International Journal of Scientific Research in Mathematical and Statistical Sciences
Vol.**11**, Issue.**6**, pp.**23-31**, December **2024**
E-ISSN: 2348-4519
Available online at: www.isroset.org

Research Article

An Efficient Numerical Scheme for the Solution of Integer-Order Delays Differential Equations

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Received: 12/Oct/2024; **Accepted**: 12/Nov/2024; **Published**: 31/Dec/2024

Abstract — This study presents the MNIM approach, which integrates the El-Kalla polynomial with a new iterative method. The linear components of delay differential equations (DDEs) are first addressed using the new iterative method, followed by a secondary iterative process to manage the complexities introduced by nonlinear terms, as detailed in Section 2.3. Essential definitions and concepts related to DDEs, the iterative methodology, and the El-Kalla polynomial are also discussed. The effectiveness and reliability of MNIM are demonstrated through three notable test cases, with absolute errors evaluated both graphically and numerically for different values of the time variable xxx. The findings emphasize MNIM's capability to produce highly accurate approximations that closely match exact solutions with minimal error. This highlights MNIM's potential as a powerful and efficient tool for solving nonlinear DDEs. Additionally, the study lays the groundwork for future research, showcasing the method's potential in addressing complex problems across various scientific and engineering fields where nonlinear DDEs play a key role in modeling.

Keywords — Delay differential equations; Fractional delay differential equations; El-kalla polynomials; Adomian polynomials.

1. Introduction

This section provides a concise overview of several well-established methods for solving fractional delay differential equations (FDDEs). It also outlines the problem statement, research motivation, identified research gap, and the study's aim, objectives, scope, and limitations.

FDDEs are a class of equations that combine fractional derivatives with time delays. Unlike ordinary derivatives, fractional derivatives are non-local, enabling them to account for memory effects, while time delays incorporate historical information about prior system states. The integration of these features improves the accuracy of models for real-world phenomena. FDDEs are applied across various fields, such as physics, chemistry, control systems, electrochemistry, bioengineering, and population dynamics [1–5]. In bioengineering, fractional derivatives contribute to a better understanding of biological tissue dynamics, which is essential for exploring nuclear magnetic resonance and magnetic resonance imaging of complex, porous, and heterogeneous materials in both living and non-living systems.

To address the computational challenges posed by the non-local nature of fractional derivatives, researchers have developed accurate, efficient, and cost-effective numerical techniques for solving nonlinear FDDEs. Diethelm et al. [6,7] extended the Adams-Bashforth method to create the Fractional Adams Method (FAM) for Fractional Differential Equations (FDEs). Building on this, Bhalekar and Daftardar-Gejji [8] introduced an efficient algorithm using FAM to address FDEs with delay terms. Another notable approach, the Numerical Predictor-Corrector Method (NPCM), was developed by Daftardar-Gejji et al. [9] based on the Daftardar-Gejji and Jafari method (DGJ method) [10–12]. This method was later extended to effectively solve FDDEs, demonstrating superior time efficiency compared to alternative techniques [13]. Kumar and Methi [14] introduced a novel approach, the Banach Contraction Method (BCM), which integrates the BCM algorithm developed by Daftardar-Gejji and Bhalekar [15] with the New Iterative Method (NIM) proposed by Daftardar-Gejji and Jafari [10]. This hybrid approach demonstrated greater accuracy and time efficiency than the Fractional Adams Method (FAM) [8] and the Numerical Predictor-Corrector Method (NPCM) [13].

Building on this foundation, the current study modifies and extends NIM to effectively solve FDDEs. While the non-local nature of fractional derivatives poses challenges for time efficiency in numerical simulations, their ability to capture memory effects in natural phenomena is invaluable. The primary objective of this research is to develop a methodology that surpasses existing techniques in precision and computational efficiency. The proposed method will exhibit rapid convergence, addressing the limitations of approaches such as NIM, VIM, FAM, and NPCM, while significantly reducing simulation time. Additionally, it aims to offer heightened accuracy, establishing itself as a robust and efficient solution for solving nonlinear FDDEs.

2. Related Work

The study of solution methodologies for Delay Differential Equations (DDEs) and their more complex variants, Fractional Delay Differential Equations (FDDEs), has seen significant advancements.

Srivastava [16] introduced the New Variational Iteration Method (NVIM), a promising technique for deriving approximate analytical solutions to FDDEs. This method has been successfully applied to both linear and nonlinear initial value problems, demonstrating its ability to produce accurate approximate solutions with relatively few iterations when compared against exact solutions.

Jhinga and Daftardar-Gejji [17] proposed an innovative predictor-corrector technique specifically designed for nonlinear FDDEs. Their comprehensive error analysis and illustrative examples highlighted its superior accuracy and time efficiency compared to established numerical approaches such as the Fractional Adams-Moulton (FAM) method and the Three-Term Numerical Predictor-Corrector Method (NPCM). A key finding was the L1-PCM method's convergence for very small values of the parameter α\alphaα, where FAM and NPCM methods diverged.

Nemah [18] combined the Mahgoub transform with the Variational Iteration Method (VIM) to solve nonlinear FDDEs, addressing unnecessary assumptions in other algorithms. Their results, summarized in Tables 2 to 4, demonstrated that this approach closely matched exact solutions and outperformed methods like the Modified Adomian Decomposition Method (MADM) and the Homotopy Analysis Method (HAM) in several scenarios. These findings underscored the Mahgoub-Variational Iteration Method's (MVIM) superior efficacy.

El-Kalla et al. [19] explored the application of the Adomian Decomposition Method (ADM) to nonlinear delay differential equations (NDDEs) using an enhanced Adomian polynomial known as the El-Kalla polynomial. The El-Kalla polynomial offers several advantages:

1. It is recursive and free of derivative terms, simplifying programming and saving processing time.
2. Solutions derived using it exhibit faster convergence compared to traditional Adomian polynomials.
3. It facilitates the estimation of the maximum absolute truncated error of the series solution.

The study analyzed convergence aspects and solved a range of numerical examples using both standard Adomian and El-Kalla polynomials, demonstrating significant promise for this approach in various applications.

Avcı [20] introduced a numerical solution method for FDDEs with Caputo fractional derivatives using a fractional integration operational matrix derived from a fractional Taylor basis. This method transforms the original equation into a system of algebraic equations, which can be efficiently solved using computational algorithms. The study provided an error bound for the approximation and validated the method through examples, showcasing its accuracy and practicality.

Anil Kumar [21] employed the Banach Contraction Method (BCM) to solve FDDEs with proportional delays. The study included convergence and error analyses, revealing that errors decrease significantly with additional iterations. BCM avoids discretization, linearization, and perturbation techniques, simplifying complex calculations and making implementation straightforward.

Al-Sawalha et al. [22] addressed pantograph delay differential equations using the Chebyshev pseudospectral method with Caputo fractional derivatives. By converting the equations into algebraic systems, this approach streamlined the solution process. Convergence was thoroughly analyzed, and accuracy was validated through examples, showing superior performance compared to other methods. This technique's simplicity, efficiency, and broader applicability to linear and nonlinear FDDEs make it a valuable contribution to the field.

3. Theory/Calculation

In this section should extend, not repeat the information To understand the core principles of the New Iterative Method (NIM), it is useful to examine a well-established functional equation, as explored in the works of Daftardar-Gejji & Bhalekar (2010), Ramadan & Al-Luhaibi (2015), Moltot & Deresse (2022), and Ashitha & Ranjini (2020). This approach starts with analyzing the nonlinear functional equation introduced by Daftardar-Gejji & Jafari (2006).

$$y(x) = g(x) + N[y(x)] \qquad \dots(1)$$

In this context, N represents the nonlinear operator, and f is a known function. The goal is to determine a solution, denoted as $y(x)$, which possesses a series representation in the following format:

$$y = \sum_{i=0}^{\infty} y_i \, . \qquad \dots(2)$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \qquad (3)$$

From Eqns. (2) and (3), Eqn. (1) is equivalent to

$$\sum_{i=0}^{\infty} y_i = g + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \qquad (4)$$

We define the recurrence relation:

$$\begin{cases} y_0 = g, \\ y_1 = N(y_0) \\ y_{m+1} = N(y_0 + ... + y_m) - N(y_0 + ... + y_{m-1}), m = 1,2,... \end{cases} \quad (5)$$

Then

$$(y_1 + ... + y_{m+1}) = N(y_0 + ... + y_m), m = 1,2,... \quad ...(6)$$

and

$$y = g + \sum_{i=0}^{\infty} y_i \, . \quad ...(7)$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to a solution of Eqn. (1).

3.2 The El-Kalla polynomial Formula

$$\overline{A}_n = f(S_n) - \sum_{i=0}^{n-1} A_i \quad (8)$$

where $\overline{A}_n$, are the El-kalla polynomials, $\overline{A}_0$, $\overline{A}_1$, $\overline{A}_2$,…, $f(S_n)$ is the substitution of the summation of dependent variable in the nonlinear term. For example the El-Kalla polynomials of the nonlinear term $y^2(x)$ and the nonlinear term $y^3(x)$ are shown above

El-Kalla polynomials of $y^2(x)$

$$\begin{cases} \overline{A}_0 = y_0^2(x) \\ \overline{A}_1 = 2y_0(x)y_1(x) + y_1^2(x) \\ \overline{A}_2 = 2y_0(x)y_2(x) + 2y_1(x)y_2(x) + y_2^2(x) \\ \overline{A}_3 = 2y_0(x)y_3(x) + 2y_1(x)y_3(x) + 2y_2(x)y_3(x) + y_3^2(x) \\ \overline{A}_4 = 2y_0(x)y_4(x) + 2y_1(x)y_4(x) + 2y_2(x)y_4(x) + 2y_3(x)y_4(x) + y_4^2(x) \end{cases} \quad ...(9)$$

El-Kalla polynomials of $y^3(x)$

$$\begin{cases} \overline{A}_0 = y_0^3(x) \\ \overline{A}_1 = 3y_1(x)y_0^2(x) + 3y_0(x)y_1^2(x) + y_1^3(x) \\ \overline{A}_2 = 3y_2y_0^2 + 6y_0y_1y_2 + 3y_2y_1^2 + 3y_1y_2^2 + y_2^3 \\ \overline{A}_3 = 3y_3y_0^2 + 6y_0y_1y_3 + 3y_0y_3^2 + 3y_3y_1^2 + 6y_1y_2y_3 + 3y_1y_3^2 + 3y_3y_2^2 + y_3^3 \end{cases} \quad ...(10)$$

3.3 The Proposed New Iterative Method (NIM)

In a prior study, the New Iterative Method (NIM) was used to approximate solutions for ordinary differential equations. In this section, we present new algorithms aimed at simplifying the resolution of Delay Differential Equations (DDEs). To ensure a clear understanding of these newly developed, generalized NIM algorithms, we will first explore the fundamental structure of Delay Differential Equations (DDEs).

$$y^{(n)}(x) + P[y(x)] + N[y(x-t)] = f(x) \,, \quad n = 1,2,3,... \quad ...(11)$$

$$y^{(k)} = \delta_i \,, \qquad i = 0,1,2,... \quad ...(12)$$

where $y^{(n)}(x)$ is the derivative of y of order n, P is the linear bounded operator, N is a nonlinear bounded operator, $f(x)$ is a given continuous function, and $y = y(x)$.

In this section, the general form of the n^{th}-order DDE Eqn. (11) with the initial value Eqn. (12) is treated using the suggested method MNIM.

Next, by isolating the term associated with the derivative, we get

$$y^{(n)}(x) = f(x) - P[y(x)] - N[y(x-t)] \quad (13)$$

Applying the J^n on both sides of Eqn. (13), we get

$$y(x) = \left. \begin{matrix} J^n\left[f(x) - P(y(x)) - N(y(x-t))\right] \\ + \sum_{i=0}^{r} y^{(i)}(0) \dfrac{x^i}{i!} \end{matrix} \right\} \quad (14)$$

Let's consider dividing this equation into two separate parts as follows:

$$y(x) = N(y(x)) + g(x) \quad ...(15)$$

where

$$N(y(x)) = J^n\left[f(x) - P(y(x)) - N(y(x-t))\right] \quad (16)$$

In typical cases, N serves as the nonlinear operator; however, when dealing with the DDE, it is employed with linear functions. Additionally, "g" represents a known function, defined as:

$$g(x) = \sum_{i=0}^{r} y^{(i)}(0) \frac{x^i}{i!}, \quad (17)$$

In our quest for a solution to Eqn. (11), we seek a representation in the form of a series:

$$y(x) = \sum_{i=0}^{\infty} y_i(x). \quad (18)$$

The operator N can be decomposed into the following

$$N\left(\sum_{i=0}^{\infty} y_i(x-t)\right) = N(y_0) + \sum_{i=1}^{\infty} \left\{ \begin{matrix} N\left(\sum_{j=0}^{i} y_j\right) \\ - N\left(\sum_{j=0}^{i-1} y_j\right) \end{matrix} \right\} (19) \,.$$

From Eqns. (11), (17) and Eqn. (18)

$$\sum_{i=0}^{\infty} y_i = g(x) + N(y_0) + \sum_{i=1}^{\infty} \left\{ \begin{matrix} N\left(\sum_{j=0}^{i} y_j\right) \\ - N\left(\sum_{j=0}^{i-1} y_j\right) \end{matrix} \right\} \quad (20) \,.$$

The El-Kalla polynomials and the classical NIM, in conjunction with the characteristics of fractional integral and fractional derivative, are integrated to derive the recurrence relation presented above:

$$y_0 = g(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!}, \qquad \text{...(21)}$$

$$y_1 = J^n\left[f(x) - P\left(y_0(x)\right) - N\left(y_0(x-t)\right)\right] \qquad (22)$$

$$\begin{cases} y_2 = N(y_0 + .y_1) - N(y_0) = \\ J^n[-P(y_0(x)) - N(y_0(x-t)) - P(y_1(x)) - N(y_1(x-t))] - J^n[y_0(x-t)] \end{cases} \qquad \text{...(23)}$$

$$\begin{cases} y_3 = N(y_0 + .y_1 + .y_2) - (y_0 + .y_1) = \\ J^n[-P(y_0(x)) - N(y_0(x-t)) - P(y_1(x)) - N(y_1(x-t)) \\ -P(y_2(x)) - N(y_2(x-t))] - J^n[y_0(x-t) + y_1(x-t)] \end{cases} \qquad \text{...(24)}$$

$$\vdots$$

We define the recurrence relation from the systems of Eqn. (20) as follows:

$$\begin{cases} y_0 = g(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!}, \\ y_1 = J^n\left[f(x) - P\left(y_0(x)\right) - N\left(y_0(x-t)\right)\right] \\ y_2 = J^n[-P(y_1(x)) - N(y_1(x-t))] \\ y_3 = J^n[-P(y_2(x)) - N(y_2(x-t))] \\ . \\ . \\ . \\ y_{n+1} = J^n \sum_{i=3}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}, \qquad i \geq 3. \end{cases} \qquad \text{...(25)}$$

Therefore, in truncated series form, the approximate analytical solution of the DDE Eqn. (11) is given by

$$y(x) = \lim_{k\to\infty}\sum_{n=0}^{k} y_n = y_0 + y_1 + y_2 + y_3 + \ldots \qquad (23)$$

3.4 Suitable Algorithm for Fractional Delay Differential Equation

In this section, we introduce a suitable algorithm for solving Fractional Delay Differential Equations using the proposed New Iterative Method (NIM). Consider the following Fractional Delay Differential Equations:

$$\begin{cases} D^\alpha y(x) + Ly(x) + N\left[y(x-t)\right] = g(x),\ x > 0, \\ y^{(i)} = \delta_i, \qquad i = 0,1,2,\ldots \end{cases} \qquad (27)$$

where L is a linear operator, N , represent a nonlinear operator, $g(x)$ is the source term, and D^α is the Caputo fractional derivative of order with $m-1 < \alpha - m$. To solve Eqn. (27) by means of the proposed modification of the NIM, we apply the operator J^α , the inverse of the operator D^α , to both sides of Eqn. (27) as follows:

$$y(x) = \sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!} + J^\alpha\begin{bmatrix} -Ly(x) - Ny(x-t) \\ +g(x) \end{bmatrix}.(28)$$

Let's consider dividing this equation into two separate parts as follows:

$$y(x) = N(y(x)) + \sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!} \qquad (29)$$

where

$$N(y(x)) = J^\alpha[g(x) - Ly(x) - Ny(x-t)] \qquad (30)$$

In our quest for a solution to Eqn. (29), we seek a representation in the form of a series:

$$y(x) = \sum_{i=0}^{\infty} y_i(x). \qquad (31)$$

The operator N can be decomposed into the following

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N\left(y_0\right) + \sum_{i=1}^{\infty}\left\{\begin{matrix} N\left(\sum_{j=0}^{i} y_j\right) \\ -N\left(\sum_{j=0}^{i-1} y_j\right) \end{matrix}\right\} \qquad (31).$$

From Eqns. (29), (31) and Eqn. (32)

$$\sum_{i=0}^{\infty} y_i = \sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!} + N\left(y_0\right) + \sum_{i=1}^{\infty}\left\{\begin{matrix} N\left(\sum_{j=0}^{i} y_j\right) - \\ N\left(\sum_{j=0}^{i-1} y_j\right) \end{matrix}\right\} \qquad (32)$$

.

We define the recurrence relation:

$$y_0 = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!}, \qquad (34)$$

$$y_1 = J^\alpha\left[g(x) - Ly_0(x) - Ny_0(x-t)\right] \qquad (35)$$

$$\begin{cases} y_2 = N(y_0 + y_1) - N(y_0) \\ = J^\alpha[g(x) - Ly_0(x) - Ny_0(x-t) - Ly_1(x) \\ - Ny_1(x-t)] \\ - J^\alpha[g(x) - Ly_0(x) - Ny_0(x-t)] = \\ J^\alpha[-Ly_1(x) - Ny_1(x-t)] \end{cases} \qquad (36)$$

$$\begin{cases} y_3 = N(y_0+y_1+y_2) - N(y_0+y_1) = \\ J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x-t) - Ly_1(x) - \\ Ny_1(x-t) - Ly_2(x) - Ny_2(x-t)] \\ -J^{\alpha}[g(x) - Ly_0(x) - Ny_0(x) - \\ Ly_1(x) - Ny_1(x-t)] = \\ J^{\alpha}[-Ly_2(x) - Ny_2(x-t)] \end{cases} \quad (37)$$

$$\begin{cases} \mathrm{y}_4 = \mathrm{N}(\mathrm{y}_0+\mathrm{y}_1+\mathrm{y}_2+\mathrm{y}_3) - \mathrm{N}(\mathrm{y}_0+\mathrm{y}_1 \\ +\mathrm{y}_2) = \\ \mathrm{J}^{\alpha}[\mathrm{g(x)} - \mathrm{Ly}_0(\mathrm{x}) - \mathrm{Ny}_0(\mathrm{x-t}) - \\ \mathrm{Ly}_1(\mathrm{x}) - \mathrm{Ny}_1(\mathrm{x}) - \mathrm{Ly}_2(\mathrm{x}) - \\ \mathrm{Ny}_2(\mathrm{x-t}) - \mathrm{Ly}_3(\mathrm{x}) - \mathrm{Ny}_3(\mathrm{x-t})] \\ -\mathrm{J}^{\alpha}[\mathrm{g(x)} - \mathrm{Ly}_0(\mathrm{x}) - \mathrm{Ny}_0(\mathrm{x-t}) - \mathrm{Ly}_1 \\ (\mathrm{x}) - \mathrm{Ny}_1(\mathrm{x-t}) - \mathrm{Ly}_2(\mathrm{x}) - \mathrm{Ny}_2(\mathrm{x-t})] = \\ \mathrm{J}^{\alpha}[-\mathrm{Ly}_3(\mathrm{x}) - \mathrm{Ny}_3(\mathrm{x-t})] \end{cases} \quad (38).$$

.

$\vdots$

Then k-term series solution will be in the form

$$y(x) = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + \ldots \quad (39)$$

From above Eqns., we can deduce the following:

$$y_0 = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!},$$

$$y_1 = J^{\alpha}\left[g(x) - Ly_0(x) - Ny_0(x-t)\right]$$

$$y_2 = J^{\alpha}[-Ly_1(x) - Ny_1(x-t)] \quad \ldots(40)$$

$$y_3 = J^{\alpha}[-Ly_2(x) - Ny_2(x-t)]$$

$$\vdots$$

$$y_{m+1} = J^{\alpha}\left[-Ly_m(x) - Ny_m(x-t)\right], \quad m \geq 3$$

4. Experimental Method/Procedure/Design

This section focuses on evaluating the performance and accuracy of the proposed approach presented in Section 2.3, particularly for solving differential-delay equations (DDEs) of integer order. Two nonlinear DDEs and one linear DDE are used in this segment to demonstrate the effectiveness and validity of the Modified New Iterative Method (MNIM).

4.1 Nonlinear Delay Differential Equations (NDDEs)

Example 1 [refer to Srivastava (2020)]: Consider the following first-order nonlinear DDE:

$$y'(x) = 1 - 2y^2\left(\frac{x}{2}\right), 0 \leq x \leq 1, \quad y(0) = 0. \quad (41)$$

The analytical solution is given by $y(x) = \sin(x)$

Based on Eqn. (14), Eqn. (41) can be approximately written as follows:

$$y(x) = J\left[1 - 2y^2\left(\frac{x}{2}\right)\right] + \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!} \quad (42)$$

The following recurrence relation is derived from Section 2.3:

$$y_0(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!} = x$$

$$y_0\left(\frac{x}{2}\right) = \frac{x}{2}$$

$$y_1(x) = J\left[-2y_0{}^2\left(\frac{x}{2}\right)\right] = -2J\left[\frac{x^2}{4}\right] = -J\left[\frac{x^2}{2}\right] = -\frac{x^3}{6}$$

$$y_1\left(\frac{x}{2}\right) = \frac{\left(-\frac{x}{2}\right)^3}{6} = -\frac{x^3}{48}$$

$$y_2(x) = -2J\sum_{n=0}^{1} y_n^2\left(\frac{x}{2}\right) - J\left[y_0^2\left(\frac{x}{2}\right)\right]$$

$$= -2J\left[\left(\frac{x}{2} - \frac{x^3}{48}\right)^2 - \left(\frac{x}{2}\right)^2\right]$$

$$= -2J\left[\frac{x^2}{4} - \frac{x^4}{48} + \frac{x^6}{2304} - \frac{x^2}{4}\right] = \frac{x^5}{120} - \frac{x^7}{8064}$$

$$y_2\left(\frac{x}{2}\right) = \frac{\left(\frac{x}{2}\right)^5}{120} - \frac{\left(\frac{x}{2}\right)^7}{8064} = \frac{x^5}{3840} - \frac{x^7}{1032192}$$

$$y_3 = -2J\sum_{n=0}^{2} y_n^2\left(\frac{x}{2}\right) - J\sum_{n=0}^{1}\left[y_n^2\left(\frac{x}{2}\right)\right]$$

$$= -2J\left[\left(\frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} - \frac{x^7}{1032192}\right)^2 - \left(\frac{x}{2} - \frac{x^3}{48}\right)^2\right] =$$

$$y(x) = x - \frac{1}{6}x^3 - \frac{1}{5040}x^7 + \frac{1}{120}x^5 - \frac{1861217}{27423919161999360000}x^{15} + \frac{64117}{408094035148800}x^{13} - \frac{8177}{326998425600}x^{11} + \frac{1}{362880}x^9 - \frac{1}{1062664199886151693758358595882188800}x^{31} + \frac{1}{2003250224799783128854752657408000}x^{29} - \frac{14029}{208366086151723132018001077862400 0}x^{27} + \frac{46639}{13913333744105444178552422400000}x^{25} - \frac{45289}{477440516121239878631424000 0}x^{23} + \frac{35411}{113226999870649773588480000}x^{21} - \frac{338999}{1010955355987944407040000}x^{19} + \frac{45990949}{2386977923860424294400 0}x^{17}$$

$$-\frac{1}{7990652436480}x^{15} + \frac{1}{12881756160}x^{13} - \frac{67}{3406233600}x^{11} + \frac{61}{23224320}x^9 - \frac{1}{13440}x^7$$

$$y_4 = -2J\sum_{n=0}^{3} y_n^2\left(\frac{x}{2}\right) - J\sum_{n=0}^{2}\left[y_n^2\left(\frac{x}{2}\right)\right] =$$

$$-\frac{1}{1062664199886151693758358595882188800}x^{31} + \frac{1}{2003250224799783128854752657408000}x^{29} - \frac{14029}{2083660861517231320180010778624000}x^{27} + \frac{46639}{13913333744105444178552422400000}x^{25} - \frac{45289}{4774405161212398786314240000}x^{23} + \frac{35411}{113226999870649773588480000}x^{21} - \frac{338999}{1010955355987944407040000}x^{19} + \frac{45990949}{23869779238604242944000}x^{17} - \frac{1518017}{27423919161999360000}x^{15} + \frac{32437}{408094035148800}x^{13} - \frac{349}{65399685120}x^{11} + \frac{1}{7741440}x^9$$

Now, in vision of Eqn. (39), the solution of Example 1 is

$$y(x) = y_0 + y_1 + y_2 + y_3 + y_4 + \ldots = \qquad \ldots(43)$$

Table 1: Four-term approximate solution using MNIM and its comparison with the exact solution of Example 1 for different time variable values, along with the corresponding absolute error $E = |y_4 - exact|$.

x	EXACT	MNIM	Error of MNIM
0.01	0.01	0.00999983	0
0.02	0.019999	0.01999867	0
0.03	0.029996	0.0299955	3.46945E-18
0.04	0.039989	0.03998933	0
0.05	0.049979	0.04997917	6.93889E-18
0.06	0.059964	0.05996401	0
0.07	0.069943	0.06994285	0
0.08	0.079915	0.07991469	1.38778E-17
0.09	0.089879	0.08987855	1.38778E-17
0.1	0.099833	0.09983342	1.38778E-17

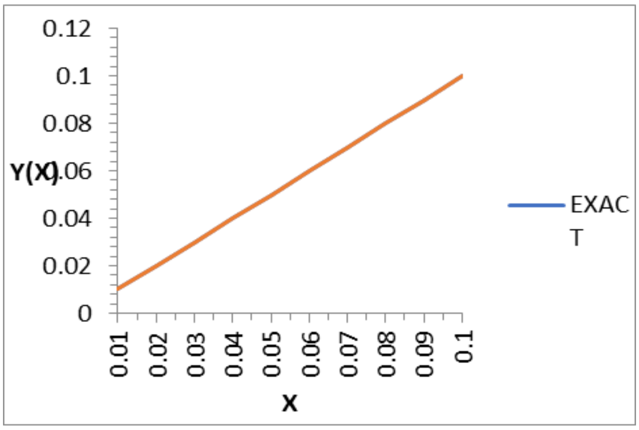

Figure 1: Solution plots for Example 1 obtained using MNIM, compared with the exact solutions

Table 2: Four-term approximate solution using MNIM compared with the eight-term approximate solution from SIM and the exact solution of Example 1 for various time variable values.

X	EXACT	MNIM	SIM	Error of MNIM	Error SIM
0.01	0.01	0.009999833	0.00999975	0.000000000	8.33333E-08
0.02	0.019999	0.019998667	0.019998	0.000000000	6.66667E-07
0.03	0.029996	0.0299955	0.02999325	3.46945E-18	2.25E-06
0.04	0.039989	0.039989334	0.039984001	0.000000000	5.33333E-06
0.05	0.049979	0.049979169	0.049968753	6.93889E-18	1.04167E-05
0.06	0.059964	0.059964006	0.059946006	0.000000000	1.8E-05
0.07	0.069943	0.069942847	0.069914264	0.000000000	2.85833E-05
0.08	0.079915	0.079914694	0.079872027	1.38778E-17	4.26667E-05
0.09	0.089879	0.089878549	0.089817799	1.38778E-17	6.075E-05
0.1	0.099833	0.099833417	0.099750083	1.38778E-17	8.33333E-05

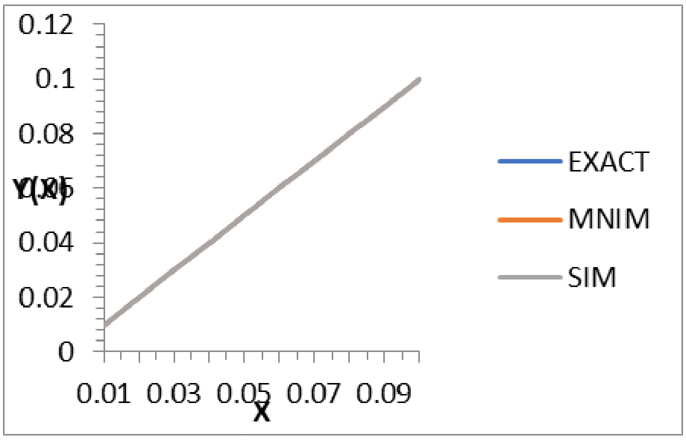

Figure 2: Solution plots for Example 1 obtained using MNIM, compared with SIM and the exact solutions.

Example 2 [see Moltot & Deresse (2022)]: Consider the following nonlinear second-order DDE:

$$y''(x) = y^2\left(\frac{x}{2}\right), x \geq 0, \qquad (44)$$

$$y(0) = 1, y'(0) = 1.$$

The analytical solution is given by

$$y(x) = e^x$$

By applying the fractional integral to both sides of Eqn. (44), and considering Eqn. (14), Eqn. (44) can be approximately expressed as follows:

$$y(x) = 1 + x + J^2\left[y^2\left(\frac{x}{2}\right)\right] \qquad (45)$$

The following recurrence relation is derived from Section 2.3:

$$y_0(x) = 1 + x$$

$$y_0\left(\frac{x}{2}\right) = 1 + \frac{x}{2}$$

$$y_1(x) = J\left[y_0^2\left(\frac{x}{2}\right)\right] = J^2\left[\left(1+\frac{x}{2}\right)^2\right] = J^2\left[1 + x + \frac{x^2}{4}\right]$$

$$= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{48}$$

$$y_1\left(\frac{x}{2}\right) = \frac{\left(\frac{x}{2}\right)^2}{2} + \frac{\left(\frac{x}{2}\right)^3}{6} + \frac{\left(\frac{x}{2}\right)^4}{48}$$

$$= \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{768}$$

$$y_2(x) = J^2\sum_{n=0}^{1} y_n^2\left(\frac{x}{2}\right) - J^2\left[y_0^2\left(\frac{x}{2}\right)\right]$$

$$= J^2\left[\left(1 + x + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{768}\right)^2 - \left(1+\frac{x}{2}\right)^2\right]$$

$$= \frac{x^4}{48} + \frac{x^5}{120} + \frac{x^6}{768} + \frac{5x^7}{32256} + \frac{x^8}{73728} + \frac{x^9}{1327104} + \frac{x^{10}}{53084160}$$

$$y_2\left(\frac{x}{2}\right) = \frac{\left(\frac{x}{2}\right)^4}{48} + \frac{\left(\frac{x}{2}\right)^5}{120} + \frac{\left(\frac{x}{2}\right)^6}{768} + \frac{5\left(\frac{x}{2}\right)^7}{32256} + \frac{\left(\frac{x}{2}\right)^8}{73728} + \frac{\left(\frac{x}{2}\right)^9}{1327104} + \frac{\left(\frac{x}{2}\right)^{10}}{53084160} =$$

$$\frac{x^4}{768} + \frac{x^5}{3840} + \frac{x^6}{49152} + \frac{10x^7}{8257536} + \frac{x^8}{18874368} + \frac{x^9}{679477248} + \frac{x^{10}}{5435818000\ 0}$$

Now, in vision of Eqn. (39), the solution of Example 2 is

$$y(x) = y_0 + y_1 + y_2 + \ldots =$$

$$y(x) = 1 + x + \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{53084160}x^{10} + \frac{1}{1327104}x^9 + \frac{1}{73728}x^8 + \frac{5}{32256}x^7 + \frac{1}{768}x^6 + \frac{1}{120}x^5$$

Table 3: Two-term approximate solution using MNIM, compared with the exact solution of Example 2 for various time variable values, along with the absolute error.

$$E = |y_2 - exact|.$$

x	EXACT	MNIM	ERROR
0.01	1.01005	1.01005	2.22E-16
0.02	1.020201	1.020201	5.55E-15
0.03	1.030455	1.030455	6.44E-14
0.04	1.040811	1.040811	3.63E-13
0.05	1.051271	1.051271	1.39E-12
0.06	1.061837	1.061837	4.17E-12
0.07	1.072508	1.072508	1.06E-11
0.08	1.083287	1.083287	2.37E-11
0.09	1.094174	1.094174	4.83E-11
0.1	1.105171	1.105171	9.13E-11

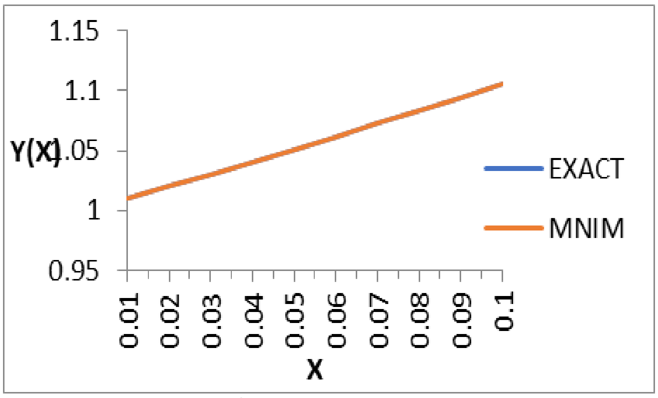

Figure 3: Solution plots for Example 2 obtained using MNIM, compared with the exact solutions.

4.2 Linear Delay Differential Equations (LDDEs)

Example 3 [see Mohyud-din & Yildirim (2010)]: Consider the second-order LDDE:

$$y''(x) = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) - x^2 + 2, 0 \leq x \leq 1, \qquad (47)$$

$$y(0) = 0, y'(0) = 0.$$

The analytical solution is given by $y(x) = x^2$

In view of Eqn. (14), the Eqn. (47) is approximately expressed as follows:

$$y(x) = J^2\left[\frac{3}{4}y(x) + y\left(\frac{x}{2}\right)\right] + x^2 - \frac{x^4}{12} + \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!} \quad (48)$$

We deduce the following recurrence relation from section 2.3

$$y_0(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!} + x^2 - \frac{x^4}{12} = x^2 - \frac{x^4}{12}$$

$$y_0\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2 - \frac{\left(\frac{x}{2}\right)^4}{12} = \frac{x^2}{4} - \frac{x^4}{192}$$

$$y_1(x) = J^2\left[\frac{3}{4}y_0(x) + y_0\left(\frac{x}{2}\right)\right] =$$

$$-\frac{13}{5760}x^6 + \frac{1}{12}x^4$$

$$y_1\left(\frac{x}{2}\right) = -\frac{13}{368640}x^6 + \frac{1}{192}x^4$$

$$y_2(x) = -\frac{91}{2949120}x^8 + \frac{13}{5760}x^6$$

$$y_2\left(\frac{x}{2}\right) = -\frac{91}{754974720}x^8 + \frac{13}{368640}x^6$$

$$y_3 = -\frac{17563}{67947724800}x^{10} + \frac{91}{2949120}x^8$$

Now, in vision of Eqn. (39), the solution of Example 1 is

$$y(x) = y_0 + y_1 + y_2 + y_3 + \ldots =$$

$$x^2 - \frac{1}{12}x^4 - \frac{13}{5760}x^6 + \frac{1}{12}x^4 - \frac{91}{2949120}x^8 + \frac{13}{5760}x^6 - \frac{17563}{67947724800}x^{10} + \frac{91}{2949120}x^8$$

$$y(x) = x^2 - \frac{17563}{67947724800}x^{10}$$

...(49)

Table 4: Three-term approximate solution using MNIM, compared with VIM, ADM, and the exact solution of Example 3 for various time variable values.

x	EXACT	MNIM (n=3)	VIM (n=8)	ADM (n=7)
0.01	0.0001	0.0001	9.9999E-05	1E-04
0.02	0.0004	0.0004	0.00039999	0.0004
0.03	0.0009	0.0009	0.00089993	0.0009
0.04	0.0016	0.0016	0.00159979	0.0016
0.05	0.0025	0.0025	0.00249948	0.0025
0.06	0.0036	0.0036	0.00359892	0.0036
0.07	0.0049	0.0049	0.004898	0.0049
0.08	0.0064	0.0064	0.00639659	0.0064
0.09	0.0081	0.0081	0.00809453	0.0081
0.1	0.01	0.01	0.00999167	0.01

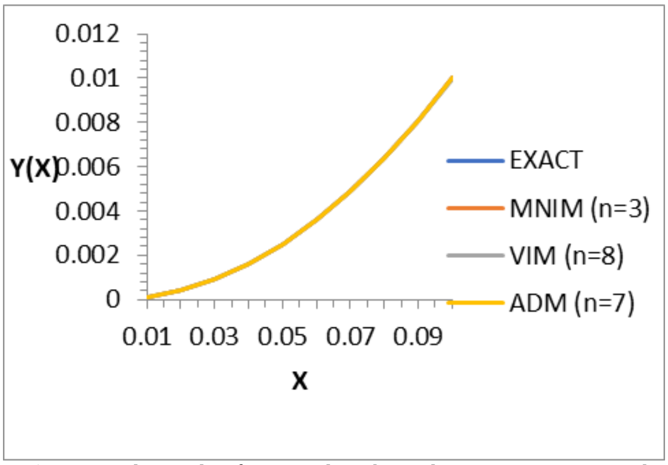

Figure 4: Solution plots for Example 3 obtained using MNIM, compared with ADM, VIM, and the exact solutions.

5. Results and Discussion

Graphs are essential for visualizing physical structures and practical applications. In this section, the obtained solutions are analyzed and presented using figures and tables. Figures 1 to 4 illustrate comparisons between the approximate solutions generated by the proposed method and the exact solutions for Examples 1 to 3 across various values of the time variable xxx. These figures highlight the effectiveness of the MNIM approach as a mathematical tool, with the solutions produced by this method closely matching the exact solutions.

Figures 1 through 4 feature 2D graphs depicting the approximate solutions for Examples 1, 2, and 3, alongside those obtained using ADM and VIM, providing a comparative analysis of the precision and efficiency of MNIM. The corresponding absolute error values are also presented to validate its accuracy. Furthermore, Tables 1 to 4 provide a detailed comparison of the approximate and exact solutions for Examples 1 to 3, including their absolute errors at different values of xxx. The results confirm that the solutions produced by the proposed method are highly accurate, with minimal errors compared to ADM, VIM, and the exact solutions.

6. Conclusion and Future Scope

This study presents MNIM, a method that combines the El-Kalla polynomial with the New Iterative Method (NIM). Initially, NIM addresses the linear component of DDEs, and to manage the complexity introduced by nonlinear terms, a post-treatment NIM is applied, as described in Section 2.3. We introduce the key definitions and terminology related to DDEs, the new iterative approach, and the El-Kalla polynomial. The validity and consistency of MNIM have been demonstrated through the analysis of three significant problems. Absolute errors for all examples are both graphically and numerically presented across different time variable values. The results clearly show that MNIM provides highly accurate approximations that closely match the exact solutions with minimal error. As a result, MNIM proves to be an effective tool, improving accuracy and efficiency in solving nonlinear DDEs. This research represents a step forward in exploring the potential of this method to address complex problems across various scientific and engineering disciplines, especially as nonlinear DDEs continue to gain prominence in modeling real-world systems.

Data Availability
Not applicable.

Conflict of Interest
All authors declare that they do not have any conflict of interest.

Funding Source
No funding source exists,

Authors' Contributions
All authors reviewed and edited the manuscript and approved the final version of the manuscript

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