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Regularity and Normality in Bitopological Spaces

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Abstract- In this paper, we define one notion of regular and two notions of normal space in bitopological spaces. We find some relationship from topological spaces and bitopological spaces of such notions. Then, we have shown that such notions given satisfy hereditary, projective and productive properties. Furthermore, we have shown some of their features.

Keywords-Topology, Bitopology, Regular property, Normal property, Hereditary property.

I. INTRODUCTION

The definition of bitopological space, regular bitopological space and normal bitopological space was first defined by Kelly [1]. Again the definition of normal property in topological space was first defined by Tietze [2]. After then Aarts[3], Hyung [4], Lal [5], Lane [6], Nour [7] turned their interest in this regard. Recently, Arunmaran [8], Dorsett [9], Jasiml [10], J.M.H [11], Mukharjee [12] introduced the generalizations of different concepts of pairwise regularity, pairwise normality and complete normality.

Section I in this paper presents the description of this field. Section II defines some of the notations which are used in the paper. Section III recalls some definitions which are ready reference in our work. In section IV, we establish some properties of separation axiom in bitopological space and some of their features. Section V is the conclusion of our research work.

II. NOTATIONS

Through this paper X, Y will be a non empty set and $\mathcal{S}, \mathcal{T}, \mathcal{W}, \mathcal{Z}$ be the topology on X. $(X, \mathcal{S}, \mathcal{T})$ and $(Y, \mathcal{W}, \mathcal{Z})$ be bitopological spaces. U, V are open sets and its elements are x, y, x_1, x_2, y_1, y_2.

III. PRELIMINARIES

Definition 3.1 A bitopological space $(X, \mathcal{S}, \mathcal{T})$ is called T_0 space if $\forall x, y \in X$ with $x \neq y$ then
$\exists U \in \mathcal{S} \cup \mathcal{T}$ such that $x \in U, y \notin U$ or $x \notin U, y \in U$ [13].

Definition 3.2 A bitopological space $(X, \mathcal{S}, \mathcal{T})$ is called T_1 space if $\forall x, y \in X$ with $x \neq y$ then
$\exists U \in \mathcal{S}$ and $V \in \mathcal{T}$ such that $x \in U, y \notin U$ and $x \notin U, y \in U$ [14].

Definition 3.3 Let x be a non empty set. A class $\mathcal{T}$ of subsets of X is a topology on X iff $\mathcal{T}$ satisfies the following axioms

(a) X and ϕ belong to $\mathcal{T}$.
(b) The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.
(c) The intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The members of $\mathcal{T}$ are then called $\mathcal{T}$ open sets or simply open sets and X together with $\mathcal{T}$ .Hence the pair $(X, \mathcal{T})$ is called a topological space [15].

Definition 3.4 Let A be a non empty subset of a topological space $(X, \mathcal{T})$ .The class $\mathcal{T}_A$ all intersections of A with $\mathcal{T}$ open subsets of X is a topology on A, it is called the relative topology on A or the relativization of $\mathcal{T}$ to A, and the topological space $(A, \mathcal{T}_A)$ is called a subspace of $(X, \mathcal{T})$ [15].

Definition 3.5 Let, $\{(X_i, \mathcal{S}_i, \mathcal{T}_i)\}$ be a collection of bitopology spaces and let X be the product of the sets X_i i.e $X = \prod i \ X_i$. The coarsest topology $\mathcal{S}$ and $\mathcal{T}$ on X with respect to which all the projections $\wedge i : X \longrightarrow X_i$ are continuous is called the product topology. The product set X with the product bitopology $\mathcal{S}$ and $\mathcal{T}$ i.e $(X, \mathcal{S}, \mathcal{T})$ is called the product topological space or simply product space [15].

Definition 3.6 A space X on which are defined two topologies $\mathcal{S}$ and $\mathcal{T}$ is called a bitopological space and denoted by $(X, \mathcal{S}, \mathcal{T})$ [1].

Definition 3.7 A mapping $f: (X, \mathcal{S}, \mathcal{T}) \longrightarrow (Y, \mathcal{W}, \mathcal{Z})$ is called P-continuous (respectively P-open, P-closed) if the induced mappings $f: (X, \ \mathcal{S}) \longrightarrow (Y, \mathcal{W})$ and $f: (X \ , \mathcal{T}) \longrightarrow (Y, \ \mathcal{Z})$ are continuous (respectively open, closed) [1].

Definition 3.8 A bitopological space $(X, \mathcal{S}, \mathcal{T})$ is called T_2 space if $\forall x, y \in X$ with $x \neq y$ then
$\exists U \in \mathcal{S}, V \in \mathcal{T}$ such that $x \in U, y \in V$ and $U \cap V = \phi$ [1].

Definition 3.9 In a space $(X, \mathcal{S}, \mathcal{T}), \mathcal{S}$ is said to be regular with respect to $\mathcal{T}$ if, for each x in X, there is a $\mathcal{S}$-neighbourhood base of $\mathcal{T}$-closed set, or, as easily seen to

be equivalent , if, for each point x in X and each $\mathcal{S}$-closed set S such that $x \notin S$, there are a $\mathcal{S}$-open set U and a $\mathcal{T}$ open set such that
$x \in U, S \subseteq V$, and $U \cap V = \phi$.
$(X, \mathcal{S}, \mathcal{T})$ is, or $\mathcal{S}$ and $\mathcal{T}$ are, pairwise regular if $\mathcal{S}$ is regular with respect to $\mathcal{T}$ and vice versa [1].

Definition 3.10 A space $(X, \mathcal{S}, \mathcal{T})$ is said to be pairwise normal if, given a $\mathcal{S}$ closed set F and a $\mathcal{T}$ closed set G with $F \cap G = \phi$, then $\exists$ a $\mathcal{T}$ open set U and a $\mathcal{S}$ open set V such that $F \subseteq U, G \subseteq V$ and $U \cap V = \phi$ [1].

Definition 3.11 A space $(X, \mathcal{S}, \mathcal{T})$ is said to be completely normal space if $F, G \subseteq X$ with $F \cap \overline{G}^{\mathcal{S}}$, $\overline{F}^{\mathcal{T}} \cap G = \phi$ then $\exists$ a $\mathcal{T}$-open set U and a $\mathcal{S}$- open set V such that $F \subseteq U, G \subseteq V$ and $U \cap V = \phi$ [1].

IV. PROPERTIES OF REGULAR, NORMAL AND COMPLETE NORMALITY IN BITOPOLOGICAL SPACES

In this section, we define one notion of regular and two notions of normal in bitopological space. We also find some relations of them.

Definition 4.1. A space $(X, \mathcal{S}, \mathcal{T})$ is said to be regular if for each $x \in X$ and a $\mathcal{T}$ closed set F such that $x \notin F$ then $\exists$ a $\mathcal{T}$ open set U and a $\mathcal{S}$ open set V such that $x \in U, F \subseteq V$ and $U \cap V = \phi$.

Definition 4.2. A space $(X, \mathcal{S}, \mathcal{T})$ is said to be normal space if a $\mathcal{T}$ closed set F and a $\mathcal{S}$ closed set G with $F \cap G = \phi$ then $\exists$ a $\mathcal{T}$ open set U and a $\mathcal{S}$ open set V such that $F \subseteq U, G \subseteq V$ and $U \cap V = \phi$.

Definition 4.3. A space $(X, \mathcal{S}, \mathcal{T})$ is said to be completely normal space if $F, G \subseteq X$ with $F \cap \overline{G}^{\mathcal{S}}$, $\overline{F}^{\mathcal{T}} \cap G = \phi$ then $\exists$ a $\mathcal{T}$-open set U and a $\mathcal{S}$- open set V such that $F \subseteq U, G \subseteq V$ and $U \cap V = \phi$.

Theorem 4.1. If $(X, \mathcal{S})$ and $(X, \mathcal{T})$ are both regular space then $(X, \mathcal{S}, \mathcal{T})$ need not be regular space.
Proof. Let, $X = \{a, b, c\}$
$\mathcal{S} = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$
$\mathcal{T} = \{X, \phi, \{a, b\}, \{c\}\}$
$(X, \mathcal{S})$ and $(X, \mathcal{T})$ are both regular space but $(X, \mathcal{S}, \mathcal{T})$ is not a regular space since $a \in X$ and $\mathcal{T}$ closed set $F = \{c\}$ such that $a \notin F = \{c\}$ then $\exists$ $\mathcal{T}$ open set $U = \{a, b\}$ and $\mathcal{S}$ open set $V = \{b, c\}$ such that $a \in U, F \subseteq V$ but $U \cap V = \{b\} \neq \phi$.

Theorem 4.2. If $(X, \mathcal{S}, \mathcal{T})$ is a regular space then need not be $(X, \mathcal{S})$ and $(X, \mathcal{T})$ be regular space.
Proof. Let, $X = \{a, b, c\}$
$\mathcal{S} = \{X, \phi, \{a\}\}$
$\mathcal{T} = \{X, \phi, \{b, c\}\}$
A space $(X, \mathcal{S}, \mathcal{T})$ is regular space but $(X, \mathcal{S})$ is not regular because $a \in X$ and a closed set $F = \{b, c\}$ such that $a \notin F$ then $\exists$ $U = \{a\}$ and $V = X$ such that $a \in U, F \subseteq V$ but $U \cap V \neq \phi$.
Similarly, $(X, \mathcal{T})$ is not regular because $b \in X$ and a closed set $F = \{a\}$ such that $b \notin F$ then $\exists$ $U = \{b, c\}, V = X$ such that $b \in U, F \subseteq V$ but $U \cap V \neq \phi$.

Theorem 4.3. If $(X, \mathcal{S})$ and $(X, \mathcal{T})$ are both normal space then $(X, \mathcal{S}, \mathcal{T})$ need not be normal space.
Proof. Let, $X = \{a, b, c\}$
$\mathcal{S} = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$
$\mathcal{T} = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$
$(X, \mathcal{S})$ and $(X, \mathcal{T})$ are both normal space but $(X, \mathcal{S}, \mathcal{T})$ is not a normal space since $\mathcal{T}$ closed set $F = \{a, b\}$ and $\mathcal{S}$ closed set $G = \{c\}$ with $F \cap G = \phi$ then $\exists$ $\mathcal{T}$ open set $U = \{c\}$ and $\mathcal{S}$ open set $V = X$ such that $F \subseteq U, G \subseteq V$ but $U \cap V = \{c\} \neq \phi$.

Theorem 4.4. Show that every closed subspace of regular space is also regular.
Proof. Suppose $(X, \mathcal{S}, \mathcal{T})$ is a regular space and A is a closed subset of X with respect to $\mathcal{S}$ and $\mathcal{T}$.We shall prove that $(X, \mathcal{S}_A, \mathcal{T}_A)$ is regular.
Let $x \in A$ and F be a $\mathcal{T}$-closed subset of A with $x \notin F$. Since $A \subseteq X$ then $x \in X$. Again since F is $\mathcal{T}$-closed then $F = \overline{F}$. Further since F is $\mathcal{T}$-closed subset of A then $\exists$ $\mathcal{T}$-closed subset F_1 such that $F_1 \cap A = F \implies F_1 \cap A = F$. Now we have $x \notin F = F_1 \cap A \implies x \notin F_1$ or $x \notin A$
$\implies x \notin F_1$ or $x \in A$. From above we have $x \in A \subseteq X$ i.e $x \in X$ and a $\mathcal{T}$-closed subset F_1 with $x \notin F_1$. Since $(X, \mathcal{S}, \mathcal{T})$ is regular space then $\exists$ $\mathcal{T}$-open set U and $\mathcal{S}$-open set V such that $x \in U, F_1 \subseteq V$ and $U \cap V = \phi$. Since U is $\mathcal{T}$-open then $U \cap A$ is $\mathcal{T}_A$-open . Again V is $\mathcal{S}$-open then $V \cap A$ is $\mathcal{S}_A$-open. Now we have $x \in A$ and $x \in U \implies x \in U \cap A$. $F_1 \subseteq V \implies F_1 \cap A \subseteq V \cap A \implies F = V \cap A$ and $(U \cap V) = \phi \implies (U \cap V) \cap A = \phi \implies (U \cap A) \cap (V \cap A) = \phi$.
Hence $(X, \mathcal{S}_A, \mathcal{T}_A)$ is a regular.

Theorem 4.5. Show that every closed subspace of normal space is also normal.
Proof. Suppose $(X, \mathcal{S}, \mathcal{T})$ is a normal space and A is a closed subset of X with respect to $\mathcal{S}$ and $\mathcal{T}$.We shall prove that $(X, \mathcal{S}_A, \mathcal{T}_A)$ is normal.

Let, F be a $\mathcal{T}$-closed subset of A and G be a $\mathcal{S}$-closed subset of A with $F \cap G = \phi$. Since F is $\mathcal{T}$-closed subset of A and G is $\mathcal{S}$-closed subset of A then $\exists$ $\mathcal{T}$-closed subset F_1 and $\mathcal{S}$-closed subset G_1 of X such that $F_1 \cap A = F$ and $G_1 \cap A = G$.Now we have ,$F \cap G = \phi \implies (F_1 \cap A) \cap (G_1 \cap A) = \phi \implies (F_1 \cap G_1) \cap A = \phi \implies F_1 \cap G_1 = \phi$.
From above we have , a $\mathcal{T}$-closed subset F_1 and a $\mathcal{S}$-closed subset G_1 with $(F_1 \cap G_1) = \phi$. Since $(X, \mathcal{S}, \mathcal{T})$ is a normal space then $\exists$ $\mathcal{T}$-open set U and $\mathcal{S}$-open set V such that $F_1 \subseteq U, G_1 \subseteq V$ and $U \cap V = \phi$. Since U is $\mathcal{T}$-open then $U \cap A$ is $\mathcal{T}_A$ open. Again since ,V is $\mathcal{S}$-open then $V \cap A$ is $\mathcal{S}_A$ open. Now we have $F_1 \subseteq U \implies F_1 \cap A \subseteq U \cap A \implies F \subseteq U \cap A$. Similarly,$G_1 \subseteq V \implies G_1 \cap A \subseteq V \cap A \implies G \subseteq V \cap A$ and $(U \cap V) = \phi \implies (U \cap V) \cap A = \phi \implies (U \cap A) \cap (V \cap A) = \phi$.

Hence $(X, \mathcal{S}_A, \mathcal{T}_A)$ is a normal.

Theorem 4.6. Show that regular space is a topological property.

Proof. Let $f: (X, \mathcal{S}, \mathcal{T}) \longrightarrow (Y, \mathcal{W}, \mathcal{Z})$ be a homeomorphism and $(X, \mathcal{S}, \mathcal{T})$ is regular space. We shall prove that $(Y, \mathcal{W}, \mathcal{Z})$ is also regular space.

Consider, $y \in Y$ and $F^C \in \mathcal{Z}$ with $y \notin F$. Since f is onto then $\exists\, x \in X$ such that $f(x) = y$. Again since f is one one then x is unique $f(x) = y \Longrightarrow f^{-1}(y) = \{x\}$. Also we have, $x \notin f^{-1}(F)$ as $f(x) \notin F$. Now we have , $f^{-1}(F)$ is closed set with $x \notin f^{-1}(F)$. Since $(X, \mathcal{S}, \mathcal{T})$ is regular space then $\exists\, U \in \mathcal{T}$ and $V \in \mathcal{S}$ with $\in U$,$f^{-1}(F) \subseteq$ V and $U \cap V = \phi$. Now we have,$f(x) \in f(U)$ and $f(f^{-1}(F)) \subseteq f(V)$ i.e $F \subseteq f(V)$ with $f(U \cap V) = f(\phi) \Longrightarrow f(U) \cap f(V) = \phi$. Further since f is open then $f(U) \in \mathcal{Z}$ and $f(V) \in \mathcal{W}$. Now we get $f(U) \in \mathcal{Z}$ and $f(V) \in \mathcal{W}$ such that $f(x) = y \in f(U)$ and $F \subseteq f(V)$ with $f(U) \cap f(V) = \phi$.

$\therefore (Y, \mathcal{W}, \mathcal{Z})$ is a regular space. i.e, every homeomorphic image of a regular space is also a regular space. Hence, regular space is a topological property.

Theorem 4.7. Show that normal space is a topological property.

Proof. Let $f: (X, \mathcal{S}, \mathcal{T}) \longrightarrow (Y, \mathcal{W}, \mathcal{Z})$ be a homeomorphism and $(X, \mathcal{S}, \mathcal{T})$ is normal space. We shall prove that $(Y, \mathcal{W}, \mathcal{Z})$ is also normal space.

Consider, $F^C \in \mathcal{Z}$ and $G^C \in \mathcal{W}$ with $F \cap G = \phi$. Since f is onto then $f^{-1}(F \cap G) = f^{-1}(\phi) \Longrightarrow f^{-1}(F) \cap f^{-1}(G) = \phi$. Since f is continuous then $f^{-1}(F)$ is $\mathcal{T}$-closed and $f^{-1}(G)$ is $\mathcal{S}$-closed with $f^{-1}(F) \cap f^{-1}(G) = \phi$ in X. Since $(X, \mathcal{S}, \mathcal{T})$ is normal then $\exists\, U \in \mathcal{T}$ and $V \in \mathcal{S}$ with $f^{-1}(F) \subseteq U$, $f^{-1}(G) \subseteq V$ and $U \cap V = \phi$.Now we have,$f(f^{-1}(F)) \subseteq f(U)$ and $f(f^{-1}(G)) \subseteq f(V)$ i.e $F \subseteq f(U), G \subseteq f(V)$ with $f(U \cap V) = f(\phi) \Longrightarrow f(U) \cap f(V) = \phi$. . Further since f is open then $f(U) \in \mathcal{Z}$ and $f(V) \in \mathcal{W}$. Now we get $f(U) \in \mathcal{Z}$ and $f(V) \in \mathcal{W}$ such that $F \subseteq f(U)$ and $G \subseteq f(V)$ with $f(U) \cap f(V) = \phi$.

$\therefore (Y, \mathcal{W}, \mathcal{Z})$ is a normal space i.e, every homeomorphic image of a normal space is also a normal space. Hence, normal space is a topological property.

Theorem 4.8. A bitopological space $(X, \mathcal{S}, \mathcal{T})$ is normal if and only if for every $\mathcal{S}$-open set G and $\mathcal{T}$-closed set F with $F \subseteq G$, then $\exists$ a $\mathcal{S}$-open set V either a $\mathcal{T}$-open set V such that $F \subseteq V \subseteq \overline{V} \subseteq G$.

Proof. Let ,$(X, \mathcal{S}, \mathcal{T})$ be a normal space. Consider, a $\mathcal{T}$-closed set F and $\mathcal{S}$-open set G such that $F \subseteq G$. Since $F \subseteq G \Longrightarrow F \cap G^C = \phi$. Now we have , $\mathcal{T}$ closed set F and $\mathcal{S}$ closed set G^C such that $F \cap G^C = \phi$. Since $(X, \mathcal{S}, \mathcal{T})$ is normal space then $\exists$ a $\mathcal{T}$ open set H and a $\mathcal{S}$ open set H_1 such that $F \subseteq H, G^C \subseteq H_1$ and $H \cap H_1 = \phi$. Since $G^C \subseteq H_1 \Longrightarrow H_1^C \subseteq G$. Again since, $H \cap H_1 = \phi \Longrightarrow H \subseteq H_1^C$ and $H_1^C \subseteq H$ and $\overline{H} \subseteq \overline{H_1}^C \Longrightarrow \overline{H} \subseteq H_1^C$. Now we get, $F \subseteq H \subseteq \overline{H} \subseteq H_1^C \subseteq G \Longrightarrow F \subseteq H \subseteq \overline{H} \subseteq G$. Put $H = V$ then we get, $F \subseteq V \subseteq \overline{V} \subseteq G$.

Conversely, let F_1 be a $\mathcal{T}$ closed set and F_2 be a $\mathcal{S}$ closed set with $F_1 \cap F_2 = \phi \Longrightarrow F_1 \subseteq F_2^C$, where F_1 is $\mathcal{T}$ closed and F_2^C is $\mathcal{S}$ open. Now by the assumption $\exists\, V \in \mathcal{T}$ or $V \in \mathcal{S}$ such that $F \subseteq V \subseteq \overline{V} \subseteq F_2^C \Longrightarrow F_1 \subseteq V$ and $\overline{V} \subseteq F_2^C \Longrightarrow (F_2)^C \subseteq \overline{V}^C$. Now we get, $\mathcal{T}$ open set V and $\mathcal{S}$ open set $W = \overline{V}^C$ such that $F_1 \subseteq V$ and $F_2 \subseteq W$. We shall prove that $V \cap W = \phi$. Now, $V \cap W = V \cap \overline{V}^C = V \cap (F_2^C) = \phi$.

Theorem 4.9. Show that every closed subspace of completely normal space is also completely normal.

Proof. Suppose $(X, \mathcal{S}, \mathcal{T})$ is a completely normal space and A is a closed subset of X with respect to $\mathcal{S}$ and $\mathcal{T}$. We shall prove that $(X, \mathcal{S}_A, \mathcal{T}_A)$ is completely normal.

Let, $F, G \subseteq A$ with $\overline{F}^{\mathcal{T}} \cap G = \phi$ and $F \cap \overline{G}^{\mathcal{S}} = \phi \Longrightarrow F \cap G = \phi$. Again let F and G be any two subset of A with $\overline{F}^{\mathcal{T}A} \cap G = \phi$ and $F \cap \overline{G}^{\mathcal{S}A} = \phi$, then F, G is also subset of X with $\overline{F}^{\mathcal{T}} \cap G = \phi$ and $F \cap \overline{G}^{\mathcal{S}} = \phi$. Since $(X, \mathcal{S}, \mathcal{T})$ is completely normal then $\exists\, H_1 \in \mathcal{T}$ and $H_2 \in \mathcal{S}$ such that $\overline{F}^{\mathcal{T}} \subseteq H_1$ and $\overline{G}^{\mathcal{S}} \subseteq H_2$. Now we have $A \cap \overline{F}^{\mathcal{T}} \subseteq H_1 \cap A$ and $A \cap \overline{G}^{\mathcal{S}} \subseteq A \cap H_2 \Longrightarrow \overline{F}^{\mathcal{T}A} \subseteq H_1 \cap A$ and $\overline{G}^{\mathcal{S}A} \subseteq A \cap H_2$ where $H_1 \cap A \in \mathcal{T}_A$, $A \cap H_2 \in \mathcal{S}_A$. Hence we get $H_1 \cap A \in \mathcal{T}_A$ and $A \cap H_2 \in \mathcal{S}_A$ such that $\overline{F}^{\mathcal{T}A} \subseteq H_1 \cap A$ and $\overline{G}^{\mathcal{S}A} \subseteq A \cap H_2$.

Hence $(X, \mathcal{S}_A, \mathcal{T}_A)$ is a completely normal space.

Theorem 4.10. Show that completely normal space is a topological property.

Proof. Let $f: (X, \mathcal{S}, \mathcal{T}) \longrightarrow (Y, \mathcal{W}, \mathcal{Z})$ be a homeomorphism and $(X, \mathcal{S}, \mathcal{T})$ is completely normal space. We shall prove that $(Y, \mathcal{W}, \mathcal{Z})$ is also a completely normal space.

Consider, $F^C \in \mathcal{Z}$ and $G^C \in \mathcal{W}$ with $\overline{F}^{\mathcal{T}} \cap G = \phi$ and $F \cap \overline{G}^{\mathcal{S}} = \phi$. Since F is $\mathcal{T}$-closed then $F = \overline{F}^{\mathcal{T}}$ and G is $\mathcal{S}$ closed then $G = \overline{G}^{\mathcal{S}}$. Again since f is onto then $f^{-1}(F \cap \overline{G}^{\mathcal{S}}) = f^{-1}(\phi)$ and $f^{-1}(\overline{F}^{\mathcal{T}} \cap G) = f^{-1}(\phi) \Longrightarrow f^{-1}(F) \cap f^{-1}(G) = \phi$. Since f is continuous then $f^{-1}(F)$ is $\mathcal{T}$-closed and $f^{-1}(G)$ is $\mathcal{S}$-closed with $f^{-1}(F) \cap f^{-1}(G) = \phi$ in X. Since $(X, \mathcal{S}, \mathcal{T})$ is completely normal then $\exists\, U \in \mathcal{T}$ and $V \in \mathcal{S}$ with $f^{-1}(F) \subseteq U$, $f^{-1}(G) \subseteq V$ and $U \cap V = \phi$.Now we have,$f(f^{-1}(F)) \subseteq f(U)$ and $f(f^{-1}(G)) \subseteq f(V)$ i.e $F \subseteq f(U), G \subseteq f(V)$ with $f(U \cap V) = f(\phi) \Longrightarrow f(U) \cap f(V) = \phi$. Further since f is open then $f(U) \in \mathcal{Z}$ and $f(V) \in \mathcal{W}$. Now we get $f(U) \in \mathcal{Z}$ and $f(V) \in \mathcal{W}$ such that $F \subseteq f(U)$ and $G \subseteq f(V)$ with $f(U) \cap f(V) = \phi$.

$\therefore (Y, \mathcal{W}, \mathcal{Z})$ is a completely normal space . i.e, every homeomorphic image of a completely normal space is also a completely normal space. Hence, completely normal space is a topological property.

Theorem 4.11. If $(X_1, \mathcal{S}_1, \mathcal{T}_1)$ and $(X_2, \mathcal{S}_2, \mathcal{T}_2)$ are both regular space then $(X_1 \times X_2, \mathcal{S}_1 \times \mathcal{S}_2, \mathcal{T}_1 \times \mathcal{T}_2)$ is regular.
Proof. Let $(x_1, x_2) \in X_1 \times X_2$ and F be a $\mathcal{T}_1 \times \mathcal{T}_2$ closed subset of $X_1 \times X_2$ with $(x_1, x_2) \notin F$. Since F is closed subset of $\mathcal{T}_1 \times \mathcal{T}_2$ then $\exists\, \mathcal{T}_1$ closed set F_1 and $\mathcal{T}_2$ closed set F_2 such that $F_1 \times F_2 = F$. Now $(x_1, x_2) \notin F \Longrightarrow (x_1, x_2) \notin F_1 \times F_2 \Longrightarrow x_1 \notin F_1$ or $x_2 \notin F_2$ or both. Consider, $x_1 \notin F_1$. Since $(X_1, \mathcal{S}_1, \mathcal{T}_1)$ is regular space then $\exists\, \mathcal{T}_1$ open set U_1 and $\mathcal{S}_1$ open set V_1 such that $x_1 \in U_1, F_1 \subseteq V_1$ with $U_1 \cap V_1 = \phi$. Again if $x_2 \notin F_2$. Since $(X_2, \mathcal{S}_2, \mathcal{T}_2)$ is regular space then $\exists\, \mathcal{T}_2$ open set U_2 and $\mathcal{S}_2$ open set V_2 such that $x_2 \in U_2, F_2 \subseteq V_2$ with $U_2 \cap V_2 = \phi$. From above we have, $x_1 \in U_1, x_2 \in U_2 \Longrightarrow (x_1, x_2) \in U_1 \times U_2 \in \mathcal{T}_1 \times \mathcal{T}_2$ and $F_1 \subseteq V_1, F_2 \subseteq V_2 \Longrightarrow F_1 \times F_2 \in V_1 \times V_2 \in \mathcal{S}_1 \times \mathcal{S}_2$ with $U_1 \cap V_1 = \phi, U_2 \cap V_2 = \phi \Longrightarrow (U_1 \times U_2) \times (V_1 \times V_2) = \phi$.
Hence, regular space is productive.

Theorem 4.12. If $(X_1 \times X_2, \mathcal{S}_1 \times \mathcal{S}_2, \mathcal{T}_1 \times \mathcal{T}_2)$ is regular then $(X_1, \mathcal{S}_1, \mathcal{T}_1)$ and $(X_2, \mathcal{S}_2, \mathcal{T}_2)$ are both regular space.
Proof. $x_1 \in X_1$ and $\mathcal{T}_1$ closed set F_1 with $x_1 \notin F_1$. Again consider F_2 be any $\mathcal{T}_2$ closed subset of X_2 then $(x_1, x_2) \notin F_1 \times F_2 \Longrightarrow x_1 \notin F_1$. Since $(X_1 \times X_2, \mathcal{S}_1 \times \mathcal{S}_2, \mathcal{T}_1 \times \mathcal{T}_2)$ is regular then $\exists\, U \in \mathcal{T}_1 \times \mathcal{T}_2$ and $V \in \mathcal{S}_1 \times \mathcal{S}_2$ such that $(x_1, x_2) \in U, F_1 \times F_2 \subseteq V$. Again since $U \in \mathcal{T}_1 \times \mathcal{T}_2$ then $\exists\, U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2$ with $U = U_1 \times U_2$ and $V \in \mathcal{S}_1 \times \mathcal{S}_2$ then $\exists\, V_1 \in \mathcal{S}_1, V_2 \in \mathcal{S}_2$ with $V = V_1 \times V_2$. Now $(x_1, x_2) \in U \Longrightarrow U_1 \times U_2 \Longrightarrow x_1 \in U_1,, F_1 \times F_2 \subseteq V_1 \times V_2 \Longrightarrow F_1 \subseteq V_1$ with $U \cap V = \phi \Longrightarrow (U_1 \times U_2) \times (V_1 \times V_2) = \phi \Longrightarrow U_1 \cap V_1 = \phi$. For any $x_1 \in X_1$ and $\mathcal{T}_1$ closed set F_1 with $x_1 \notin F_1$ then $\exists\, \mathcal{T}_1$ open set U_1 and $\mathcal{S}_1$ open set V_1 such that $x_1 \in U_1, F_1 \subseteq V_1$ with $U_1 \cap V_1 = \phi$.
$\therefore$ $(X_1, \mathcal{S}_1, \mathcal{T}_1)$ is regular. Similarly $(X_2, \mathcal{S}_2, \mathcal{T}_2)$ is also regular. Hence, regular property is projective.

Theorem 4.13. If $(X_1, \mathcal{S}_1, \mathcal{T}_1)$ and $(X_2, \mathcal{S}_2, \mathcal{T}_2)$ are both normal space then $(X_1 \times X_2, \mathcal{S}_1 \times \mathcal{S}_2, \mathcal{T}_1 \times \mathcal{T}_2)$ is normal.
Proof. Let $(x_1, x_2) \in X_1 \times X_2$ and F be a $\mathcal{T}_1 \times \mathcal{T}_2$ closed subset of $X_1 \times X_2$ and G be a $\mathcal{S}_1 \times \mathcal{S}_2$ closed subset of $X_1 \times X_2$. Since F is closed subset of $\mathcal{T}_1 \times \mathcal{T}_2$ then $\exists\, \mathcal{T}_1$ closed set F_1 and $\mathcal{T}_2$ closed set F_2 such that $F_1 \times F_2 = F$. Similarly, Since G is closed subset of $\mathcal{S}_1 \times \mathcal{S}_2$ then $\exists\, \mathcal{S}_1$ closed set G_1 and $\mathcal{S}_2$ closed set G_2 such that $G_1 \times G_2 = G$. Since, $F \cap G = \phi \Longrightarrow (F_1 \times F_2) \times (G_1 \times G_2) = \phi$. Since $(X_1, \mathcal{S}_1, \mathcal{T}_1)$ is normal space then $\exists\, \mathcal{T}_1$ open set U_1 and $\mathcal{S}_1$ open set V_1 such that $F_1 \subseteq U_1$, $G_1 \subseteq V_1$ with $U_1 \cap V_1 = \phi$. Again Since $(X_2, \mathcal{S}_2, \mathcal{T}_2)$ is regular space then $\exists\, \mathcal{T}_2$ open set U_2 and $\mathcal{S}_2$ open set V_2 such that $F_2 \subseteq U_2, G_2 \subseteq V_2$ with $U_2 \cap V_2 = \phi$. From above we have, $F_1 \subseteq U_1, F_2 \subseteq U_2 \Longrightarrow F_1 \times F_2 \in U_1 \times U_2 \in \mathcal{T}_1 \times \mathcal{T}_2$ and $G_1 \subseteq V_1, G_2 \subseteq V_2 \Longrightarrow G_1 \times G_2 \in V_1 \times V_2 \in \mathcal{S}_1 \times \mathcal{S}_2$ with $U_1 \cap V_1 = \phi, U_2 \cap V_2 = \phi \Longrightarrow (U_1 \times U_2) \times (V_1 \times V_2) = \phi$.
Hence, normal space is productive.
Theorem 4.14. If $(X_1 \times X_2, \mathcal{S}_1 \times \mathcal{S}_2, \mathcal{T}_1 \times \mathcal{T}_2)$ is normal then $(X_1, \mathcal{S}_1, \mathcal{T}_1)$ and $(X_2, \mathcal{S}_2, \mathcal{T}_2)$ are both normal space.
Proof. Let, $\mathcal{T}_1$ closed set F_1 of X_1 and $\mathcal{T}_2$ closed set F_2 of X_2 with $F_1 \cap F_2 = \phi$. Again let, $\mathcal{S}_1$ closed set G_1 of X_1 and $\mathcal{S}_2$ closed set G_2 of X_2 with $G_1 \cap G_2 = \phi$. We have, $F_1 \cap F_2 = \phi$ and $G_1 \cap G_2 = \phi$ which implies that $F_1 \times F_2$ is $\mathcal{T}_1 \times \mathcal{T}_2$ closed subset of $X_1 \times X_2$ and $G_1 \times G_2$ is $\mathcal{S}_1 \times \mathcal{S}_2$ closed subset of $X_1 \times X_2$ with $(F_1 \times F_2) \cap (G_1 \times G_2) = \phi$ such that $, F_1 \times F_2 \subseteq U, G_1 \times G_2 \subseteq V$ with $U \cap V = \phi \Longrightarrow (U_1 \times U_2) \times (V_1 \times V_2) = \phi$. Since $U \in \mathcal{T}_1 \times \mathcal{T}_2$ then $\exists\, U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2$ with $U = U_1 \times U_2$ and $V \in \mathcal{S}_1 \times \mathcal{S}_2$ then $\exists\, V_1 \in \mathcal{S}_1, V_2 \in \mathcal{S}_2$ with $V = V_1 \times V_2$.Now $F_1 \times F_2 \subseteq U_1 \times U_2$ and $G_1 \times G_2 \subseteq V_1 \times V_2 \Longrightarrow F_1 \subseteq U_1, F_2 \subseteq U_2$ and $G_1 \subseteq V_1, G_2 \subseteq V_2$ with $U \cap V = \phi \Longrightarrow (U_1 \times U_2) \times (V_1 \times V_2) = \phi \Longrightarrow U_1 \cap V_1 = \phi$ and $U_2 \cap V_2 = \phi$. For any $\mathcal{T}_1$ closed set F_1 , $\mathcal{T}_2$ closed set F_2 then $\exists\, \mathcal{T}_1$ open set U_1, $\mathcal{T}_2$ open set U_2 such that $F_1 \subseteq U_1 F_2 \subseteq U_2$ and $\mathcal{S}_1$ closed set G_1, $\mathcal{S}_2$ closed set G_2 then $\exists \mathcal{S}_1$ open set V_1, $\mathcal{S}_2$ open set V_2 such that $G_1 \subseteq V_1, G_2 \subseteq V_2$ with $U_1 \cap V_1 = \phi$ and $U_2 \cap V_2 = \phi$.
$\therefore$ $(X_1, \mathcal{S}_1, \mathcal{T}_1)$ is normal. Similarly $(X_2, \mathcal{S}_2, \mathcal{T}_2)$ is also normal. Hence, normal property is projective.

V. CONCLUSIONS

In this paper, we have studied some notations of regular, normal, complete normality in bitopological space. We have seen that our concept is different from classical topological concept but these satisfied some properties of topology.

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