to serve as ready reference.

Theorem 5.1 : A sequence may be αβ-convergent and c-limit may not satisfy conditions (i) or (ii) or both for αβ-limit.That is c-limit of αβ-convergent sequence is not necessarily αβ-limit.

Observation : In fact we show this fact by constructing following examples.

Example 1**.** Let $x_k^{(n)} = (\frac{n}{n+1})^k$ Then $\sum_{k=1}^{\infty} |x_k^{(n)}|^r = \sum(\frac{n}{n+1})^{kr}$ $= (\frac{n}{n+1})^r + (\frac{n}{n+1})^{2r} + (\frac{n}{n+1})^{3r} + \ldots\ldots\ldots\ldots$

Which is a geometric series with common ratio $(\frac{n}{n+1})^r<1$ as r>0.So $x^n \epsilon \sigma_r$. we take r such that r is not an even integer.

Let u be a sequence in σ Then

$\sum_{k=1}^{\infty}[u_k x_k^{(n)}]^r = \sum_{k=1}^{k_0} + \sum_{k_0+1}^{\infty}[u_k x_k^{(n)}]^r$

Now $\sum_{k_0+1}^{\infty}[u_k x_k^{(n)}]^r$

$=u_{k_0+1}^r[x_{k_0+1}^{(n)}]^r + u_{k_0+1}^r(-1)^r[x_{k_0+2}^{(n)}]^r + u_{k_0+1}^r(-1)^{2r}[x_{k_0+3}^{(n)}]^r + \ldots\ldots\ldots\ldots\ldots$

$=u_{k_0+1}^r[\{x_{k_0+1}^{(n)}\}^r + (-1)^r\{x_{k_0+2}^{(n)}\}^r + (-1)^{2r}\{x_{k_0+3}^{(n)}\}^r + \ldots\ldots]$

$=u_{k_0+1}^r \ [\sum_{s=1}^{\infty}(-1)^{r(s-1)}\{x_{k_0+s}^{(n)}\}^r]$

$=u_{k_0+1}^r[\sum_{s=1}^{\infty}(-1)^{r(s-1)}(\frac{n}{n+1})^{(k_0+s)r}]$

$=u_{k_0+1}^r(-1)^{2n-r}[\sum_{s=1}^{\infty}(-1)^{rs}(\frac{n}{n+1})^{rs}](\frac{n}{n+1})^{k_0 r}$

[since$(-1)^r = (-1)^{2n-r}$]

$=u_{k_0+1}^r(-1)^{2n-r}(\frac{n}{n+1})^{k_0 r}\{(-1)^r\left(\frac{n}{n+1}\right)^r + (-1)^{2r}\left(\frac{n}{n+1}\right)^{2r} + \ldots\}$

$=\boldsymbol{u_{k_0+1}^r(-1)^{2n-r}} \ \frac{(\frac{n}{n+1})^{k_0 r}(-1)^r(\frac{n}{n+1})^r}{1-(-1)^r(\frac{n}{n+1})^r}$

$= \frac{(-1)^{2n}u_{k_0+1}^r(\frac{n}{n+1})^{r(k_0+1)}}{1-(-1)^r(\frac{n}{n+1})^r} = \frac{u_{k_0+1}^r(\frac{n}{n+1})^{r(k_0+1)}}{1-(-1)^r(\frac{n}{n+1})^r}$

Hence $\sum_{k=1}^{\infty}[u_k x_k^{(n)}]^r = \sum_{k=1}^{k_0} u_k^r (\frac{n}{n+1})^{kr} + \frac{u_{k_0+1}^r(\frac{n}{n+1})^{r(k_0+1)}}{1-(-1)^r(\frac{n}{n+1})^r}$

As n→∞, $(\frac{n}{n+1})^{kr} = \frac{1}{(1+\frac{1}{n})^{kr}}$ →1, for all k

$\therefore \lim \sum_{k=1}^{\infty}\left[u_k x_k^{(n)}\right]^r = \sum_1^{k_0} u_k^r + \frac{u_{k_0+1}^r}{1-(-1)^r}$

Which exists for every u in β, provided 1-$(-1)^r \neq 0$

⇒ $(-1)^r \neq 1$ that is r is not an even integer.

Thus if r is not an even integer, then x^n is $\sigma_r\sigma$-convergent.

Since $\sigma \le \sigma_r^* = L_\infty$ Now $\lim_{n\to\infty} x_k^{(n)} = \lim_{n\to\infty}(\frac{n}{n+1})^k = 1$

Hence c-limit $x^{(n)}=x$ where $x_k=1$ for every k

In this case $\sum_{k=1}^{\infty}|u_k x_k|^r = \sum_{k=1}^{\infty}|u_k|^r$ is not condition (1) is not satisfied by c-limit $x^{(n)}$ .Thus x is not $\sigma_r\sigma$-limit $x^{(n)}$

Example 2**:** This example shows the fact that c-limit $x^{(n)}$ does not αβ-limit. For let us consider the sequence $x^{(n)}=e^{(n)}$ in σ_r (thus $\alpha=\sigma_r$) Taking $\beta=\Upsilon^{(s)}$, the space of all cnvergent sequences. Let $u\epsilon\beta \Rightarrow u\epsilon\Upsilon^{(s)}$

Then$\sum_{k=1}^{\infty}[u_k x_k^{(n)}]^r = \sum_{k=1}^{\infty}[u_k e_k^{(n)}]^r = u_n^r$[since$e_n^{(n)}=1, e_p^{(n)}=0$ (p≠n)] Since $u\epsilon\Upsilon^{(s)}$, $\lim_{n\to\infty} u_n$ exists and $\{u_k\}$ is bounded.

Then $\sum_{k=1}^{\infty}[u_k x_k^{(n)}]^r$ converges for every u in $\Upsilon^{(s)}$

Therefore $x^{(n)}$ is σ_r $\Upsilon^{(s)}$-convergent $(\Upsilon^{(s)} \le \sigma_r^* = \sigma_\infty)$

Also $\lim_{n\to\infty} x_k^{(n)} = \lim_{n\to\infty} e_k^{(n)} = 0 = x_k = 0$, for all k. Thus c-lim is zero .Thus $\sum_{k=1}^{\infty} x_k^r u_k^r = 0$

But $\lim_{n\to\infty}\sum_{k=1}^{\infty}[u_k x_k^{(n)}]^r = \lim_{n\to\infty} u_n^r$

If we take $u_k=1$ for every k, $u\epsilon\Upsilon^{(s)}$ but then the above limit is not 0 as it is.Thus the condition (ii) is not satisfied. When

$\beta=\phi(S)$ ($\phi(s)\leq\alpha^*$, for any α) we have the following simple result, in case r is not even.

Theorem 5.2 : $\alpha\phi(s)$ convergence coincides with c-convergence and $\alpha\phi(s)$-limit is coextensive with c-limits of c-convergent sequences in α, in case r is not an even integer.

Proof : Let $x^{(n)}$ in α be c-convergent. Let $x_k^{(n)} \to x_k$ as $n\to\infty$. Then c-limit $x^{(n)}=x$.Let $u\epsilon\phi(s)$ and $u_k=0$ for k>p. Then $\lim_{n\to\infty}\sum_{k=1}^{\infty}\{x_k^{(n)}u_k)\}^r=\lim_{n\to\infty}\sum_{k=1}^{p}\{x_k^{(n)}u_k\}^r=\sum_{k=1}^{p}x_k^r u_k^r$

So limit exists for every u in $\phi(s)$. Hence $x^{(n)}$ is $\alpha\phi$-convergent. So every c-convergent sequence in α is $\alpha\phi(s)$-convergent.We know that $\alpha\phi(s)$-convergence implies c-convergence and so the two convergences coincide.

Now $\sum_{k=1}^{\infty}|u_k^r x_k^r|=\sum_{k=1}^{p}|u_k^r x_k^r|$ Which converges for all u (in $\phi(s)$) $\lim_{n\to\infty}\sum_{k=1}^{p}\{x_k^{(n)}u_k\}^r=\sum_{k=1}^{\infty}x_k^r u_k^r$ for every u in β. Thus c-limit of c-convergent sequence in α is its $\alpha\phi(s)$-limit. Also $\alpha\phi(s)$-limit is c-limit as r is not even.Hence the result is proved.

In order to make a comparative study of sequence and function spaces, we extend the results established above to the case of function and function spaces.We shall also observe that the forms of the results for sequence spaces are the same for function space or quite different.As a matter of fact we recall here that in the case of sequence spaces we deal with integers where as in the case of function spaces we deal with continuous variables. Due to this change in sequence and function spaces their remains a fair chance of change in the form of the results for two spaces.

We now need the definitions of dual space of a function space, parametric convergent, projective convergent, parametric limit and projective limit for function spaces so I am giving here.

Def-4.6 Dual Space : Let α be a function space then its dual space is denoted by α^* and is defined the space of all functions f such that $\int |f(x)\, g(x)|dx<\infty$ for every function g in α. Also α^* is a function space and α^* is the dual space of the function space α only.

Def-4.7 Perfect Space : A function space α is said to be perfect when $\alpha^{**}=\alpha$. Also L_1,L_∞, E_s,E_s are perfect.

Def-4.8 Parametric Convergent (OR t-Convergent): Let $f_t(x)$ be a family of functions of x defined for all t in $[0,\infty]$, where t is a parameter.

If to every $\boldsymbol{\epsilon}>0$, there corresponds a positive number $T(\boldsymbol{\epsilon})$, independent of x, such that,for almost all $x\geq0$,$|f_t(x)-f_{t'}(x)|\leq\boldsymbol{\epsilon}$ for all $t,t'\geq T(\boldsymbol{\epsilon})$, then the family $f_t(x)$ is said to be parametric convergent (t-convergent).

Def-4.9 Parametric Limit (t-limit) : If , to given any $\boldsymbol{\epsilon}>0$, there corresponds a number $t(\boldsymbol{\epsilon})$, independent of x, such that for almost all $x\geq0$, $|f_t(x)-\boldsymbol{\Psi}(x)|\leq\boldsymbol{\epsilon}$ for all $t\geq T(t)$, then $\boldsymbol{\Psi}(x)$ is called the parametric limit (t-limit) of $f_t(x)$ and we write t-limit of $f_t(x)=\boldsymbol{\Psi}(x)$.

Here we observe that any function equal to $\boldsymbol{\Psi}(x)$, for almost all $x\geq0$, is also a t-limit of $f_t(x)$.

Therefore when we say that $\boldsymbol{\Psi}(x)$ is the parametric limit (t-limit) of $f_t(x)$, we mean that $\boldsymbol{\Psi}(x)$ is a t-limit of $f_t(x)$ and all functions equivalent to $\boldsymbol{\Psi}(x)$ in $[0,\infty]$ are t-limits of $f_t(x)$. [A function $\boldsymbol{\Theta}$ is said to be equivalent to $\boldsymbol{\Psi}$ in $[0,\infty]$ when $\boldsymbol{\Theta}(x)=\boldsymbol{\Psi}(x)$ almost everywhere in $[0,\infty]$].

Def-4.10 Projective Convergence(or αβ-Convergence or p-Convergence : Let $\beta\subseteq\alpha^*$ and $F_g(t)=\int f_t(x)\, g(x)\, dx$ Where f_t is in α and g is in β then if $F_g(t)$ tends to a definite finite limit as t-tends to ∞ for every g in β then we say that f_t is projective convergent (or p-convergent)relative to β, Also $f_t(x)$ is called $\alpha\beta$-convergent. also $f_t(x)$ is called p-convergent in α or α-convergent when $\beta=\alpha^*$that is when $\beta\supseteq\alpha^*$ [see sharan, (1)]

Def-4.11 Projective Limit [p-limit or αβ-limit]: A function $\boldsymbol{\Psi}$, in α or outside α, is called a projective limit (p-limit) of $f_t(x)$ in α relative to β and we write $\boldsymbol{\Psi}(x)=\alpha\beta$-limit of $f_t(x)$ when

(i) $\int |g(x)\,\boldsymbol{\Psi}(x)|dx<\infty$ for every g in β, and (ii) $\lim \int f_t(x)\, g(x)dx=\int\boldsymbol{\Psi}(x)\, g(x)\, dx$ for every g in β.

When $\beta=\alpha^*$, $\boldsymbol{\Psi}$ is called a projective limit (p-limit) of $f_t(x)$ in α and we write $\boldsymbol{\Psi}(x)=\alpha$-limit of $f_t(x)$. Different $\alpha\beta$-limits of $f_t(x)$ can differ only in a set of x of measure zero. Hence where we say that $\boldsymbol{\Psi}(x)$ is the $\alpha\beta$-limit of $f_t(x)$, we mean that $\boldsymbol{\Psi}(x)$ is an $\alpha\beta$-limit of $f_t(x)$and other $\alpha\beta$-limits of $f_t(x)$ are equivalent to $\boldsymbol{\Psi}(x)$ .It follows from the definitions that every $\alpha\beta$-limit belongs to β^*.

Theorem 5.3 : Every parametric convergent family $f_t(x)$ in Υ is Υ-convergent.

Proof : Let f_t be in Υ and f_t is t-convergent in Υ then,to every given $\epsilon>0$,there exists, a positive number T(t), independent of x, such that, for almost all $x\geq0$, $|f_t(x)-f_{t'}(x)|\leq\epsilon$ ……..(3.11)

for all $t,t'\geq T(\epsilon)$ .Now let g(x) be any function in L_1.

Hence $\int|g(x)|dx<\infty$ ………………………………(3.12)

Also g(x) is in L_1 implies g(x) is in Υ^* ……(3.13)

Now since, $|\int g(x)\{f_t(x)-f_{t'}(x)\}dx|\leq\int|g(x)||f_t(x)-f_{t'}(x)|dx\leq$ $\epsilon\int|g(x)|dx$ [by (3.11)] $\leq\epsilon\, k(g)$ [by(3.12)]

for every $t,t'\geq T(\epsilon)$, for every $\epsilon>0$, where k(g) is a constant depending on g but independent of t in $E=[0,\infty]$.Thus $f_t(x)$ is $\Upsilon\Upsilon^*$-convergent .That is $f_t(x)$ is Υ-convergent.Thus the theorem is established.

Theorem 5.4: Every t-convergent family $f_t(x)$ in α is αL_1-convergent provided $\alpha\supseteq L_1$.

Proof : Let $f_t(x)$ be a family of functions of x and t be a parameter where t is in $[0,\infty]$.Let f_t in α be t-convergent then, To every $\epsilon>0$, there exists a positive number $T(\epsilon)$, independent of x, such that, for almost all $x\geq0$,

$|f_t(x)-f_{t'}(x)|\leq\epsilon$ ……………………(3.14)

For all $t,t'\geq T(\epsilon)$ .Also let g(x) be any function in L_1

Hence $\int|g(x)|dx<\infty$ …………………………..(3.15)

Thus by hypothesis $g(x)\epsilon\alpha^*$.

Now since, $|\int g(x)\{f_t(x)-f_{t'}(x)\}dx|\leq\int|g(x)||f_t(x)-f_{t'}(x)|dx$ $\leq\epsilon\int|g(x)dx$ [by(3.14)] $\leq\epsilon k(g)$ [by (3.15)]

For every $t,t'\geq T(\epsilon)$, for every $\epsilon>0$, where k(g) is a constant depending on g but independent of t in $E=[0,\infty]$.

Thus $f_t(x)$ is αL_1-convergent.Thus the theorem is proved.

Theorem 5.5: Every t-convergent family $f_t(x)$ in α is αβ-convergent provided $\beta \supseteq L_1$

Proof : Let $f_t(x)$ in α be a family of functions of x and t be a parameter where t is in [0,∞].By hypothesis $f_t(x)$ in α is t-convergent then to every ϵ>0,There exists a positive number T(ϵ), independent of x,Such that, for almost all x≥0,

$|f_t(x)-f_{t'}(x)|\le\epsilon$ …………………………..(3.16)

For all t,t'≥T(ϵ)

Now let g(x)be in L_1 then $\int|g(x)|dx<\infty$ ……..(3.17)

Now since, $|\int g(x)\{f_t(x)-f_{t'}(x)\}dx|\le\int|g(x)||f_t(x)-f_{t'}(x)|dx \le\epsilon k(g)$ by [(3.16) and (3.17)]

Constant depending on g but independent of t in E=[o,∞).

Thus $f_t(x)$ is αβ-convergent.Thus the theorem is proved.

Theorem 5.6: every t-convergent family $f_t(x)$ is in α is αβ-convergent provided $\beta\subseteq L_1$.

Proof : Let g(x) be in β.Hence g(x) is integral That is

$\int|g(x)|dx<\infty$ ………………………(3.18)

Also let $f_t(x)$ be a family of functions of x and t be a parameter where t is in [0,∞].Since $f_t(x)$ is t-convergent so by definition to every ϵ>0,there exists a positive number T(ϵ), independent of x, such that ,for almost all x≥0,

$|f_t(x)-f_{t'}(x)|\le\epsilon$ ……………………(3.19)

For all t,t'≥T(ϵ),Now since, $|\int g(x)\{f_t(x)-f_{t'}(x)\}dx| \le\int|g(x)||f_t(x)-f_{t'}(x)|dx|\le\epsilon k(g)$ by [(3.18) and (3.19)]

For every t,t'>T(ϵ), for every ϵ>0, where k(g) is a constant depending on g but independent of t in E=[o,∞).

Thus $f_t(x)$ is αβ-convergent .Thus the theorem is proved.

Thus with the help of a few theorems established above it is exhibited that in a vis-à-vis to sequence spaces t-convergent necessarily implies projective convergence.We now establish a few results to establish a connection between t-limit and p-limit.

Theorem 5.7: Let (i) f_t in α be αβ-convergent (ii) $\psi(x)$ be t-limit of $f_t(x)$ then t-limit is the αβ-limit of $f_t(x)$ provided $\beta\subseteq L_1$.

Proof: Let f_t be a family of functions in α.Also let $\psi(x)$=the t-limit of $f_t(x)$.Then by definition, for a given ϵ>0, there exists a positive number T(ϵ), independent of x, such that, for almost all x≥0, $|f_t(x)-\psi(x)|\le\epsilon$ ……………………(3.20)

Again let E=[0,∞].Now $\int|g(x)\psi(x)|dx$ For g is in β,ϵ>0;t≥T(ϵ) $=\int|g(x)||\psi(x)-f_t(x)+f_t(x)|dx=\int|g(x)||\psi(x)-f_t(x)|dx+\int|g(x)f_t(x)|dx <\epsilon.k(x)+\infty$

Since g is in β so g is in L_1 and hence g is integrable

Also$\int|g(x)f_t(x)|dx<\infty$ Since g is in β implies that g is in α^*. Since $\alpha^*\supseteq\beta$ and $f_t\epsilon\alpha$ Therefore $\int|g(x)\psi(x)|dx<\infty$ …..(3.21)

Again $|\int f_t(x)g(x)dx-\int\psi(x)g(x)dx|\le\int|g(x)||f_t(x)-\psi(x)|dx \le.\epsilon.k(x)$ by (3.20)

For since g is in $\beta\subseteq L_1$

Therefore,$\lim_{t\to\infty}\int f_t(x)g(x)dx=\int\psi(x)g(x)dx$ …..(3.22)

Thus by [(3.21) and (3.22)] it follows that $\psi(x)$ is an αβ-limit of $f_t(x)$

Theorem 5.8 : Let (i) f_t in α be αβ-convergent(ii) $\psi(x)$ be t-limit of $f_t(x)$ then t-limit is the αβ-limit of $f_t(x)$ provided $\alpha^*\supseteq\beta\supseteq L_1$.

proof : Let $f_t(x)$ be a fimily of functions of x and t be a parameter. Such that t is in [0,∞)=E. Also let t-limit of $f_t(x)=\psi(x)$,Then to give any ϵ>0, there exists a positive number T(ϵ) independent of x, such that , for almost all x≥0,

$|f_t(x)-\psi(x)|\le\epsilon$ …………….(3.23) for all t≥T(ϵ).

Now choosing any ϵ>0 and any t≥T(ϵ), we have $\int|g(x)(x)|dx=\int|g(x)||\psi(x)-f_t(x)+f_t(x)|dx \le\int|g(x)||\psi(x)-f_t(x)|dx+\int|g(x)f_t(x)|dx\le\epsilon.k(g)+A(g)$

For every g in $L_1\subseteq\beta\le\alpha^*$, where k and A are constants depending on g but independent of t in E.

Thu $\int|g(x)(x)|dx<\infty$ ……………..(3.24)

For every g in L_1 and hence in α^*.

Also, $|\int f_t(x)g(x)dx-\int\psi(x)g(x)dx|\le\int|g(x)||f_t(x)-\psi(x)|dx\le\epsilon.k(g)$

For every t≥T(ϵ)and every g in L_1 and is true for every ϵ>0.

Therefore $\lim_{t\to\infty}\int f_t(x)g(x)dx=\int\psi(x)g(x)dx$ ….(3.25)

For every g in L_1.Now by [(3.24) and (3.25)] it follows that $\psi(x)$ is an αβ-limit of $f_t(x)$ .That is, t-limit of $f_t(x)$ is αβ–limit of $f_t(x)$.Thus the theorem is established.

Theorem 5.9 : Let (i) $f_t(x)$ in α be α-convergent,(ii) $\psi(x)$ be t-limit of $f_t(x)$,(iii) $L_1\supset\alpha^*$ and (iv) $\psi(x)$ be the t-limit of $f_t(x)$ in α. Then $\psi(x)$= α-limit of $f_t(x)$.

Proof: Let $f_t(x)$ in α be a family of functions of x where t is a parameter and t is in E=[0,∞].Also let g(x) be any function in α^*.Thus $\int|g(x)|dx<\infty$ ……………………(3.26)

By hypothesis,(x)=t-limit of $f_t(x)$,then to give any ϵ>0, there exists a positive number T(ϵ) independent of x, such that, for almost all x≥0, $|f_t(x)-\psi(x)|\le\epsilon$ …(3.27) for all t≥T(ϵ).

Now choosing any ϵ>0, and any t≥T(ϵ) we have $\int|g(x)(x)|dx=\int|g(x)||\psi(x)-f_t(x)+f_t(x)|dx \le\int|g(x)||\psi(x)-f_t(x)|dx+\int|g(x)f_t(x)|dx\le\epsilon k.(g)+A(g)$ [by (3.26) and (3.27)and the fact that g is in α^*] Where k and A are constants depending on g but independent of t in E.

Thus $\int|g(x)(x)|dx<\infty$ ………(3.28) for every g in α^*

Again by [(3.26) and (3.27)] $|\int f_t(x)g(x)dx-\int\psi(x)g(x)dx| \le\int|g(x)||f_t(x)-\psi(x)|dx\le\epsilon.k(g)$. For every t≥T(ϵ) and every g in α^* and is true for every ϵ>0. Therefore, $\lim_{t\to\infty}\int f_t(x)g(x)dx=\int\psi(x)g(x)dx$ ..…(3.29)for every g in α^*.Now with the joint effort of [(3.28) and (3.29)] it follows that $\psi(x)$ is an $\alpha\alpha^*$-limit of $f_t(x)$.That is t-limit of $f_t(x)$ is α-limit of $f_t(x)$.Thus the theorem is established.

Theorem 5.10 : If (i) $\alpha\ge\boldsymbol{\phi}$ (ii) $\beta\le L_\infty$ then every function in β^* is αβ-limit of its section.

Proof: Let $\psi(x)\epsilon\beta^*$ and let $f_t(x)$={ $\psi(x)$ for 0≤x≤t 0 for x>t

Then $f_t(x)$ is a section of $\psi(x)$, and is in $\boldsymbol{\phi}$. But since $\beta\le L_\infty$

Therefore $f_t(x)\epsilon\alpha\ge\boldsymbol{\phi}$ .Hence $f_t(x)\epsilon\boldsymbol{\phi}$ .Since $\psi(x)$ is in β^* then

$\int|\psi(x)g(x)|dx<\infty$ …………(3.30) for every g in β. Also ,

$\lim_{t\to\infty}\int f_t(x)g(x)dx=\lim_{t\to\infty}\int_0^t\psi(x)g(x)dx=\int\psi(x)g(x)dx$ ………(3.31).Thus from [(3.30) and (3.31)]

$\psi(x)$ is the αβ-limit of $f_t(x)$.Thus the result is proved.

VI. CONCLUSION

Thus we find that in the case of function spaces for the results through which t-limits are αβ-limits there not necessary for β to be normal as it essential for the case of sequence spaces for which we refer to theorem(5.1).

Also t-limits are αβ-limits to the cases of function spaces whereas this is not possible in the case of sequence spaces.

Also in the light of the theorem (5.10) established above it is found that the same can be proved in the case of sequence space for which we refer to theorem(3.3,II).

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AUTHOR'S PROFILE

Dr.A.N.Kumar pursed ,M.Sc. and Ph.D. in Pure Mathematics from Patna University. Patna & V.K.S.University , Ara in 2007 & 2018 respectively. He is currently working as Assistant Professor cum Head in Department of Mathematics of Deo Sharan Women's Evening College, Sohsarai , Nalanda ,Bihar, India under Patliputra University , Patna since 2007. He is a life member of ISCA since 2013.He has published more than 8 and 9 research papers in reputed international journals including international conferences and national journal including national conferences respectively. His main research work focuses on function Spaces.