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On Multivariate Bayesian frailty models: A Review

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Abstract—In this paper we have reviewed three models one on: Bayesian frailty Bivariate Normal model, second on: Bayesian frailty estimation of Multivariate Normal model with prior Multivariate Normal model and third on: Bayesian frailty estimation of Compound Poisson model with prior frailty model as Poisson. In all these models Bayesian frailty estimators have been obtained.

Keywords—*Bayesian frailty model, Bivariate Normal model, Compound Poisson model, Multivariate Normal model, Prior frailty model.*

I. INTRODUCTION

Some authors like Clayton [1], Haugaard [2, 3], Sahu et al. [4], Hangal [5], Parekh et al. [6] have used Bivariate frailty distributions and obtained Bayesian estimates. Yashin A.I. et al. [7] have analysed bivariate data of population studies by using multivariate models. Yashin A.I. and Iachine, I [8] also discussed dependent hazards in multivariate survival problems. Iachine I [9] analysed twins in the family and used statistical methods based on multivariate Bayesian frailty models. Wienke A et al. [10] have discussed bivariate frailty models for the analysis of survival data of multivariate lifetime data. Also Ibrahim et al. [11], kheiri et al. [12], Santosh et al. [13] and Parekh et al. [14] have discussed some Bayesian frailty estimation. Parekh and Patel [15] have obtained estimates of multivariate frailty models.

In section II we have estimated (i) Bayesian frailty estimation of Bivariate normal model with prior model of $\beta_1$and β_2 in Cox regression model (ii) Bayesian frailty estimation of multivariate normal model with prior multivariate normal frailty model (iii) Bayesian frailty estimation of Compound Poisson model with prior frailty model as Poisson. Section III deals with conclusion.

II. SOME MULTIVARIATE BAYESIAN FRAILTY MODELS

(A): Bayesian model of frailty Bivariate Normal **distribution**

Let Cox bivariate linear regression model be

$y = b_1 X_1 + b_2 X_2 + \epsilon$, $0\leq b_1, b_2 \leq 1$

and let $\underline{y} = (y_1, y_2, \dots, y_n)'$ and $X = \begin{pmatrix} X_{11} & X_{21} \\ \dots & \dots \\ X_{1n} & X_{2n} \end{pmatrix}$ be samples from the model and let ϵ_i be frailty variables having $N(0,1)$ distributions that is $y_i \sim N(b_1 X_{1i} + b_2 X_{2i}, 1)$,

$i = 1, 2, \dots, n$ be independently distributed.

Further frailty non informative prior distribution is

$[\pi(b_1, b_2)]^2 = 1$, if $0\leq b_1 \leq 1$, $0\leq b_2 \leq 1$ then the posterior frailty mean of (b_1, b_2) is

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sim N_2 \left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, (X'X)^{-1} \right)$$

were

$$X = \begin{pmatrix} X_{11} & X_{21} \\ \dots & \dots \\ X_{1n} & X_{2n} \end{pmatrix}$$

If $V = (v_{ij}) = (X'X)^{-1} = \begin{pmatrix} v_{11}^2 & v_{12} \\ v_{12} & v_{22}^2 \end{pmatrix}$

then the conditional distribution of b_1 given b_2 is

$$f(b_1 \mid b_2) = N\left(\widehat{b_1} + \frac{v_{12}}{v_{22}^2}(b_2 - \hat{b}_2),\ v_{11}^2 - \frac{v_{12}^2}{v_{22}^2}\right)$$

where $\widehat{b_1}$ and $\hat{b}_2$ are estimates of b_1 and b_2 from posterior distribution.

Using these, one can get Bayesian frailty estimates of $\hat{b}_i$ as

$$\delta_i^\pi(y_1, y_2, \dots, y_n) = \hat{b}_i - v_{ii}\frac{exp\left\{-\frac{(1-\hat{b}_i)^2}{2v_{ii}^2}\right\} - exp\left\{-\frac{\hat{b}_i{}^2}{2v_{ii}^2}\right\}}{\sqrt{2\pi}\left\{\Phi\left(\frac{(1-\hat{b}_i)}{v_{ii}}\right) - \Phi\left(-\frac{\hat{b}_i}{v_{ii}}\right)\right\}}, \text{ for}$$

$i = 1,2.$

Were $\Phi(x)$ is cumulative distribution function of $N(0,1)$

(B) Bayesian frailty estimation of Multivariate Normal distribution with prior Multivariate Normal distribution.

Let the baseline distribution of $\underline{y}$ be $N_p(\theta, \Sigma)$ and let $\underline{\theta}$ have prior frailty distribution, $\pi(\theta)$ as $N_p(\underline{\mu}, A)$ where $\underline{\mu}$ is known. Σ and A are $(p \text{ x } p)$ known positive definite matrices. Then the Bayesian frailty estimate of θ is

$$\theta_y^\pi = \underline{y} - \Sigma(A+\Sigma)^{-1}\left(\underline{y} - \underline{\mu}\right).$$

We give below the salient features of the above result.

The exponent of the joint density of $\underline{y}$ and $\underline{\theta}$

$\left(\underline{y} - \underline{\theta}\right)' \Sigma^{-1}\left(\underline{y} - \underline{\theta}\right) + \left(\underline{\theta} - \underline{\mu}\right)' A^{-1}\left(\underline{\theta} - \underline{\mu}\right)$ transfers to

$$\left[\underline{\theta} - C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)\right]' C\left[\underline{\theta} - C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)\right] - \left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)'.\ C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right) + \underline{y}'\Sigma^{-1}\underline{y} + \underline{\mu}'A^{-1}\underline{\mu}$$

where $C^{-1} = A - A(A+\Sigma)^{-1}A$ which when substituted the above expression and simplifying, we get

$$\underline{y}'\Sigma^{-1}\underline{y} + \underline{\mu}'A^{-1}\underline{y} - \left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)' C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)$$

$$= \left(\underline{\mu} - \underline{y}\right)' (A+\Sigma)^{-1}(\underline{\mu} - \underline{y})$$

so that the joint density of $\underline{y}$ and $\underline{\theta}$ will be

$$f\left(\underline{y}, \underline{\theta}\right) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}|A|^{\frac{1}{2}}} exp\left\{-\frac{1}{2}\left[\underline{\theta} - C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)\right]' C\left[\underline{\theta} - C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)\right]\right\}.exp\left\{-\frac{1}{2}(\underline{y} - \underline{\mu})'(A+\Sigma)^{-1}(\underline{\mu} - \underline{y})\right\}$$

which shows that the Bayesian frailty estimate of $\underline{\theta}$ being mean of the posterior distribution of $\underline{\theta}$ is

$$\theta_y^\pi = C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)$$

$$= \underline{y} - \Sigma(A+\Sigma)^{-1}\left(\underline{y} - \underline{\mu}\right)$$

and the covariance matrix is $A - A(A+\Sigma)^{-1}A$.

Taking $\underline{y}_1, \underline{y}_{2,\dots,}\underline{y}_n$ as sample from $N_p(\underline{\theta}, \Sigma)$ and if $\underline{\theta}$ has frailty distribution $N_p(\underline{\mu}, A)$ where $\underline{\mu}, A, \Sigma$ are known. $\overline{\underline{y}} = (\overline{\underline{y}}_1, \overline{\underline{y}}_2, \dots, \overline{\underline{y}}_n)$ being sufficient for $\underline{\theta}$ and the distribution of $\overline{\underline{y}}$ has $N_p\left(\underline{\theta}, \frac{1}{n}\Sigma\right)$.

The frailty Bayesian estimate of $\underline{\theta}$ will turns out as

$\theta_y^\pi = \overline{\underline{y}} - \Sigma(nA+\Sigma)^{-1}(\overline{\underline{y}} - \underline{\mu})$ with frailty covariance matrix as $A - nA(nA+\Sigma)^{-1}A$.

Note that the estimator is obtained here under quadratic loss function.

(C) Bayesian frailty estimation of Compound Poisson distribution with prior frailty distribution as Poisson

Consider $Y_1, Y_2, \dots, Y_N$ as identically independently distributed as Gamma, $G(\alpha, \beta)$ variates with known scale parameter β and known shape parameter α and consider N as prior frailty variate having Poisson distribution with known mean μ. Then the compound Poisson variate Z is defined as

$$Z = \begin{cases} Y_1 + Y_2 + \cdots + Y_N & ; if\ N > 0 \\ 0 & ; if\ N = 0 \end{cases}$$

As $Z = \sum_{i=1}^N Y_i \sim G(N\alpha, \beta)$ with p.d.f.

$$f(z \mid N) = \frac{\beta^{N\alpha}}{\Gamma(\alpha N)} z^{N\alpha - 1} e^{-\beta z}, \quad \alpha > 0, \beta > 0$$

and N has p.d.f. as

$$\pi(N) = \frac{e^{-\mu}\mu^N}{N!}, \quad N = 0,1,2, \dots, \infty$$

so that the joint p.d.f. of Z and N will be

$$h(z,N) = \frac{\beta^{N\alpha}\mu^{N}}{\Gamma(\alpha N)N!} z^{N\alpha-1} e^{-(\mu+\beta z)}$$

Evaluating the marginal density of Z the posterior frailty distribution of N has p.d.f. as

$$\pi(N \mid z) = \frac{\frac{(\beta z)^{N\alpha}\mu^{N}}{\Gamma(\alpha N)N!}}{\sum_{N=0}^{\infty} \frac{(\beta z)^{N\alpha}\mu^{N}}{\Gamma(\alpha N)\ N!}}$$

For squared error loss function, the Bayesian frailty estimate, $\delta^{\pi}(z) = E(N \mid z)$ will be

$$\delta^{\pi}(z) = \mu(\beta z)^{\alpha}.$$

III. CONCLUSION

Considering Bayesian frailty Bivariate Normal model, Bayesian frailty estimation of Multivariate Normal model with prior Multivariate Normal model and Bayesian frailty estimation of Compound Poisson model with prior frailty model as Poisson, we have estimated the frailty parameters in these models with squared error loss function. Instead of squared error loss function if absolute loss function is taken than the Bayesian frailty estimators for the above models can be studied for the Bayesian frailty estimation.

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