International Journal of Scientific Research in Mathematical and Statistical Sciences **Research Paper**
Vol.6, Issue.1, pp.276-281, February (2019) **E-ISSN:** 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i1.276281

About Secured Isolate Inclusive Sets in Graphs

D.K. Thakkar[1*] , N.J. Savaliya[2*]

[1,2]Department of Mathematics, Saurashtra University, Rajkot-360005, Gujarat, India

[*]*Corresponding Author: nsavaliya24@gmail.com*

Available online at: www.isroset.org

Received: 20/Sep/2018, Accepted: 10/Oct/2018, Online: 28/Feb/2019

Abstract—In this paper we introduce in new concept call secured isolate inclusive sets in graphs. We also define maximum secured isolate inclusive set and we define the cardinality of a maximum secured isolate inclusive set to be the secured isolate inclusive number of the graph G. We observe that a 1-maximal isolate inclusive set cannot be a secured isolate inclusive set. We also observe that if a secured isolate inclusive set contains a pendent vertex then it is an isolate in this set. It $\boldsymbol{v}$ & u are pendent vertices which are adjacent and M is a maximum secured isolate inclusive set of G then $v \in M$ & $u \notin M$ or $u \in M$ & $v \notin M$.

Keywords—*isolate inclusive set, maximum isolate inclusive set, 1-maximal isolate inclusive set, secured isolate inclusive set, maximum secured isolate inclusive set, maximal secured isolate inclusive set, external private neighbourhood, maximum independent set, independent number.*

AMS Subject Classification (2010): 05C69

I. INTRODUCTION

The concept of isolate inclusive set was studied in [1]. We defined the concept of isolate inclusive set in [1].
In this paper, we further study isolate inclusive sets in graphs. It is prove that a 1-maximal isolate inclusive set cannot ba a secured inclusive set in any graphs. We observe that if a secured isolate inclusive set contains a pendent vertex then it is an isolate in this set. We further see that if u & v are pendent vertices which are adjacent then

(1) There is a maximum secured isolate inclusive set M such that $v \in M$ & $u \notin M$.
(2) There is a maximum secured isolate inclusive set M_1 such that $u \in M_1$ & $v \notin M_1$.

The paper contains five sections in which Section I contains the introduction of Secured Isolate Inclusive Sets in graphs. Section II contains preliminaries and notations. In Sections III basic definitions and examples have been given. In Section IV the operation of vertex removal from the graph is considered and its effect on the Secured Isolate Inclusive Number have been proved. In Section V concluding remarks indicates future directions and further scope of results.

II. PRELIMINARIES AND NOTATIONS

If G is a graph then $V(G)$ denotes the vertex set of the graph G and $E(G)$ denotes the edge set of the graph G. If v is vertex of the graph G then $G - v$ is the subgraph of G induced by all the vertices different from v. If x is a vertex of G then will denote the degree $d(x)$ of the vertex x in the graph G.
We will consider only simple undirected graphs with finite vertex set.

III. DEFINITIONS AND EXAMPLES

Definition 3.1: (isolate inclusive set) [2]

Let G be a graph and S be a nonempty subset of V(G) then the S is said to be an isolate inclusive set if the $< S >$ has an isolated vertex. An isolate inclusive set will be also called isoinc set.
An isoinc with maximum cardinality is called a maximum isoinc set and its cardinality is denoted as $\beta_{is}(G)$.
Let G be a graph and S be a isoinc set of G. Then S is said to be 1-maximal isoinc set if $S \cup \{v\}$ is not an isoinc set.

Definition 3.2: (secured isolate inclusive set)

Let G be a graph and $S \subset V(G)$ be an isoinc set. Then S is said to be a secured isolate inclusive set if for every $v \in S$ there is a vertex $u \in V(G) - S \ni u$ is adjacent to v and $T = S - \{v\} \cup \{u\}$ is an isoinc set of G. An secure isolate inclusive set will be also called secured isoinc set.

Definition 3.3: (maximum secured isoinc set)

A secured isoinc set with maximum cardinality is called a maximum secured isoinc set of G. The cardinality of

maximum secured isoinc set is called the secured isoinc number of the graph and it is denoted as $\beta_{sis}(G)$.
It is obvious that $\beta_{sis}(G) \leq \beta_{is}(G)$, for any graph .

Example 1: Consider the path graph P_4 with 4 vertices $\{1, 2, 3, 4\}$

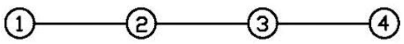

Figure 1. Path graph P_4

Let $S = \{1, 4\}$ then S is a secured isoinc set of G.
Consider $S_1 = \{1, 2, 4\}$ then S_1 is an isoinc set but it is notsecuredisoincset.

Remark:
Let G be a graph and $S \subset V(G)$ be a 1-maximal isoinc set of G. Let $u \in V(G) - S$ then u is adjacent to every isolate of S. Let $v \in S$ be an isolate of S. If u is any neighbour of v in $V(G) - S$ then $S - \{v\} \cup \{u\}$ cannot have an isolate by the above statement.
Thus S is not a secured isoinc set of G.
Thus every 1-maximal isoinc set is not a secured isoinc set.

Definition 3.4: (External private neighborhood) [1]

Let v be a vertex of the graph G and $S \subset V(G)$ containing v then external private neighbourhood of v with respect to S, i.e. $E_{pex}[v, S] = \{w \in V(G) - S : N(w) \cap S = \{v\}\}$.

Definition 3.5: (independent set) [7]

A set of vertices in a graph G is said to be an independent set or an internally stable set if no two vertices in the set are adjacent.

Definition 3.6: (maximum independent set)[7]

A independent set with maximum cardinality is called a maximum independent set of G.
The cardinality of maximum independent set is called the independent number of the graph and it is denoted as $\beta_0(G)$.

IV. MAIN RESULTS

Proposition 4.1: Let S be an isoinc set of G which contains atleast 2 isolates. Then S is a secured isoinc set if and only if for every $v \in S$, there is a neighbour u of v in $V(G) - S \ni u$ is not adjacent to some isolate of S different from v.
Proof: Suppose S is a secured isoinc set of G.
Let $v \in S$.
There is a neighbour u of v in $V(G) - S \ni T = S - \{v\} \cup \{u\}$ is an isoinc set of G. Now T has an isolate. If u is not adjacent with any vertex of $S - \{v\}$. In particular u is not adjacent with any isolate of S (except possibly v). This means that u is not adjacent with any isolate S different from v.
Suppose u is not isolate in T. Since T is an isoinc set of G, it contains an isolate say v'. Then v' is not adjacent to u and v' is different from v.
Thus the condition is satisfied.
Conversely, the condition is satisfy .
Let $\in S$. By the given condition there is a neighbour u of v in $V(G) - S$ such that u is not adjacent to some isolate x of S.
Obviously $x \neq v$.
Therefore x is an isolate in $S - \{v\} \cup \{u\}$.
Thus $S - \{v\} \cup \{u\}$ is an isoinc set of G.

Theorem 4.2: Let G be a graph and S be an isoinc set of G which has exactly one isolate. Then S is a secured isoinc set if and only if for each $\in S$, the following condition is satisfied.
(1) $E_{pex}[v, S] \neq \emptyset$ if v is an isolate of the S.
(2) v is not isolate of S and there is a neighbour u of v in $V(G) - S$ such that u is not adjacent to the isolate of S or there is a neighbour of v which is a pendent vertex of $< S \cup \{u\} >$.
Proof: Suppose S is a secured isoinc set of G.
Let $\in S$.
(1) suppose v is the isolate of S. Since S is a secured isoinc set of G, there is a neighbour u of v in $V(G) - S$ such that $S - \{v\} \cup \{u\}$ is an isoinc set. Now $S - \{v\}$ does not contain any isolate of S but $S - \{v\} \cup \{u\}$ contains an isolate. This isolate must be u. Thus $u \in V(G) - S$, u is adjacent v, u is not adjacent to any vertex of $S - \{v\}$. Therefore $u \in P_{extn}[v, S]$.
Thus $P_{extn}[v, S] \neq \emptyset$.
(2) suppose v is not an isolate of S. Let x be the isolate of S . Science S is a secured isoinc set of G, there is a neighbour u of v in $V(G) - S$ such that $T = S - \{v\} \cup \{u\}$ is an isoinc set of G.
(i) suppose x is an isolate of T. Then u is not adjacent to x.
(ii) Suppose u is an isolate of T. Then u is not adjacent to x.
Suppose neighbour u or x is an isolate of T. Let z be an isolate of T. Then $z \in S, z \neq v$ and z is not an isolate of S but z is an isolate of T. Therefore z must be adjacent to v in S and z is not adjacent to any other vertex of S that is z is a neighbour of v and z is a pendent vertex of $< S \cup \{u\} >$.
Conversely, suppose the condition is satisfy.
Let $v \in S$.
(1) Suppose v is isolate of S. Then $P_{extn}[v, S] \neq \emptyset$.
Let $u \in P_{extn}[v, S]$. Then u is an isolate in $S - \{v\} \cup \{u\}$.
(2) Suppose v is not isolate of S.
Let x be an isolate of S.
Suppose there is a neighbour u of v in $V(G) - S$ such that u is not adjacent to x. Then $S - \{v\} \cup \{u\}$ contains an isolate namely x. Suppose there is a neighbour z of v in S such that v is a pendent vertex in the $< S \cup \{u\} >$. Then z is an adjacent to only one vertex of $S \cup \{u\}$ namely v.
Therefore z is an isolate in $S - \{v\} \cup \{u\}$.

Thus in both the cases there is a neighbour u of v in $V(G) - S$ such that $S - \{v\} \cup \{u\}$ is has an isolate.
Therefore S is a secured isoinc set of G.

Proposition 4.3: Let G be a graph with atleast 1 non isolated vertex, $\beta_o(G) \geq 2$ and $\Delta(G) < \beta_o(G)$ then every maximum independent subset of $V(G)$ is a secured isoinc set of G.
Proof: Suppose S be a maximum independent subset of $V(G)$.
Obviously S is an isoinc set of G.
Let $v \in S$.
Let u be any neighbour of v in $V(G) - S$. Since $d(u) \leq \Delta(G) < \beta_o(G)$, u is adjacent to atmost $\beta_o(G) - 2$ vertices of S which are different from v.
There u is adjacent to atmost $\beta_o(G) - 2$ vertices of $T = S - \{v\} \cup \{u\}$.
Thus T is an isoinc set of G.
Thus S is a secured isoinc set G.

Corollary 4.4: Let G be a graph with $\beta_o(G) \geq 2$ and having atleast 1 non isolated vertex and suppose $\Delta(G) < \beta_o(G)$ then $\beta_o(G) \leq \beta_{sis}(G)$.
Proof: From the above proposition every maximum independent set is a secured isoinc set of G.
Therefore $\beta_o(G) \leq \beta_{sis}(G)$.
Converse of the above theorem did not be true.

Example 2: Consider the cycle graph C_5 with 5 vertices {1 , 2 , 3 , 4, 5}

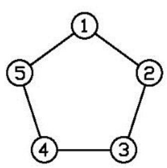

Figure 2. Cycle graph C_5
Now $\beta_o(G) = 2$ and $\Delta(G) = 2$.
Thus $\Delta(G) \nless \beta_o(G)$.
However every maximum independent subset of G is a secured isoinc set of G.

Example 3: Consider the cycle graph C_5 with 5 vertices {1 , 2 , 3 , 4, 5}

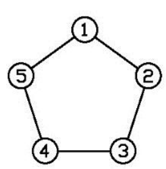

Figure 3. Cycle graph C_5
Here $\beta_{is}(G) = 3$ and $\beta_{sis}(G) = 2$.
Therefore $\beta_{sis}(G) < \beta_{is}(G)$.

Example 4: Consider the complete graph K_5 with 5 vertices {1 , 2 , 3 , 4, 5}

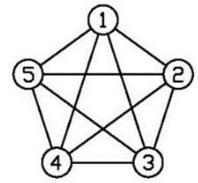

Figure 4. Complete graph K_5
Here $\beta_{is}(G) = 1$ and $\beta_{sis}(G) = 1$.
Therefore $\beta_{sis}(G) = \beta_{is}(G)$.

Theorem 4.5:
(1) Let G be a graph with $\beta_{is}(G) = 1$ then $\beta_{sis}(G) = \beta_{is}(G) = 1$.
(2) If G is a graph with at least 1 non isolated vertex and $\beta_{is}(G) \geq 2$ then $\beta_{sis}(G) < \beta_{is}(G)$.

Proof: (1) Suppose $\beta_{is}(G) = 1$.
Now $\beta_{sis}(G) < \beta_{is}(G)$.
Therefore $\beta_{sis}(G) = 0$ or 1. It is impossible $\beta_{sis}(G) = 0$ and $\beta_{sis}(G) = 1$.

(2) Suppose G has at least 1 non isolated vertex and $\beta_{is}(G) \geq 2$.
Now every maximum isoinc set is not a secured isoinc set and therefore it cannot be a maximum secured isoinc set of G. Therefore $\beta_{sis}(G) < \beta_{is}(G)$.

Proposition 4.6: Let G be a graph and suppose $S_1, S_2, S_3, \dots \dots \dots, S_k$ are all the isoinc sets of G, all having the same cardinality. Also suppose that atleast one of them is secured isoinc set then $S_1 \cap S_2 \cap S_3 \cap \dots \dots \dots \cap S_k = \emptyset$.
Proof: Suppose $T = S_1 \cap S_2 \cap S_3 \cap \dots \dots \dots \cap S_k$ and $T \neq \emptyset$.
Let S_j be a secured isoinc set among above sets.
Let $v \in T$ then $v \in S_j$.Since S_j be a secured isoinc set of G.
There is a vertex u is adjacent to v and $V(G) - S_j$ such that u is adjacent to v and $S_j' = S_j - \{v\} \cup \{u\}$ is an isoinc set of G.
Note that $S_j' \neq S_j$. Therefore $S_j' \in \{S_1, S_2, S_3, \dots \dots \dots, S_k\}$.
However $v \notin S_j'$.
This is a contradiction.
Therefore $T = \emptyset$.
Thus $S_1 \cap S_2 \cap S_3 \cap \dots \dots \dots \cap S_k = \emptyset$.

Proposition 4.7: Let G be a graph and $v \in V(G)$ then every secured isoinc set of $G - v$ is a secured isoinc set of G.
Proof: Let S be a secured isoinc set of $G - v$. Obvious S is isoinc set of G.
Let $x \in S$ then there is a vertex y of $G - v$ such that $y \notin S$ and $S - \{x\} \cup \{y\}$ is an isoinc set of $G - v$.
Now $S - \{x\} \cup \{y\}$ is also an isoinc set of G.
Thus S is a secured isoinc set of G.

Theorem 4.8: Let G be a graph and $v \in V(G)$ then $\beta_{sis}(G - v) \leq \beta_{sis}(G)$.
Proof: Let M be a maximum secured isoinc set of $G - v$ then M is an secured isoinc set of G.
Therefore $\beta_{sis}(G) \geq |M| = \beta_{sis}(G - v)$.
Thus $\beta_{sis}(G - v) \leq \beta_{sis}(G)$.

Example 5: Consider the path graph P_5 with 5 vertices $\{1, 2, 3, 4, 5\}$

Figure 5. Path graph P_5

Now $\beta_{sis}(G) = 3$.
Let $v = 5$ then $G - v$ is the path graph P_4 with 4 vertices $\{1, 2, 3, 4\}$.

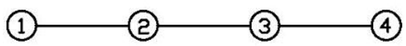

Figure 6. Path graph P_4

Here $\beta_{sis}(G - v) = 2$.
Thus $\beta_{sis}(G - v) < \beta_{sis}(G)$ in this graph.

Example 6: Consider the cycle graph C_5 with 5 vertices $\{1, 2, 3, 4, 5\}$

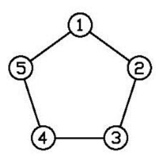

Figure 7. Cycle graph C_5

Here $\beta_{sis}(G) = 2$.
Let $v = 5$ then $G - v$ is the path graph P_4 with 4 vertices $\{1, 2, 3, 4\}$.

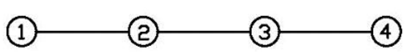

Figure 8. Path graph P_4

Here $\beta_{sis}(G - v) = 2$.
Thus $\beta_{sis}(G - v) = \beta_{sis}(G)$ in this example.

Theorem 4.9: Let G be a graph and v be an isolated vertex of G then $\beta_{sis}(G - v) = \beta_{sis}(G)$.
Proof: Let S be a maximum secured isoinc set of G then $v \notin S$.
Then S is a set of vertices of $G - v$ and obiously S is a secured isoinc set of $G - v$.
Therefore $\beta_{sis}(G - v) = \beta_{sis}(G)$.

Proposition 4.10: Let G be a graph and v be a pendent vertex of G. Let S be a secured isoinc set of G such that $v \in S$ then v must be an isolate of S.
Proof: Suppose v is not an isolate of S then v is adjacent to some vertex v' of S. Also since S is a secured isoinc set of G there is a vertex u in $V(G) - S$ which is adjacent to v.
Therefore $d(v) \geq 2$.
Which contradict the fact that v is a pendent vertex of G.

Theorem 4.11: Let G be a graph and v be a pendent vertex of G. Let S be a maximum secured isoinc set of $G - v$. If there is a neighbour u of v in $V(G) - S$ such that u is not adjacent to some isolate of S. Then $S \cup \{v\}$ is a secured isoinc set of G and $\beta_{sis}(G - v) < \beta_{sis}(G)$.
Proof: Let $T = S \cup \{v\}$ then v is isolate of T.
Thus T is a isoinc set of G.
Let $x \in T$.
Case(i): $x \neq v$
Then $x \in S$.
Since S is a secured isoinc set of $G - v$. There is a vertex y of $G - v$ such that $y \notin S$, y is adjacent to x and $S - \{x\} \cup \{y\}$ is an isoinc set of $G - v$.
Therefore T$-\{x\} \cup \{y\}$ is also isoinc set of G.
Case(ii): suppose $x = v$.
By the assumption there is a vertex u of v in $V(G) - T$ such that u is not adjacent to some isolate of S. Then T$-\{x\} \cup \{y\}$ is an isoinc set of G.
From case (i) and case (ii), it follow that T is a secured isoinc set of G.
Futher, $\beta_{sis}(G) \geq |T| > |S| = \beta_{sis}(G - v)$.
Thus $\beta_{sis}(G) > \beta_{sis}(G - v)$.

Theorem 4.12: Let G be a graph and v be a pendent vertex of G, u be its pendent neighbour. Then for every maximum secured isoinc set M of G either $u \in M \,\&\, v \notin M$ or $v \in M \,\&\, u \notin M$.
Proof: Suppose there is a maximum isoinc set S of G such that $u \notin S \,\&\, v \notin S$.
Let $S_1 = S \cup \{v\}$. Then S_1 is a secured isoinc set of G and $|S_1| > |S|$.
Which is a contradiction.
Therefore $u \in S$ or $\in S$.
If $u \in S$ then obviously $v \notin S$ because S is a secured isoinc set of G.
Similarly if $v \in S$ then $u \notin S$.

Theorem 4.13: Let G be a graph and v be a pendent vertex of G and suppose u is neighbour is also pendent. Then
(1) There is a maximum secured isoinc set M of G such that $u \in M$ and $v \notin M$.
(2) There is a maximum secured isoinc set M_1 of G such that $v \in M_1$ and $u \notin M_1$.
Proof: By the above theorem,
There is a maximum secured isoinc set M of G such that that $u \in M$ and $v \notin M$ or $v \in M$ and $\notin M$.
(a) Suppose $u \in M$ and $v \notin M$.Then (1) is satisfy.
(b) Now let $M_1 = M - \{u\} \cup \{v\}$ then obviously M_1 is a maximum secured isoinc set of G.
Similarly, If there is a maximum secured isoinc set M_1 of G such that $v \in M_1$ and $u \notin M_1$.

Then $M = M_1 - \{v\} \cup \{u\}$ is a maximum secured isoinc set of G with $v \in M$ and $u \notin M$.

Corollary 4.14: Let G be a graph and v be a pendent vertex of $G$and u be its pendent neighbour. Then G contains at least two maximum secured isoinc set of G.
Proof: Is obvious.

Theorem 4.15: Let G be a graph and v be a pendent vertex of G and u be its pendent neighbour. Suppose $\beta_{sis}(G) \geq 2$. Then $\beta_{sis}(G - v) < \beta_{sis}(G)$.
Proof: Let S be a maximum secured isoinc set of $G - v$. If $u \in S$ then u does not have any neighbour outside S in $G - v$. This is not possible for a secured isoinc set. Therefore $u \notin S$.
Let $S_1 = S \cup \{u\}$.Then obviously S_1 is a secured isoinc set of G
Therefore $\beta_{sis}(G) \geq |S_1| > |S| = \beta_{sis}(G - v)$.
Thus $\beta_{sis}(G - v) < \beta_{sis}(G)$.

Theorem 4.16: Let G be a graph and v be a pendent vertex of G such that its neighbour u is not a pendent vertex of G. Suppose there is a maximum secured isoinc set S_1 of $G - v$ such that $u \notin S_1$ and u is not adjacent to some isolate of S_1 then
(1) $S_1 \cup \{v\}$ is a secured isoinc set of G.
(2) $\beta_{sis}(G) > \beta_{sis}(G - v)$.
Proof: Let $S = S_1 \cup \{v\}$.
We prove that S is a secured isoinc set of G for $\in S$.
Consider the set $T = S - \{v\} \cup \{u\}$.
Since u is not adjacent to some isolate of S_1, T has an isolate and therefore T is an isoinc set of G.
Let $x \in S$ such that $x \neq v$ then $x \in S_1$
Since S_1 is a secured isoinc set of $G - v$.
There is a vertex y of $G - v$ such that $y \notin S_1$, $S_1 - \{x\} \cup \{y\}$ is an isoinc set of $G - v$.
If $y = u$ then the S_1 has isolate which is not adjacent to u will remain isolate in $S - \{x\} \cup \{y\}$.
If $y \neq u$ then an isolate of $S_1 - \{x\} \cup \{y\}$ is also an isolate of $S - \{x\} \cup \{y\}$.
Thus S is a secured isoinc set of G containg v.
Further $\beta_{sis}(G) \geq |S_1| > |S| = \beta_{sis}(G - v)$.
Thus $\beta_{sis}(G) > \beta_{sis}(G - v)$.

Theorem 4.17: Let G be a graph and v be a pendent vertex of G. If $\beta_{sis}(G) > \beta_{sis}(G - v)$ then there is a maximum secured isoinc set M of G such that $v \in M$.
Proof: Let M_1 be a maximum secured isoinc set of G.
If $v \in M_1$ then let $M = M_1$ and therefore its proved.
Suppose $v \notin M_1$
Suppose $(v) \subset V(G) - M_1$.
Then M_1 is a secured isoinc set of $G - v$ also and this will implies that $\beta_{sis}(G - v) \geq |M_1| = \beta_{sis}(G)$.
Which is a contradiction.
Therefore there is a neighbour u of v such that $u \in M_1$.
Since M_1 is a secured isoinc set of G. There is a vertex $u'$of G such that $u' \notin M_1$, u' is adjacent to u and $M_1 - \{u\} \cup \{u'\}$ is an isoinc set of G.
If $u' \neq v$ then it implies that M_1 is a secured isoinc set of $G - v$.
This would implies that $\beta_{sis}(G - v) \geq |M_1| = \beta_{sis}(G)$.
Which is again a contradiction.
Therefore there is a neighbour u of v such that $u \in M_1$ and v is the only vertex outside M_1 such that $M_1 - \{u\} \cup \{v\}$ is an isoinc set of G.
Let $M = M_1 - \{u\} \cup \{v\}$ then M is a maximum secured isoinc set of G such that $v \in M$.

Theorem 4.18: Let G be a graph with $\beta_{sis}(G) \geq 3$ and v be a pendent vertex and u be its neighbour which is not a pendent vertex then $\beta_{sis}(G - v) = \beta_{sis}(G)$ if and only if for every maximum secured isoinc set M of G, $u \in M$ & $v \notin M$ or $v \in M$ & $u \notin M$.
Proof: Suppose $\beta_{sis}(G - v) \leq \beta_{sis}(G)$.
Suppose for some maximum secured isoinc set M of G, $u \notin M$ & $v \notin M$.(It cannot happen that both $u, v \in M$ because v is a pendent vertex). Then M is a secured isoinc set of $G - v$ also this implies that
$\beta_{sis}(G - v) \geq |M| = \beta_{sis}(G)$
Which is a contradiction.
Therefore $u \in M$ & $v \notin M$ or $\in M$ & $u \notin M$.
Conversely, suppose $\beta_{sis}(G - v) = \beta_{sis}(G)$.
Let S_1 be a maximum isoinc set of $G - v$. Then S_1 is a also maximum isoinc set of G.
Claim: $u \notin S_1$
Proof of the claim: suppose $u \in S_1$.
There is a neighbour w of u in $G - v$ such that $w \notin S_1$ and $S_1 - \{u\} - \{v, w\}$.
Now we prove that S is a secured isolate inclusive set of G.
Now $v \in S$.Consider the set $T = S - \{v\} \cup \{u\}$. Since u is not adjacent to some isolate of S_1, u is also not adjacent to some isolate of $S - \{v\} \cup \{u\}$.
Thus $S - \{v\} \cup \{u\}$ is an isoinc set of G.
Again $w \in S$. Now u is a neighbour of w in G and $u \notin S$.
Now consider the set $S - \{w\} \cup \{u\}$. Then since u is not adjacent to some isolate $S - \{w\} \cup \{u\}$.
Thus S is a secured isolate inclusive set of G such that $|S| > |S_1| = \beta_{sis}(G - v) = \beta_{sis}(G)$.
This is a contradiction.
Therefore $u \notin S_1$.
Let $T = S_1 \cup \{u\}$ then T is a secured isolate inclusive set of G.
Therefore $\beta_{sis}(G) \geq |T| > |S_1| = \beta_{sis}(G - v)$.
Thus $\beta_{sis}(G - v) < \beta_{sis}(G)$.

Theorem 4.19: Let G be a graph with $\beta_{sis}(G) = 2$. Let $v \in V(G)$ be a pendent vertex and u be its neighbour which is not a pendent vertex then $\beta_{sis}(G - v) < \beta_{sis}(G)$ if and

only if u is not adjacent to some non isolated vertex of $G-v$.
Proof: Suppose $\beta_{sis}(G-v) \leq \beta_{sis}(G)$.
Let M be a maximum secured isoinc set of G.
Suppose $u \in M$ & $v \notin M$.
Now M has only two vertices say u & y then u & y are non adjacent vertices. Since $y \in M$, y cannot be an isolated vertex in $G-v$.
Thus u is non adjacent to y which a non isolated vertex in $G-v$.
Suppose $\in M$ & $u \notin M$.
Now $M-\{v\} \cup \{u\}$ is an isoinc set of G.
Therefore u cannot be adjacent to some isolate of M different from v.
So there is a vertex x of M such that $x \neq v$ and u is not adjacent to x.
Conversely, suppose there is a non isolated x of $G-v$ such that u is not adjacent to x.
Now consider the set $M=\{x,v\}$ then M is a maximum secured isoinc set of G.
Thus $\beta_{sis}(G) > \beta_{sis}(G-v)$.

Proposition 4.20: Let G be the path graph with $v_n \geq 4$ vertices $v_1, v_2, \dots \dots \dots, v_n$ and with $d(v_1)=1$ & $d(v_n)=1$ then every maximum secured isoinc set of G contains v_1 & v_n.
Proof: For $n=4$ the path with 4 vertices $\{v_1, v_2, v_3, v_4\}$ has only one maximum secured isoinc set namely $\{v_1, v_4\}$.
Suppose for $n>4$, the theorem is true for path graph with n vertices. Now consider the path graph P_{n+1} with $n+1$ vertices $v_1, v_2, \dots \dots \dots, v_{n+1}$. Also consider the path graph P_n with n vertices $v_1, v_2, \dots \dots \dots, v_n$.
First note that $\beta_{sis}(P_n) \leq \beta_{sis}(P_{n+1})$. (see theorem 16)
Let M be maximum secured isoinc set of P_{n+1}.
Suppose $v_1 \notin M$ and $v_{n+1} \notin M$.
Claim : $v_n \in M$.
Proof of the claim : suppose $v_n \notin M$.
Then M is a secured isoinc set of P_n.
Then $\beta_{sis}(P_{n+1}) = |M| \leq \beta_{sis}(P_n)$.
Therefore $\beta_{sis}(P_n) = \beta_{sis}(P_{n+1}) = |M|$.
Thus M is a maximum secured isoinc set of P_n such that $v_n \notin M$.
This contradict the induction hypothesis for n.
Thus $v_n \in M$.
Thus we have proved that if M is a maximum secured isoinc set of P_{n+1} then $v_n \in M$ if $v_{n+1} \notin M$.
It $v_n \notin M$ then let $M_1 = M-\{v_n\} \cup \{v_{n+1}\}$ then M_1 is a maximum secured isoinc set of P_{n+1} and $v_{n+1} \in M_1$.
Similarly, we can prove that v_1 belong to every maximum secured isoinc set of P_{n+1}.
Thus the theorem is proved.

V. CONCLUSION AND FUTURE SCOPE

In this paper we have considered the effect of removing a vertex on the secured isolate inclusive number of the graph. Further, we can consider the operations of edge removal and edge addition and their effect on secured isolate inclusive number can be studies.

ACKNOWLEDGMENT

The Authors would like to acknowledge their thanks to the Editor-in-chief and Referees for their valuable suggestions and comments to improve the quality of this paper.

REFERENCES

[1] D.K.Thakkar and J.C.Bosamiya, "*Graph critical with respect to independent domination",* International Journal of Discrete Mathematics Sciences and Cryptography, Vol. 16, pp. 179-186, 2013.

[2] D.K.Thakkar and N.J.Savaliya, "*On Isolate Inclusive sets in Graphs ",* International Journal of Innovation in Science and Mathematical, Vol. 5, Issue 3, pp. 74-76, 2017.

[3] D.K.Thakkar and N.J.Savaliya, "*About Isolate Domination in Graphs ",* International Journal of Mathematical Archive, Vol. 8, Issue 11, pp. 171-178, 2017.

[4] Michel A. Henning, Anders Yeo, *"Total Domination In graphs"*, Springer, New York, 2013.

[5] I. Sahul Hamid and S. Balamurugan*, "Isolate Domination Numberand Maximum Degree*" bulletin of the international mathematical virtual institute, Vol. 3, pp. 127-133, 2013.

[6] I. Sahul Hamid and S. Balamurugan, "*Isolate Domination In graphs"*, Arab Journal of Mathematical Science Math Sci, Vol. 22, pp. 232-241, 2016.

[7] T.W.Haynes, S.T.Hedetniemi, P.J.Slater, "*Fundamental of Domination In graphs*", New York, 1998.

AUTHORS PROFILE

Dr. D. K. Thakkar is a retired professor of Mathematics, Saurashtra University, Rajkot. His areas of interest are Graph Theory and Topology. He has published more than 90 research papers in various journals.

Ms Niketa J. Savaliya is a young research student who is working in the area of Domination in graphs. She is a Research scholar in Department of Mathematics, Saurashtra University, Rajkot.