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A Review on Univariate Bayesian Frailty Models

S.G. Parekh[1*], S.R. Patel[2]

[1*]Faculty of Business Administration, Dharmsinh Desai University, Nadiad, India
[2]Department of Statistics, Sardar Patel University, Vallabh Vidyanagar, India

[*]Corresponding Author: sg_parekh@yahoo.com

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Abstract—The main idea of this review article is to review different univariate Bayesian frailty model, discussed by some authors, such as estimation of negative binomial distribution with Bayesian prior distribution as beta, Bayesian frailty estimation of Poisson distribution with gamma as frailty prior distribution, Bayesian frailty estimation of gamma distribution with gamma as frailty prior distribution, Bayesian frailty estimation of univariate normal distribution with frailty prior distribution as gamma and Bayesian estimation of inverse Gaussian distribution with uniform prior frailty model. In all these Bayesian estimation squared error loss function is used.

Keywords—Bayesian frailty model, Bayesian prior distribution, Beta, Negative Binomial, Gamma, Inverse Gaussian, Normal, Poisson, Uniform distribution.

I. INTRODUCTION

Clayton [1], Vaipul et.al [2] and many other authors have introduced survival models and Cox [3] has proposed proportional hazard (PH) regression model. Haugaard [4, 5] has taken some models as failure time models. Many authors have dealt the classical Bayesian approach for estimation of frailty models, amongst which, Akaike [6], Le cam [7], Joshi [8], Geyer and Thompson [9] are pioneer. Aalen [10] has discussed modelling heterogeneity in survival analysis using compound Poisson distribution. Mc Gilchrist [11] obtain REML estimation for survival models with frailty. Yasin et al.[12] have worked on advantages approach to survival analysis of bivariate data for correlated individual frailty. Yasin AI et al. [13] have discussed multivariate frailty models and estimation strategies in Genetic and environment factors in duration studies. Xue X and Brookmeyer R [14] have discussed bivariate frailty model for the analysis of multivariate survival time. Sahu et al. [15] utilized prior distribution similar to normal distribution with mean zero and large variance. Yasin et al. [16] have done comparative analysis of bivariate survival models on Genetic factors in susceptibility to death. Ibrahim et al. [17] have discussed many Bayesian frailty models in their book. Maximum likelihood inference for multivariate frailty models using an automated MCEM algorithm discussed by Ripatti S [18]. Iachine I [19] used survival data of twins in the family and used the statistical methods based on multivariate frailty survival models. A bivariate frailty model with a cure fraction for modelling familial correlations in diseases was discussed by Wienke A et.al. [20]. Santosh et al. [21] have used Weibull as baseline distribution and Gamma, Lognormal as Bayesian frailty distribution, while kheiri et al. [22] have used the inverse Gaussian Bayesian frailty model. Hangal [23] used Weibull distribution as baseline. In the Cox model and gamma distribution as frailty Bayesian distribution. Parekh et al. [24] used linear hazard function and exponential baseline distribution. Parekh et al. [25] have discussed the Bayesian frailty models and obtained Bayesian frailty estimators.

In section II we have considered different Univariate Bayesian frailty models with different prior frailty models such as Negative Binomial-Beta, Poisson-Gamma, Gamma-Gamma, Normal-Gamma, Inverse Gaussian-Uniform respectively. Section III deals with conclusion and future work.

II. UNIVARIATE BAYESIAN FRAILTY MODELS

We consider the following univariate Bayesian frailty models with different base line distributions and different prior frailty distributions.

(A) Estimation of Negative Binomial distribution with Bayesian prior distribution as Beta.

Let $x_1, x_2, \ldots x_n$ are n independent observations from negative binomial baseline distribution with parameters k, θ; $Neg(k, \theta)$, and if frailty prior distribution of θ is Beta with parameters α, β i.e. $\pi(\theta) \sim Be(\alpha, \beta)$ where α and β are known , then the Bayesian frailty estimate, $\delta^{\pi}(x)$ of θ is

$$\delta^{\pi}(x) = \frac{kn + \alpha}{kn + \alpha + \sum x_i + \beta}$$

Further considering loss function $L(\theta, \delta) = (\delta - \frac{1}{\theta})^2$, the Bayesian frailty estimate, $\delta_1^{\pi}(x)$ of $\frac{1}{\theta}$ is then

$$\delta_1^{\pi}(x) = \binom{k+\sum x_i+1}{\sum x_i} \frac{1}{\beta(\alpha,\beta)} \int_0^1 \frac{1}{\theta} \theta^{nk+\alpha-1} (1-\theta)^{\sum x_i+\beta-1} d\theta$$

$$= \frac{nk+\alpha-1}{nk+\alpha+\sum x_i+\beta}$$

(B) Bayesian Frailty Estimation of Poisson distribution with Gamma as frailty prior distribution

Let X have Poisson baseline distribution, with mean θ and let θ have frailty prior distribution as Gamma with parameters α, β both known $\pi(\theta)$ of θ be $\theta \sim G(\alpha, \beta)$ where α, β be known then the Bayesian frailty estimate, $\delta^{\pi}(x)$ of θ is

$$\delta^{\pi}(x) = \frac{x + \alpha}{\beta + 1}$$

Further if the loss function is squared error one, $L(\theta, \delta) = (\delta - \frac{1}{\theta})^2$, then the Bayesian frailty estimate, $\delta_1^{\pi}(x)$ of $\frac{1}{\theta}$ is then

$$\delta_1^{\pi}(x) = \frac{(\beta+1)^{x+\alpha}}{\Gamma(x+\alpha)} \int_0^{\infty} \frac{1}{\theta} \theta^{x+\alpha-1} e^{-(\beta+1)\theta} d\,\theta$$

$$= \frac{\beta+1}{x+\alpha-1}$$

Also if $x_1, x_2, \ldots, x_n$ are n independent observations from Poisson baseline distribution and if the frailty prior distribution of θ is $\theta \sim G(\alpha, \beta)$ then the Bayesian frailty estimates of θ, $\delta_2^{\pi}(x)$ and the Bayesian frailty estimates of $\frac{1}{\theta}$, $\delta_3^{\pi}(x)$ are

$$\delta_2^{\pi}(x) = \frac{\sum x_i + \alpha}{\beta + 1}$$

and $$\delta_3^{\pi}(x) = \frac{\beta+1}{\sum x_i+\alpha-1}.$$

(C) Bayesian frailty estimation of Gamma distribution with Gamma as frailty prior distribution

If X have Gamma baseline distribution with parameters, ν and θ; $G(\nu, \theta)$, where ν is shape parameter (known) and the frailty prior distribution, $\pi(\theta)$ of θ be $\theta \sim G(\alpha, \beta)$ where α, β be known then the Bayesian frailty estimate $\delta^{\pi}(x)$ of θ is

$$\delta^{\pi}(x) = \frac{\alpha + \nu}{x + \beta}$$

Instead of squared error loss function if the loss function is $L(\theta, \delta) = (\delta - \frac{1}{\theta})^2$

Then the Bayesian frailty estimate, $\delta_1^{\pi}(x)$ of $\frac{1}{\theta}$ is then

$$\delta_1^{\pi}(x) = \frac{\beta+x}{\alpha+\nu-1}$$

Also if $x_1, x_2, \ldots, x_n$ are n observations from Gamma distribution, $G(\nu, \theta)$ and if θ has Gamma distribution $G(\alpha, \beta)$ then the Bayesian frailty estimates of θ, $\delta_2^{\pi}(x)$ and the Bayesian frailty estimates of $\frac{1}{\theta}$, $\delta_3^{\pi}(x)$ are

$$\delta_2^{\pi}(x) = \frac{\alpha + \nu}{\beta + \sum x_i}$$

and

$$\delta_3^{\pi}(x) = \frac{\beta + \sum x_i}{\alpha + \nu - 1}$$

(D) Bayesian frailty estimation of Univariate Normal distribution with frailty prior distribution as Gamma.

If X has $N(\mu, \frac{1}{\theta})$ with known μ and variance $\frac{1}{\theta}$ and if frailty prior distribution $\pi(\theta)$ is $G(\frac{\alpha}{2}, \frac{\beta}{2})$ were α, β are known then the Bayesian frailty estimate, $\delta^{\pi}(y)$ of θ is

$$\delta^{\pi}(x) = \frac{\alpha + 1}{(x - \mu)^2 + \beta}$$

Further instead of squared error loss function if the loss function is $L(\theta, \delta) = (\delta - \frac{1}{\theta})^2$ then the Bayesian frailty estimate, $\delta_1^{\pi}(x)$ of $\frac{1}{\theta}$ is

$$\delta_1^{\pi}(y) = \frac{(x-\mu)^2+\beta}{\alpha-1}$$

Also let $x_1, x_2, \dots, x_n$ be n observations from Normal distribution, $N(\mu, \frac{1}{\theta})$, and if θ has Gamma distribution, $G(\frac{\alpha}{2}, \frac{\beta}{2})$. Then the Bayesian frailty estimates of θ, $\delta_2^{\pi}(x)$ and the Bayesian frailty estimates of $\frac{1}{\theta}$, $\delta_3^{\pi}(x)$ are

$$\delta_2^{\pi}(x) = \frac{\alpha+1}{(\sum x_i - \mu)^2 + \beta}$$

and $$\delta_3^{\pi}(x) = \frac{(\sum x_i - \mu)^2 + \beta}{\alpha - 1}.$$

(E) Inverse Gaussian distribution: Bayesian Frailty model and Uniform prior frailty model.

If X have Inverse Gaussian baseline distribution, $IG(x \mid \theta)$ with p.d.f.

$$f(x \mid \theta) = \left(\frac{\theta}{2\pi x^3}\right)^{\frac{1}{2}} exp\left\{-\frac{\theta(x-1)^2}{2x}\right\}; \qquad x > 0,\ \theta > 0$$

and the frailty prior distribution, $\pi(\theta)$ of θ is $\pi(\theta) = \frac{k}{\theta}$, where k is a known.then the Bayesian frailty estimate, $\delta^{\pi}(x)$ of θ is

$$\delta^{\pi}(x) = \frac{x}{(x-1)^2}$$

Let $x_1, x_2, \dots, x_n$ be observations from Gaussian distribution and let the frailty posterior distribution of θ be Gamma distribution with shape parameter $\frac{n}{2}$ and scale parameter $\frac{(\sum x_i - 1)^2}{2\sum x_i}$ and then Bayesian frailty estimate of θ is

$$\delta_1^{\pi}(x) = \frac{n\sum x_i}{(\sum x_i - 1)^2}, \qquad \sum x_i \neq 1$$

Also when the observations $x_1, x_2, \dots, x_n$ from the Inverse Gaussian distribution, the Bayesian frailty estimate, $\delta_2^{\pi}(x)$ of $\frac{1}{\theta}$ is

$$\delta_2^{\pi}(x) = \frac{(\sum x_i - 1)^2}{(n-2)\sum x_i} \qquad , \qquad n > 2$$

Remark: For the detail proofs of all above results kindly refer Parekh and Patel [26].

III. CONCLUSION

In this paper we have given mostly all Univariate Bayesian frailty models and with their Bayesian estimates. Many authors have contributed researches on different Bayesian frailty models with different baseline distribution with different frailty prior distribution such as Negative Binomial and Beta, Poisson and Gamma, Gamma and Gamma, Normal and Gamma, Inverse Gaussian and Uniform. By generating observations for the baseline and frailty prior distribution one can verify the Bayesian estimates for the above distributions as further research work.

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