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Article on the applications of the Jensen's Inequality in Alternative proofs and Problems

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Abstract — Since as it is mentioned in the title that through this article there, we will deal with various applications of the classic Jensen's Inequality more clearly Weighted Jensen's Inequality of both kinds of a convex function and concave function. Under the title applications, we will mainly discuss two aspects of it which are the application of Jensen inequality in proving well-known inequalities such as AM-GM inequality, GM-HM inequality, Cauchy-Schwarz Inequality, Nesbitt's inequality, weighted AM-GM inequality, Power Mean Inequality etc. and another aspect is the application of Jensen's Inequality in solving and simplifying problems proposals published in different journals on the basis of it there are example problems for readers.

Keywords — Jensen's Inequality, Concave function, convex function, Weighted Jensen's Inequality etc.

I. INTRODUCTION

From several days I'd noticing applications of Jensen's inequality in different journals, magazine's problem solution corner. Here in this article, I'm sharing my collection of problems from journals and magazines like SSMA, Fibonacci Quarterly, Mathematical Reflection etc. so, before considering problems, examples and their solutions we are looking Jensen's Inequality first.

Jensen's Inequality If the function is convex for all $x_1, x_2, \cdots, x_n$ then

$$\frac{1}{n}\sum_{j=1}^{n} f(x_j) \geq f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)$$

And this inequality reverses when function is concave.
As few applications we are looking for proofs of some well-known inequalities using Jensen's Inequality

Johan Jensen

II. Nesbitt's inequality

As

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{x}{(x+y+z)-x}+\frac{y}{(x+y+z)-y}+\frac{z}{(x+y+z)-z}$$

Let $(x+y+z)=s$ then,

$$\frac{x}{(x+y+z)-x}+\frac{y}{(x+y+z)-y}+\frac{z}{(x+y+z)-z}=\frac{x}{s-x}+\frac{y}{s-y}+\frac{z}{s-z}$$

Now, consider a function

$$f(t)=\frac{t}{s-t}\Rightarrow f'(t)=\frac{s}{(s-t)^2}\Rightarrow f''(t)=\frac{2s}{(s-t)^3}>0\forall t\in\mathbb{N}$$

Then we can say its convex in nature for all t belongs to natural number. So, here with using Jensen's Inequality for three variables we can say that

$$\frac{f(x)+f(y)+f(z)}{3}\geq f\left(\frac{x+y+z}{3}\right)$$

$$\Rightarrow\frac{x}{s-x}+\frac{y}{s-y}+\frac{z}{s-z}\geq 3\frac{\left(\frac{x+y+z}{3}\right)}{s-\left(\frac{x+y+z}{3}\right)}$$

$$\Rightarrow\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq 3\frac{\left(\frac{x+y+z}{3}\right)}{(x+y+z)-\left(\frac{x+y+z}{3}\right)}$$

$$\Rightarrow\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}$$

Here we obtain Nesbitt's inequality.

III. ALTERNATIVE PROOFS TO WELL KNOW INEQUALITIES

3.1. Weighted AM-GM inequality

<u>The Weighted AM-GM inequality</u>. If $b_1,b_2,\cdots,b_n\in\mathbb{N}$, $0\leq\lambda_1,\lambda_2,\cdots,\lambda_n\leq 1$ and $\sum_{k=1}^{n}\lambda_k=1$ then only

$$\lambda_1 b_1+\lambda_2 b_2+\cdots+\lambda_n b_n\geq {b_1}^{\lambda_1}{b_2}^{\lambda_2}\cdots{b_n}^{\lambda_n}$$

<u>Proof</u>: - As we know Jensen's inequality i.e.

<u>Jensen's Inequality</u> If the function f is concave for $b_1,b_2\cdots,b_n$. Then

$$f\left(\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{a_1+a_2+\cdots+a_n}\right)\geq\frac{a_1f(b_1)+a_2f(b_2)+\cdots+a_nf(b_n)}{a_1+a_2+\cdots+a_n}$$

Let $f(x)=\ln x\Rightarrow f'(x)=\frac{1}{x}\Rightarrow f''(x)=-\frac{1}{x^2}<0\forall x\in\mathbb{N}$. Then

$$\ln\left(\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{a_1+a_2+\cdots+a_n}\right)\geq\frac{a_1\ln(b_1)+a_2\ln(b_2)+\cdots+a_n\ln(b_n)}{a_1+a_2+\cdots+a_n}$$

$$\ln\left(\frac{a_1}{a_1+a_2+\cdots+a_n}b_1+\frac{a_2}{a_1+a_2+\cdots+a_n}b_2+\cdots+\frac{a_n}{a_1+a_2+\cdots+a_n}b_n\right)$$
$$\geq\frac{a_1}{a_1+a_2+\cdots+a_n}\ln(b_1)+\frac{a_2}{a_1+a_2+\cdots+a_n}\ln(b_2)+\cdots+\frac{a_n}{a_1+a_2+\cdots+a_n}\ln(b_n)$$

Let $\frac{a_k}{a_1+a_2+\cdots+a_n}=\lambda_k$ then, $\sum_{k=1}^{n}\lambda_k=1$

$$\ln(\lambda_1b_1+\lambda_2b_2+\cdots+\lambda_nb_n)\geq\lambda_1\ln(b_1)+\lambda_2\ln(b_2)+\cdots+\lambda_n\ln(b_n)$$
$$\ln(\lambda_1b_1+\lambda_2b_2+\cdots+\lambda_nb_n)\geq\ln\left({b_1}^{\lambda_1}\right)+\ln\left({b_2}^{\lambda_2}\right)+\cdots+\ln\left({b_n}^{\lambda_n}\right)$$
$$\ln(\lambda_1b_1+\lambda_2b_2+\cdots+\lambda_nb_n)\geq\ln\left({b_1}^{\lambda_1}{b_2}^{\lambda_2}\cdots{b_n}^{\lambda_n}\right)$$
$$\Rightarrow\lambda_1b_1+\lambda_2b_2+\cdots+\lambda_nb_n\geq{b_1}^{\lambda_1}{b_2}^{\lambda_2}\cdots{b_n}^{\lambda_n}$$

Where $0\leq\lambda_1,\lambda_2,\cdots,\lambda_n\leq 1$ must follow some of us are more familiar with the above inequality for two variables and that inequality is looking like as

$$a^{\lambda}b^{1-\lambda}\leq\lambda a+(1-\lambda)b$$

3.2. Young's Inequality

This inequality was first given by William Henry Young. Inequality is

$$ab\leq\frac{a^p}{p}+\frac{b^q}{q}$$

Where $a,b>0$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}$. It can be easily proven using weighted AM-GM inequality for two variables. So,

$$a^{\lambda}b^{1-\lambda}\leq\lambda a+(1-\lambda)b$$
$$(a^p)^{\lambda}(b^q)^{1-\lambda}\leq\lambda a^p+(1-\lambda)b^q$$

And suppose $\lambda=\frac{1}{p}$ and $1-\lambda=\frac{1}{q}$. Then

$$(a^p)^{\frac{1}{p}}(b^q)^{\frac{1}{q}}\leq\frac{a^p}{p}+\frac{b^q}{q}$$

$$\Rightarrow ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

Hence Proof.

3.3. Power Mean Inequality

Since $f(x) = x^p$ is convex if $p \ge 2$ because $f'(x) = px^{p-1} \Rightarrow f(x) = p(p-1)x^{p-2} > 0$. Now using Jensen's Inequality for $a_1, a_2 \cdots, a_n \ge 0$. then

$$f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \le \frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n}$$

$$\Rightarrow \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p \le \frac{{a_1}^p + {a_2}^p + \cdots + {a_n}^p}{n}$$

there we have another inequality for which we are considering $f(x) = x^{\frac{1}{p}} \Rightarrow f'(x) = \frac{1}{p} x^{\left(\frac{1}{p}-1\right)} \Rightarrow f''(x) = -\frac{1}{p}\left(1 - \frac{1}{p}\right) x^{\frac{1}{p}-2} < 0$ so, we can say the considered function is concave in nature as a result after using Jensen's inequality we can say that

$$f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \ge \frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n}$$

$$\Rightarrow \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^{\frac{1}{p}} \ge \frac{{a_1}^{\frac{1}{p}} + {a_2}^{\frac{1}{p}} + \cdots + {a_n}^{\frac{1}{p}}}{n}$$

In conclusion we can express power mean inequality as for all $p \ge 1$

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p \le \frac{{a_1}^p + {a_2}^p + \cdots + {a_n}^p}{n}$$

This inequality reverses for $p \le 1$.

3.4. AM-HM Inequality

Consider a function $f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3} > 0 \forall x \in \mathbb{N}$ so, $f(x)$ is convex then using Jensen's inequality

$$f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \le \frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n}$$

$$\frac{1}{\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)} \le \left(\frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n}\right)$$

$$\Longrightarrow \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \ge \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)}$$

Some of us are more familiar with

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge n^2$$

3.5. AM-GM Inequality

Consider a function $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f''(x) = -\frac{1}{x^2} < 0 \forall x \in \mathbb{N}$ so, $f(x)$ is concave then using Jensen's inequality

$$f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \ge \frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n}$$

$$\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \ge \frac{\ln a_1 + \ln a_2 + \cdots + \ln a_n}{n}$$

$$\ln\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \ge \ln \sqrt[n]{a_1 a_2 \cdots a_n}$$

$$\Rightarrow \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \ge \sqrt[n]{a_1 a_2 \cdots a_n}$$

3.6. Cauchy-Schwarz Inequality

This inequality is

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \le ({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2)$$

This inequality can also be written as

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \le \left(\sum_{j=1}^{n} {a_j}^2\right)\left(\sum_{j=1}^{n} {b_j}^2\right)$$

Proof

As we know Am-Gm inequality for two variables

$$\left(\frac{x_j + y_j}{2}\right) \geq \sqrt{x_j y_j}$$

$$\Rightarrow \sum_{j=1}^{n}\left(\frac{x_j + y_j}{2}\right) \geq \sum_{j=1}^{n}\sqrt{x_j y_j}$$

Let $A = \sqrt{\sum_{j=1}^{n} a_j^{\,2}}$ likewise $B = \sqrt{\sum_{j=1}^{n} b_j^{\,2}}$ then suppose $x_j = \frac{a_j^{\,2}}{A^2}$ similarly $y_j = \frac{b_j^{\,2}}{B^2}$

$$\frac{1}{2}\sum_{j=1}^{n}\left(\frac{a_j^{\,2}}{A^2} + \frac{b_j^{\,2}}{B^2}\right) \geq \sum_{j=1}^{n}\sqrt{\frac{a_j^{\,2}}{A^2}\frac{b_j^{\,2}}{B^2}}$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{A^2}\sum_{j=1}^{n} a_j^{\,2} + \frac{1}{B^2}\sum_{j=1}^{n} b_j^{\,2}\right) \geq \frac{1}{AB}\sum_{j=1}^{n} a_j b_j$$

Since $A = \sqrt{\sum_{j=1}^{n} a_j^{\,2}}$ and $B = \sqrt{\sum_{j=1}^{n} b_j^{\,2}}$ hence

$$\sqrt{\sum_{j=1}^{n} a_j^{\,2}}\sqrt{\sum_{j=1}^{n} b_j^{\,2}} \geq \sum_{j=1}^{n} a_j b_j$$

$$\Rightarrow \left(\sum_{j=1}^{n} a_j^{\,2}\right)\left(\sum_{j=1}^{n} b_j^{\,2}\right) \geq \left(\sum_{j=1}^{n} a_j b_j\right)^2$$

Proof 2

From weighted Jensen's inequality we have

$$f\left(\frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{a_1 + a_2 + \cdots + a_n}\right) \leq \frac{a_1 f(b_1) + a_2 f(b_2) + \cdots + a_n f(b_n)}{a_1 + a_2 + \cdots + a_n}$$

Suppose a function $f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 > 0$ hence its convex in nature so,

$$\left(\frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{a_1 + a_2 + \cdots + a_n}\right)^2 \leq \frac{a_1 b_1^{\,2} + a_2 b_2^{\,2} + \cdots + a_n b_n^{\,2}}{a_1 + a_2 + \cdots + a_n}$$

$$\Rightarrow (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1 + a_2 + \cdots + a_n)\left(a_1 b_1^{\,2} + a_2 b_2^{\,2} + \cdots + a_n b_n^{\,2}\right)$$

Now from chebyshev's sum inequality we can say that

$$\left(\frac{a_1 b_1^{\,2} + a_2 b_2^{\,2} + \cdots + a_n b_n^{\,2}}{n}\right) \leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)\left(\frac{b_1^{\,2} + b_2^{\,2} + \cdots + b_n^{\,2}}{n}\right)$$

Then

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1 + a_2 + \cdots + a_n)\left(a_1 b_1^{\,2} + a_2 b_2^{\,2} + \cdots + a_n b_n^{\,2}\right)$$

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq n(a_1 + a_2 + \cdots + a_n)\left(\frac{a_1 b_1^{\,2} + a_2 b_2^{\,2} + \cdots + a_n b_n^{\,2}}{n}\right)$$

$$\leq n\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2\left(b_1^{\,2} + b_2^{\,2} + \cdots + b_n^{\,2}\right)$$

Using the power mean inequality, we had already established

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2 \leq \frac{a_1^{\,2} + a_2^{\,2} + \cdots + a_n^{\,2}}{n}$$

Using it

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq n\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2\left(b_1^{\,2} + b_2^{\,2} + \cdots + b_n^{\,2}\right)$$

$$\Rightarrow (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq n\left(\frac{a_1^{\,2} + a_2^{\,2} + \cdots + a_n^{\,2}}{n}\right)\left(b_1^{\,2} + b_2^{\,2} + \cdots + b_n^{\,2}\right)$$

$$\Rightarrow (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq \left(a_1^{\,2} + a_2^{\,2} + \cdots + a_n^{\,2}\right)\left(b_1^{\,2} + b_2^{\,2} + \cdots + b_n^{\,2}\right)$$

Hence proof

IV. EXAMPLE PROBLEMS

1. Prove that

$$n^{m-1}\sum_{k=1}^{n} {F_k}^{2m} \geq {F_n}^m {F_{n+1}}^m$$

For any positive integers n and m.[1]

Solution Since $f(x) = x^m$ is convex in nature under $(0, \infty)$ and $\sum_{k=1}^{n} {F_k}^2 = F_n F_{n+1}$. Now, using Jensen's Inequality

$$\frac{1}{n}\sum_{k=1}^{n} f\left({F_k}^2\right) \geq f\left(\frac{1}{n}\sum_{k=1}^{n} {F_k}^2\right)$$

$$\Rightarrow n^{m-1}\sum_{k=1}^{n} {F_k}^{2m} \geq (F_n F_{n+1})^m = {F_n}^m {F_{n+1}}^m$$

2. Let α, β, γ be angles of an arbitrary triangle. Prove the inequality

$$\alpha \cot \alpha + \beta \cot \beta + \gamma \cot \gamma \leq \frac{\pi}{\sqrt{3}}$$

When does equality occur? [2]

Solution Consider a function $f(x) = x \cot x$ then for $0 \leq x \leq \pi$

$$f''(x) = 2(x \cot x - 1) \leq 0$$

So, we can say considered function is concave then using Jensen's Inequality

$$\frac{f(a) + f(b) + f(c)}{3} \leq f\left(\frac{a+b+c}{3}\right)$$

$$\Rightarrow \frac{\alpha \cot \alpha + \beta \cot \beta + \gamma \cot \gamma}{3} \leq \left(\frac{\alpha+\beta+\gamma}{3}\right) \cot\left(\frac{\alpha+\beta+\gamma}{3}\right)$$

As $\alpha + \beta + \gamma = \pi$

$$\alpha \cot \alpha + \beta \cot \beta + \gamma \cot \gamma \leq \pi \cot\left(\frac{\pi}{3}\right) = \frac{\pi}{\sqrt{3}}$$

3. Let x, y and z be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{1}{5+x^2} + \frac{1}{5+y^2} + \frac{1}{5+z^2} \leq \frac{1}{2}$$

See in [3]

Solution

Since $f(t) = \frac{1}{5+t^2}$ is concave under the range $t \in (0,1)$ because of

$$f''(t) = \frac{2(3t^2 - 5)}{(t^2+5)^3} < 0$$

Now, using Jensen's Inequality

$$\frac{1}{5+x^2} + \frac{1}{5+y^2} + \frac{1}{5+z^2} \leq \frac{3}{5 + \left(\frac{x+y+z}{3}\right)^2} \leq \frac{3}{5 + \frac{x^2+y^2+z^2}{3}} \leq \frac{3}{5 + \frac{xy+yz+zx}{3}}$$

$$\Rightarrow \frac{1}{5+x^2} + \frac{1}{5+y^2} + \frac{1}{5+z^2} \leq \frac{1}{2}$$

4. For $x, y \geq 5$ show that

$$\left(\frac{1}{x}\right)^{\frac{1}{x}} \left(\frac{1}{y}\right)^{\frac{1}{y}} \leq \left(\frac{4}{x^2+y^2}\right)^{\frac{2}{x+y}}$$

See in [4]

Solution

Let $f(x) = -\frac{\ln x}{x}$ then, $f'(x) = -\frac{1-\ln x}{x^2}$ and $f''(x) = -\frac{2\ln x - 3}{x^3} < 0$ for all $x \geq 5$ then we can say that $f(x)$ is concave function then using Jensen's inequality

$$\frac{f(x) + f(y)}{2} \leq f\left(\frac{x+y}{2}\right)$$

$$\frac{\frac{1}{x}\ln\left(\frac{1}{x}\right) + \frac{1}{y}\ln\left(\frac{1}{y}\right)}{2} \leq \frac{1}{\frac{x+y}{2}} \ln\left(\frac{1}{\frac{x+y}{2}}\right)$$

$$\ln\left(\frac{1}{x}\right)^{\frac{1}{x}} + \ln\left(\frac{1}{y}\right)^{\frac{1}{y}} \leq \ln\left(\frac{2}{x+y}\right)^{\frac{4}{x+y}}$$

$$\ln\left(\frac{1}{x}\right)^{\frac{1}{x}}\left(\frac{1}{y}\right)^{\frac{1}{y}} \leq \ln\left(\frac{4}{(x+y)^2}\right)^{\frac{2}{x+y}} \leq \ln\left(\frac{4}{x^2+y^2}\right)^{\frac{2}{x+y}}$$

$$\left(\frac{1}{x}\right)^{\frac{1}{x}}\left(\frac{1}{y}\right)^{\frac{1}{y}} \leq \left(\frac{4}{x^2+y^2}\right)^{\frac{2}{x+y}}$$

5. For any positive integer n. Prove that it holds

$$\sqrt{\left(\frac{F_n-1}{F_n}\right)^{F_n}\left(\frac{L_n-1}{L_n}\right)^{L_n}} \leq \left(\frac{F_{n+1}-1}{F_{n+1}}\right)^{F_{n+2}}$$

See in [5]

Solution the function $f(x) = x\ln\left(1-\frac{1}{x}\right)$ is concave in nature for $x > 1$. Since $f''(x) = -\frac{1}{x(x-1)^2} < 0$. Therefore, using Jensen's Inequality for concave function

$$\frac{f(a)+f(b)}{2} \leq f\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \frac{a}{2}\ln\left(1-\frac{1}{a}\right)+\frac{b}{2}\ln\left(1-\frac{1}{b}\right) \leq \frac{a+b}{2}\ln\left(1-\frac{2}{a+b}\right)$$

$$\sqrt{\left(1-\frac{1}{a}\right)^{a}\left(1-\frac{1}{b}\right)^{b}} \leq \left(1-\frac{2}{a+b}\right)^{\frac{a+b}{2}}$$

Let $a = F_n$ and $b = L_n$.

$$\sqrt{\left(\frac{F_n-1}{F_n}\right)^{F_n}\left(\frac{L_n-1}{L_n}\right)^{L_n}} \leq \left(\frac{F_n+L_n-2}{F_n+L_n}\right)^{\frac{F_n+L_n}{2}}$$

As we know $F_n + L_n = 2F_{n+1}$ then,

$$\sqrt{\left(\frac{F_n-1}{F_n}\right)^{F_n}\left(\frac{L_n-1}{L_n}\right)^{L_n}} \leq \left(\frac{F_{n+1}-1}{F_{n+1}}\right)^{F_{n+2}}$$

6. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}} \geq \frac{3}{2}$$

See in [6]

Solution Consider a function $f(x) = \frac{1}{e^t+e^{\frac{3}{4}t}}$ then $f''(x) = \frac{16e^{2t}+23e^{\frac{7}{4}t}+9e^{\frac{3}{2}t}}{16\left(e^t+e^{\frac{3}{4}t}\right)^3} > 0\forall t \in \mathcal{R}$ hence this function is strictly convex then using Jensen's Inequality

$$f(x)+f(y)+f(z) \geq 3f\left(\frac{x+y+z}{3}\right)$$

$$\frac{1}{e^x+e^{\frac{3}{4}x}}+\frac{1}{e^y+e^{\frac{3}{4}y}}+\frac{1}{e^z+e^{\frac{3}{4}z}} \geq 3\frac{1}{e^{\left(\frac{x+y+z}{3}\right)}+e^{\frac{3}{4}\left(\frac{x+y+z}{3}\right)}}$$

If $x = \ln\left(\frac{b}{a}\right), y = \ln\left(\frac{c}{b}\right)$ and $z = \ln\left(\frac{a}{c}\right)$ then,

$$\frac{1}{\left(\frac{b}{a}\right)+\left(\frac{b}{a}\right)^{\frac{3}{4}}}+\frac{1}{\left(\frac{c}{b}\right)+\left(\frac{c}{b}\right)^{\frac{3}{4}}}+\frac{1}{\left(\frac{a}{c}\right)+\left(\frac{a}{c}\right)^{\frac{3}{4}}} \geq 3\frac{1}{e^{\frac{1}{3}\left(\ln\left(\frac{b}{a}\frac{c}{b}\frac{a}{c}\right)\right)}+e^{\frac{1}{4}\left(\ln\left(\frac{b}{a}\frac{c}{b}\frac{a}{c}\right)\right)}}$$

$$\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}} \geq \frac{3}{2}$$

4. Problems

Below problems are given to readers for independent study

1. See in [7]. Let a, b and c be positive with $a+b+c = 3$. Prove that

$$\frac{1}{5+a^3}+\frac{1}{5+b^3}+\frac{1}{5+c^3} \leq \frac{1}{2}$$

2. See in [8]. Prove that for all positive real numbers x, y

$$x^x+y^y \geq 2\left(\frac{x+y}{2}\right)^{\frac{x+y}{2}}$$

3. See in [9]. Let x, y, z, t be positive real numbers such that $x+y+z+t = 2$. Show that

$$\left(\frac{4}{x^2}-1\right)\left(\frac{4}{y^2}-1\right)\left(\frac{4}{z^2}-1\right)\left(\frac{4}{t^2}-1\right) \geq 15^4$$

4. See in [10]. Let a, b, c be real numbers such that $abc \geq -1$, and $a+b+c = 3$. Then

$$\left(\frac{a+1}{a+3}\right)^2+\left(\frac{b+1}{b+3}\right)^2+\left(\frac{c+1}{c+3}\right)^2\leq\frac{3}{4}$$

5. See in [11]. Let $x_1, x_2, \cdots, x_n$ be positive real numbers with $x_i < 64$ such that $\sum_{i=1}^{n} x_i = 16n$. Prove that

$$\sum_{i=1}^{n}\frac{1}{8-\sqrt{x_i}}\geq\frac{n}{4}$$

6. See in [12]. Let a, b, c be positive real numbers with $a + b + c = 3$. Prove that

$$\sqrt{\frac{ab}{2a+b+c}}+\sqrt{\frac{bc}{2b+c+a}}+\sqrt{\frac{ca}{2c+a+b}}\leq\frac{3}{2}$$

7. See in [13]. Let F_n and $L_n$be Fibonacci and Lucas Numbers. Prove that

$$\frac{{F_n}^{F_n}+{F_{n+1}}^{F_{n+1}}+{L_n}^{L_n}+{L_{n+1}}^{L_{n+1}}}{4}\geq\left(\frac{F_{n+3}}{2}\right)^{\frac{F_{n+3}}{2}}$$

8. See in [14]. Let $a, b, c > 0$. Prove that

$$\left(\frac{a}{b+c}\right)^{\frac{a}{b+c}}+\left(\frac{b}{c+a}\right)^{\frac{b}{c+a}}+\left(\frac{c}{a+b}\right)^{\frac{c}{a+b}}\geq 3^{\frac{2}{3}}$$

V. CONCLUSION AND FUTURE SCOPE

As in this article we have deals with two aspects of applications of the Jensen's inequality one is related to the alternative proofs to the well-known inequalities and another is related to the problems proposals published in the journals and magazines but there would also be some space for application of the Jensen's inequality such as someone can find the applications in proving convergence and divergence of the infinite series or even in non-elementary integrals, there could be some applications in physics respectively.

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