International Journal of Scientific Research in Mathematical and Statistical Sciences **Research Paper**
Volume-5, Issue-6, pp.278-281, December (2018) **E-ISSN:** 2348-4519

Convergence of Weighted (0, 2; 0)-Interpolation on the Unit Circle

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Available online at: www.isroset.org

Received: 01/Dec/2018, Accepted: 16/Dec/2018, Online: 31/Dec/2018

Abstract— In this paper, we study the convergence of weighted Lacunary Pál-type interpolation on two pairwise disjoint sets of nodes obtained by projecting vertically the zeros of Legendre polynomial and its second derivative with boundary points of real line on the unit circle. Here the Lacunary data is prescribed on first set of nodes, whereas the Lagrange data on the other one.

Keywords— Legendre polynomial, Weighted interpolation, Pál-type interpolation, Explicit forms.

I. INTRODUCTION

The problem of Lacunary interpolation was initiated by P. Turán [1] on the zeros of

$$\prod_n(x) = (1-x^2)P'_{n-1}(x),$$

where $P_{n-1}(x)$ is the Legendre polynomial of degree $(n-1)$. The $(0,2)-$ interpolation on the unit circle was initiated by O. Kiš [2], he established the convergence theorem for that interpolatory polynomial. After that several Mathematician considered $(0,2)-$ interpolation viz. on the unit circle, infinite interval and on the real line. H.P. Dikshit [3] considered Pál type interpolation on non – uniformly distributed nodes on the unit circle.In 1996, S. Xie [4] considered (0, 1, 3) - interpolation on the vertically projected nodes onto unit circle. He obtained its regularity, explicit representation and convergence of the some sets of nodes. In 2011, author[1] (with K. K. Mathur) [5] established a convergence theorem for the weighted (0, 2)*-interpolation on the unit circle. After that she (with M. Shukla) [6] considered $(0,2)$ - interpolation on the nodes which are obtained by projecting vertically the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$stands for Jacobi polynomial, obtained the explicit forms and establish a convergence theorem for the same. Recently, authors [7] considered $(0,2)-$interpolation on the unit circle. Also, author[1] [8,9] considered $(0;0,2)$ and $(0,2;0)-$ interpolation on the nodes by projecting vertically the zeros of $(1-x^2)\ P_n(x)$ and $P'_n(x)$ onto the unit circle. In this paper we consider $(0,2;0)-$ interpolation on two pairwise disjoint set of nodes on the unit circle, in which the Lagrange data is prescribed on the first set of nodes, whereas Lacunary data on the other one .We obtained regularity and explicit forms of interpolatory polynomials.

In this paper, we consider two pairwise disjoint set of the nodes Z_n and T_n, such that

$$(1.1)\qquad Z_n = \left\{\begin{matrix} z_0 = 1, \quad z_{2n+1} = -1, \\ z_k = cos\theta_k + i\ sin\theta_k, \\ z_{n+k} = \bar{z_k}\ ,\ k = 1(1)n \end{matrix}\right\}$$

$$(1.2)\qquad T_n = \left\{\begin{matrix} t_k = cos\varphi_k + i\ sin\varphi_k, \\ t_{n+k} = \bar{t_k}\ ,\ k = 1(1)n-2 \end{matrix}\right\}$$

be two set of nodes.

In section 2, we give some Preliminaries, in section 3, we present the explicit forms, in section 4 and 5, we give estimates and convergence of weighted $(0,2;0)-$ interpolation on the unit circle respectively.

II. RELATED WORK

Recently, authors [11] considered weighted $(0;0,2)-$ interpolation by projecting vertically the zeros of $(1-x^2)\,P_n(x)$ and $P''_n(x)$ onto the unit circle, where the Lagrange data is prescribed on first set of nodes and the Lacunary data is on the other one.

III. PRELIMINARIES

In this section, we shall give some well-known results, which we shall use.

The differential equation satisfied by $P_n(x)$ is

$$(2.1)\quad (1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0.$$

(2.2) $W(z) = \prod_{k=1}^{2n}(z-z_k) = K_n\, P_n\left(\frac{1+z^2}{2z}\right)z^n.$

(2.3) $R(z) = (z^2-1)\,W(z).$

(2.4) $H(z) = \prod_{k=1}^{2n-4}(z-t_k) = K_n^*\ P_n''\left(\frac{1+z^2}{2z}\right)z^{n-2}.$

We shall require the fundamental polynomial of Lagrange interpolation based on the zeros of $W(z)$ and $H(z)$ are respectively given as

(2.5) $l_k(z) = \frac{W(z)}{(z-z_k)\ W'(z_k)}, \qquad k = 1(1)2n.$

(2.6) $L_k(z) = \frac{H(z)}{(z-t_k)\,H'(t_k)}, \quad k = 1(1)2n-4.$

(2.7) $J_k(z) = \int_{-1}^{z} t\ l_k(t)\ dt\,.$

(2.8) $J(z) = \int_{-1}^{z} W(t)\ dt,$

which satisfies

(2.9) $J(-z) = -J(z).$

(2.10) $W'(z_k) = \frac{K_n}{2}(z_k^2-1)z_k^{n-2}P_n'(x_k)\,.$

(2.11) $W''(z_k) = K_n[(n-1)(z_k^2-1)-1]z_k^{n-3}P_n'(x_k).$

(2.12) $H'(t_k) = \frac{K_n^*}{2}(t_k^2-1)t_k^{n-4}P_n'''(x_k^*).$

We will also use the following well known inequalities:
For, $-1 < x < 1$

(2.13) $(1-x^2)^{1/4}|P_n(x)| \le \sqrt{\frac{2}{\pi}}\,n^{-1/2},$

(2.14) $(1-x^2)^{3/4}|P_n'(x)| \le \sqrt{2}\,n^{1/2},$

(2.15) $|P_n(x)| \le 1.$

Let

$$x_k = cos\theta_k\ (k = 1,2,\dots\dots,n)$$

are the zeros of n^{th} Legendre polynomial $P_n(x)$, with

$$1 > x_1 > x_2 > \cdots\dots\dots\dots\dots x_n > -1,$$

then

(2.16) $(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2},$

(2.17) $\left|P_n^{(r)}(x_k)\right| \sim k^{-r-\frac{1}{2}}\ n^{2r}, \qquad r = 0,1,2,3.$
For more details, one can see [10].

IV. THE PROBLEM

Let $\{z_k\}_{k=0}^{2n+1}$ and $\{t_k\}_{k=1}^{2n-4}$ be two disjoint set of nodes obtained by projecting vertically the zeros of $(1-x^2)\ P_n(x)$ and $P_n''(x)$ onto the unit circle respectively, where $P_n(x)$ stands for n^{th} Legendre polynomial. Here we are interested to determine the convergence of following polynomial $R_n(z)$ of degree $\le 6n-3$ satisfying the conditions

$$(3.1)\begin{cases} R_n(z_k) & = \alpha_k, \quad k = 0(1)2n+1 \\ [p(z)R_n(z)]''_{z=z_k} & = \beta_k, \quad k = 1(1)2n \\ R_n(t_k) & = \gamma_k, \quad k = 1(1)2n-4. \end{cases}$$

where α_k, β_k and γ_k are arbitrary complex constants and $p(z)$ is a weight function

$$p(z) = z^{-(n^2+n+2)/2}(z^2-1)^{9/2}(z^2+1)^{-\{2-n(n+1)\}/2}\,.$$

V. EXPLICIT REPRESENTATION OF INTERPOLATORY POLYNOMIALS

We shall write $R_n(z)$ satisfying (3.1) as

$$(4.1)\quad R_n(z) = \sum_{k=0}^{2n+1}\alpha_k A_k(z) + \sum_{k=1}^{2n}\beta_k B_k(z) + \sum_{k=1}^{2n-4}\gamma_k C_k(z),$$

where $A_k(z)$, $B_k(z)$ and $C_k(z)$ are unique polynomial, each of degree at most $6n-3$ satisfying the conditions :

For $k = 0(1)2n+1$

$$(4.2)\begin{cases} A_k(z_j) & = \delta_{jk}, \quad j = 0(1)2n+1 \\ [p(z)A_k(z)]''_{z=z_j} & = 0, \qquad j = 1(1)2n \\ A_k(t_j) & = 0, \qquad j = 1(1)2n-4. \end{cases}$$

For $k = 1(1)2n$

$$(4.3)\begin{cases} B_k(z_j) & = 0, \qquad j = 0(1)2n+1 \\ [p(z)B_k(z)]''_{z=z_j} & = \delta_{jk}, \quad j = 1(1)2n \\ B_k(t_j) & = 0, \qquad j = 1(1)2n-4. \end{cases}$$

For $k = 1(1)2n-4$

$$(4.4)\begin{cases} C_k(z_j) & = 0, \qquad j = 0(1)2n+1 \\ [p(z)C_k(z)]''_{z=z_j} & = 0, \qquad j = 1(1)2n \\ C_k(t_j) & = \delta_{jk}, \quad j = 1(1)2n-4. \end{cases}$$

Theorem 4.1: For $k = 1(1)2n-4$, we have

$$(4.5) \quad C_k(z) = \frac{R(z)W(z)}{R(t_k)\,W(t_k)} L_k(z) + \frac{H(z)\,W(z)}{R(t_k)W^2(t_k)H'(t_k)} \{S_k(z) + c_{1k}\, J(z)\},$$

where

$$(4.6)\ S_k(z) = \int_{-1}^{z} \frac{(t_k^2-1)W(t)W'(t_k)}{(t-t_k)}\, dt - \int_{-1}^{z} \frac{(t^2-1)\,W'(t)\,W(t_k)}{(t-t_k)}\, dt$$

$$(4.7)\ c_{1k} = -\frac{S_k(1)}{J(1)},$$

and $J(z)$ is defined in (2.8).

Theorem 4.2: For $k = 1(1)2n$, we have

$$(4.8)\ B_k(z) = b_k\, W(z)H(z)\{J_k(z) + b_{1k}\, J(z)\},$$

where

$$(4.9)\ b_k = \frac{1}{2z_k\, p(z_k)W'(z_k)H(z_k)},$$

$$(4.10)\ b_{1k} = -\frac{J_k(1)}{J(1)},$$

$J_k(z)$ defined in (2.7).

Theorem 4.3: For $k = 1(1)2n$, we have

$$(4.11)\ A_k(z) = \frac{(z^2-1)H(z)}{(z_k^2-1)H(z_k)} l_k^2(z) + \frac{H(z)W(z)}{(z_k^2-1)H(z_k)W'(z_k)}\{T_k(z) + a_1\, J(z)\} + a_k B_k(z),$$

where

$$(4.12)\ T_k(z) = -\int_{-1}^{z} (t^2-1)\, \frac{\{l_k'(t) - l_k'(z_k)l_k(t)\}}{(t-z_k)}\, dt,$$

$$(4.13)\quad a_1 = -\frac{T_k(1)}{J(1)},$$

$$(4.14)\ a_k = -\frac{\{(z^2-1)p(z)H(z)\}''_{z=z_k}}{(z_k^2-1)\,H(z_k)} - 4p(z_k)l_k'(z_k)\frac{W'(z_k)}{(z_k^2-1)} - 4\,p(z_k)\,{l_k'}^2(z_k).$$

For $k = 0$ and $2n+1$, we have

$$(4.15)\quad A_{0,0}(z) = \frac{W(z)H(z)}{W(1)H(1)} \frac{\int_{-1}^{z} W(t)\,dt}{\int_{-1}^{1} W(t)dt}$$

$$(4.16)\ A_{0,2n+1}(z) = \frac{W(z)H(z)}{W(-1)H(-1)} \frac{\int_{z}^{1} W(t)\,dt}{\int_{-1}^{1} W(t)dt}.$$

VI. ESTIMATION OF FUNDAMENTAL POLYNOMIALS

LEMMA: $A_k(z), B_k(z)$ and $C_k(z)$ be defined in section 4. Then for $|z| \le 1$, we have

$$(5.1)\ \sum_{k=0}^{2n+1} |p(z)A_k(z)| \le cn^2 \log n,$$

$$(5.2)\ \sum_{k=1}^{2n} |p(z)B_k(z)| \le c \log n,$$

$$(5.3)\ \sum_{k=1}^{2n-4} |p(z)C_k(z)| \le c\, n^2 \log n,$$

where c is a constant independent of n and z.

PROOF: Using the conditions from (2.10) - (2.17), we get the result.

VII. CONVERGENCE

In this section, we prove the main Theorem

THEOREM: Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$. Let the arbitrary numbers β_k's be such that

$$(6.1)\ |\beta_k| = O\left(n^2\, \omega\left(f, \frac{1}{n}\right)\right),\ k = 1(1)2n-4.$$

Then $\{Q_n(z)\}$ defined by

$$(6.2)\quad Q_n(z) = \sum_{k=0}^{2n+1} f(z_k)\, A_k(z) + \sum_{k=1}^{2n} \beta_k B_k(z)$$

$$+\sum_{k=1}^{2n-4} f(t_k)\, C_k(z)$$

satisfies the relation,

(6.3) $|p(z)\{\, Q_n(z) - f(z)\}| = O(\omega(f, n^{-1}) \log n)$,

where $\omega(f, n^{-1})$ be the modulus of continuity of $f(z)$.

To prove the theorem, we shall need the followings:
REMARK: Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$ and $f'' \in Lip\alpha,\ \alpha > 0$, then the sequence $\{Q_n(z)\}$ converges uniformly to $f(z)$ in $|z| \le 1$, which follows from (6.3) as
(6.4) $\omega(f, n^{-1}) = O(n^{-2-\alpha}),\ \alpha > 0$.

There exist a polynomial $F_n(z)$ of degree $\le 6n - 3$ atisfying Jackson's inequality
(6.5) $|f(z) - F_n(z)| \le c\ \omega(f, n^{-1}), z = e^{i\theta} (0 \le \theta < 2\pi)$.

And also an inequality due to O. Kiš [2].
(6.6) $\left|F_n^{(m)}(z)\right| \le c\ n^m\, \omega(f, n^{-1})\ ,\ m\epsilon I^+$.

PROOF: Since $Q_n(z)$ be is uniquely determined polynomial of degree $\le 6n - 3$ and the polynomial $F_n(z)$ satisfying (6.5) and (6.6) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n+1} F_n(z_k) A_k(z) + \sum_{k=1}^{2n} F_n''(z_k) B_k(z) + \sum_{k=1}^{2n-4} F_n(t_k)\, C_k(z).$$

Then

$$|p(z)\{Q_n(z) - f(z)\}| \le |p(z)\{Q_n(z) - F_n(z)\}| + |p(z)||\{F_n(z) - f(z)\}|$$

$$\le \sum_{k=0}^{2n+1} |f(z_k) - F_n(z_k)|\, |p(z) A_k(z)| + \sum_{k=1}^{2n} \{|\beta_k| + |F_n''(t_k)|\} |p(z) B_k(z)| + \sum_{k=1}^{2n-4} |f(t_k) - F_n(t_k)|\, |p(z) C_k(z)| + |p(z)||F_n(z) - f(z)|,$$

using (6.1), (6.4) - (6.6) and Lemma in section 5, we get (6.3).

VIII. Conclusion and Future Scope

In this paper, we defined the weighted Pál-type interpolation on two disjoint sets of nodes which convergence uniformly to $f'' \in Lip\alpha,\ \alpha > 0$.

Acknowledgment

The authors are extremely thankful to the Editor-in –chief and reviewers of their comments and suggestions for improving the paper.

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