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Common Fixed Point Results for Generalized Contraction under the Rational Expressions in Complex Valued Metric spaces

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Abstract-In this paper we prove the existence and uniqueness of common fixed point theorems satisfying contractive conditions involving rational expression for two pairs of weakly compatible self-mappings and further some results using (E.A) property and (CLR) property are obtained in complex valued metric spaces. An illustrative example is provided to support the results obtained. The presented results improve and generalize the main results of Öztűrk in (Öztűrk, 2014).

***Keywords*:** Complex valued metric space, weakly compatible mapping, (E.A) property, (CLR) property.
Mathematics Subject Classification: 47H10

I. INTRODUCTION

Fixed point theory is one of the most fruitful and applicable field of nonlinear analysis, which is very interesting area of research. Banach contraction principle [1] plays the major role to prove the existence and uniqueness of fixed point mappings. It is indeed the most famous results of metric fixed point theory. According to this principle, if T is a contraction on a Banach space X, then T has a unique fixed point in X, in other words if (X, d) is a complete metric space and T: X$\rightarrow$ X is contraction mapping (i.e. d (Tx, Ty) $\leq k$ d(x, y) for all x, yϵ X, where k is a non-negative real number such that $k < 1$), then T has a unique fixed point. Many researchers have investigated the Banach fixed point theorem in many directions and presented generalizations, extensions and applications of their findings.

On the other hand Azam et al. [2] introduced and studied complex valued metric spaces and proved a common fixed point theorem for a pair of contractive type mappings involving rational expressions. In 2002, Aamri and Moutawakil [3] introduced some common fixed point theorems under strict contractive conditions along with a new property (E.A). Sintunavarat and Kumam [4] also introduced the notion of CLR property for a pair of weakly compatible mapping in fuzzy metric spaces. Bhatt et al. [5] studied a common fixed point theorem for weakly compatible mapping satisfying rational inequality, without exploiting any type of commutative condition in complex valued metric spaces. Verma and Pathak [6] also have obtained the fixed point theorems in complex valued metric spaces for mappings satisfying the (E.A) and the common limit range (CLR) properties, which were more generalized then the result obtained by Aamri and Moutawakil [3]. Various fixed point theorems have been proved by the notion of (CLR)-property ([7], [8], [9], [10]). Singh et al. [11] proved the common fixed point results for a pair of mappings satisfying the contraction conditions by rational expressions having point-dependent control functions. Rao et al. [12] discussed common coupled fixed point for four maps using property (E.A.) in complex valued b-metric spaces. Recently, Dubey et al. [13] discussed common fixed point theorems in complex valued b-metric space. More recently Saluja [14] introduced the fixed point theorems under the rational contraction in the setting of complex valued metric space.

Many researchers have been discussed fixed point, common fixed point, coupled common fixed point theorems in metric spaces, Hilbert spaces, complex valued metric spaces and many other spaces. Öztűrk [15] has obtained some common fixed point results satisfying contractive type conditions with (E.A) and (CLR)-properties for two pairs of weakly compatible mappings in complex valued metric spaces, which were more, generalized and extended some of the results given by Pachpatte [16].

The purpose of this paper is to study some common fixed point theorems satisfying contractive type conditions involving the rational expression for two pairs of weakly compatible self-mappings in complex valued metric spaces using (E.A) property and the property of common limit in the range.

II. PRELIMINARIES

Let C be the set of complex numbers and let $z_1,\ z_2 \in C$

Define a partial order $\precsim$ on C as follows:

$z_1 \precsim z_2$ iff $\mathrm{Re}(z_1) \precsim \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$. If follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(iii) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(iv) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \ne z_2$ and one of (ii), (iii) and (iv) is satisfied and we will write $z_1 \prec z_2$ if only (iv) is satisfied. Note that the following conditions hold:

(i) $0 \precsim z_1 \precnsim z_2$ implies that $|z_1| < |z_2|$,

(ii) $z_1 \precsim z_2$ and $z_2 \prec z_3$ implies that $z_1 < z_3$,

(iii) $0 \precsim z_1 \precsim z_2$ implies that $|z_1| \le |z_2|$,

(iv) $a_1, a_2 \in \mathbb{R}$ and $a_1 \le a_2$ implies that $a_1 z \precsim a_2 z$, for all $z \in \mathbb{C}$.

Definition 2.1 [2] Let X be a non-empty set. Suppose that the mapping d: $X \times X \to C$ is called a complex valued metric on X if the following conditions are holds:

(a) $0 \precsim d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$;

(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(c) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$;

Then d is called complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 2.2 [2] Let (X, d) be a complex valued metric space and let x_n be a sequence in X and $x \in X$.

(1) If for every $c \in C$ with $0 \prec c$, there exists $n_0 \in N$ such that $d(x_n, x) \prec c$ for all $n > n_0$, then $\{x_n\}$ is said to be converges to x and x is a limit point of $\{x_n\}$. We denote this by $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

(2) If for every $c \in C$ with $0 \prec c$, there exists $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$ where $m \in N$, then $\{x_n\}$ is said to be Cauchy sequence.

(3) If for every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space.

Lemma 2.1 [2] Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in *X*. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.

Lemma 2.2 [2] Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in *X*. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$ where $m \in \mathbb{N}$.

Definition 2.3 [5] Let *u* and *v* be two self-maps defined on a set X. Then *u* and *v* are said to be weakly compatible if they commute at coincidence points.

Definition 2.4[3] Let S,T : X → X be two self-mappings of a complex valued metric space (X, d).The pair (S,T) is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$in X such that

$$\lim_{n\to\infty} d(Sx_n,u) = \lim_{n\to\infty} d(Tx_n,u) = 0\ , \quad for\ some\ u \in X.$$

Example 2.1 Let $X = \mathbb{C}$ be endowed with the complex valued metric $d: \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ as

$d(z_1,z_2) = |x_1 - x_2| + i\,|y_1 - y_2|$, Where $z_1 = x_1 + i\,y_1$ *and* $z_2 = x_2 + i\,y_2$. Then $(\mathbb{C},d)$ is a complete complex valued metric space.

Define the mappings $T,S : X \to X$ *as* $Tz = 4z^2 - 4z + 1,\ \ Sz = 2 - z^2$ *for all* $z \in X$

and consider the sequence $\{z_n\} = \left(1 + \frac{i}{e^{2n}}\right)$. Thus we obtain

$$\lim_{n\to\infty} d(Tz_n,z) = \lim_{n\to\infty} d(Sz_n,z)\ ,$$

$$\lim_{n\to\infty} d\left(4\left(1+\frac{i}{e^{2n}}\right)^2 - 4\left(1+\frac{i}{e^{2n}}\right)+1,1\right) = \lim_{n\to\infty} d\left(2-\left(1+\frac{i}{e^{2n}}\right)^2,1\right) = 0,$$

Where z = 1is the limit of sequence $\{z_n\}$.

Hence the pair (S, T) satisfy property (E.A).

Definition 2.5[4] Let S,T : X → X be two self-mappings of a complex valued metric space X, S and T is said to satisfy the common limit in the range of S property if

$$\lim_{n\to\infty} d(Sx_n,Sx) = \lim_{n\to\infty} d(Tx_n,Sx) = 0\ ,\ \ for\ some\ x \in X.$$

Example 2.2 Let $X = \mathbb{C}$ be endowed with the complex valued metric $d: \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ as

$d(z_1,z_2) = |x_1 - x_2| + i\,|y_1 - y_2|$, Where $z_1 = x_1 + i\,y_1$ *and* $z_2 = x_2 + i\,y_2$. Then $(\mathbb{C},d)$ is a complete complex valued metric space.

Define the mappings $T,S : X \to X$ *by* $Tz = 3x + 3iy,\ Sz = 2x + 1 + 2iy$ *for all* $x,y \in R.$

Consider the sequence $\{z_n\} = \{x_n + iy_n\} = \left\{1 + \frac{i}{2n}\right\}.$

Then for z=1, with an easy calculation, we see that

$$\lim_{n\to\infty} d(Tz_n,Sz) = \lim_{n\to\infty} d(Sz_n,Sz)\ ,$$

$$\lim_{n\to\infty} d\left(3+\frac{3i}{2n},3\right) = \lim_{n\to\infty} d\left(3+\frac{i}{n},3\right) = 0.$$

Hence T, S satisfy the common limit in the range of S property ((CLR_S)-property).

III. MAIN RESULTS

In this section, we have proved common fixed point results for the pairs, weakly compatible and satisfy properties (E.A) and (CLR), reconstructing contraction conditions described by rational expressions given in [13].

Theorem 3.1 Let (X,d) be a complex valued metric space and let $S,T,I,J:X\to X$ be four self mappings satisfying the following:

(i) $T(X) \subseteq I(X),\ S(X) \subseteq J(X)$;

(ii) $$[d(Sx,Ty)]^3 \precsim \alpha\,[d(Ix,Jy)\ d(Ix,sx)\ d(Jy,Ty)] + \beta\frac{[d(Sx,Jy)\ d(Ix,Jy)\ d(Jy,Ty)\ d(Ix,Sx)]}{d(Ix,Ty)+d(Sx,Jy)} \quad ....(1)$$

for all $x,y\in X$, where $\alpha,\beta\in(0,1)$;

(iii) the pairs $\{S,I\}$ and $\{T,J\}$ are weakly compatible;

(iv) one of the pairs $\{S,I\}$ or $\{T,J\}$ satisfies property $(E.A)$

If the range of one of the mappings J(X) or I(X) is a complete subspace of X, then the mappings I, J, S and T have a unique common fixed point in X.

Proof: Suppose that the pair {T, J} satisfies property (E.A) then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n = z, \quad \text{for some } z\in X.$$

Further, since $T(x)\subseteq I(x)$, there exists a sequence $\{y_n\}$ in X such that

$$Tx_n = Iy_n. \text{ Hence } \lim_{n\to\infty} Iy_n = z .$$

our claim is $\lim_{n\to\infty} Sy_n = z$, using condition (1), we have

$$[d(Sy_n,Tx_n)]^3 \le \alpha[d(Iy_n,Jx_n)\,d(Iy_n,Sy_n)\,d(Jx_n,Tx_n)] + \beta\frac{[d(Sy_n,Jx_n)\,d(Iy_n,Jx_n)\,d(Jx_n,Tx_n)\,d(Iy_n,Sy_n)]}{d(Iy_n,Tx_n)+d(Sy_n,Jx_n)}$$

$$=\alpha[d(Tx_n,Jx_n)\,d(Tx_n,Sy_n)\,d(Jx_n,Tx_n)] + \beta\frac{[d(Sy_n,Jx_n)\,d(Tx_n,Jx_n)\,d(Jx_n,Tx_n)\,d(Tx_n,Sy_n)]}{d(Tx_n,Tx_n)+d(Sy_n,Jx_n)}$$

$$=\alpha[d(Tx_n,Jx_n)\,d(Sy_n,Tx_n)\,d(Tx_n,Jx_n)] + \beta[d(Tx_n,Jx_n)\,d(Tx_n,Jx_n)\,d(Sy_n,Tx_n)]$$

By dividing two sides of the above inequality with $d(Sy_n,Tx_n)$ we get

$$[d(Sy_n,Tx_n)]^2 \precsim \alpha\,d(Tx_n,Jx_n)^2 + \beta d(Tx_n,Jx_n)^2$$

Thus $$|d(Sy_n,Tx_n)| \le \sqrt{\alpha+\beta}\,|d(Tx_n,Jx_n)| = 0$$

and letting $n\to\infty$ we have

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = z .$$

Now, suppose that $I(X)$ is complete subspace of X, then z = Iu for some $u\in X$.

Subsequently, we obtain.

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n == \lim_{n\to\infty} Iy_n = z = Iu \qquad ....(2)$$

we claim that Su = Iu. To prove this, in (1)

$$[d(Sx,Ty_n)]^3 \precsim \alpha[d(Iu,Jx_n)\,d(Iu,Su)\,d(Jx_n,Tx_n)] + \beta\frac{[d(Su,Jx_n)\,d(Iu,Jx_n)\,d(Jx_n,Tx_n)\,d(Iu,Su)]}{d(Iu,Tx_n)+d(Su,Jx_n)}$$

and letting $n \to \infty$ and using (2), we have

$$\left|[d(Su, Tx_n)^3\right| \le \alpha \left|[d(z,z)\, d(z,Su)\, d(z,z)]\right| + \beta \frac{\left|[d(Su,z)\, d(z,z)\, d(z,z)\, d(z,Su)]\right|}{\left|d(z,z)+d(Su,z)\right|} = 0$$

and consequently Su = z = Iu.

Thus z is a coincidence point of $\{S, I\}$ .Weak compatibility of the pair $\{S, I\}$ implies that

$$SIu = ISu = Sz = Iz.$$

Conversely, since $S(X) \subseteq J(X)$, there exists $v \in X$ such that $Su = Jv$.

Hence Su = Iu = Jv = z.

Now show that v is a coincidence point of $\{T, J\}$; that is $Tv = Jv = z.$ Putting $x = u,\ y = v$ in (1) we get

$$[d(Su,Tv)]^3 \precsim \alpha [d(Iu,Jv)\, d(Iu,Su)\, d(Jv,Tv)] + \beta \frac{[d(Su,Jv)\, d(Iu,Jv)\, d(Jv,Tv)\, d(Iu,Su)]}{d(Iu,Tv)+d(Su,Jv)}$$

$$\left|[d(z,Tv)]^3\right| \le \alpha \left|[d(z,Jv)\, d(z,z)\, d(Jv,Tv)]\right| + \beta \left|\frac{[d(z,Jv)\, d(z,Jv)\, d(Jv,Tv)\, d(z,z)]}{d(z,Tv)+d(z,Jv)}\right|$$

$$\left|[d(z,Tv)]^3\right| \le 0$$

Thus Tv = z. Hence Tv = Jv = z and v is a coincidence point of T and J.

Weak compatibility of the pair {T, J} implies that

$$TJv = JTv = Tz = Jz.$$

Therefore, z is a common coincidence point of S, T, I and J.

In order to show that z is a common fixed point of these mappings, we write in (1)

$$[d(z,Tz)]^3 = [d(Su,Tz)]^3 \precsim \alpha [d(Iu,Jz)\, d(Iu,Su)\, d(Jz,Tz)] + \beta \frac{[d(Su,Jz)\, d(Iu,Jz)\, d(Jz,Tz)\, d(Iu,Su)]}{d(Iu,Tz)+d(Su,Jz)}$$

$$= \alpha [d(z,Jz)\, d(z,z)\, d(Jz,Tz,)] + \beta \frac{[d(z,Jz)\, d(z,Jz)\, d(Jz,Tz)\, d(z,z)]}{d(z,Tz)+d(z,Jz)}$$

i.e. $\left|[d(z,Tz)]^3\right| \le 0.$

Thus $Sz = Iz = Jz = Tz = z$.

A similar argument derives if we assume that J(X) is a complete substance of X and also using the property (E.A) of the pair {S, I} gives us the same result.

Uniqueness: To prove that z is a unique common fixed point, let us suppose that z* is another common fixed point of I, J, S and T. In (1) take $x = z^*$ and y = z; then

$$[d(z^*,z)]^3 = [d(Sz^*,Tz)]^3 \precsim \alpha [d(Iz,Jz^*)\, d(Iz,Sz)\, d(Jz^*,Tz^*)] + \beta \frac{[d(Sz,Jz^*)\, d(Iz,Jz^*)\, d(Jz^*,Tz^*)\, d(Iz,Sz)]}{d(Iz,Tz^*)+d(Sz,Jz^*)}$$

$[d(z^*,z)]^3 \le 0$ is contradiction. Thus $z = z^*$. Consequently, $Sz = Iz = Tz = Jz = z$ and z is the unique common fixed point of I, J, S and T.

Putting J = I in theorem (3.1) we have the following corollary.

Corollary (3.1) Let S, T, and I be three self -mappings of a complex valued metric space (X, d) satisfying the inequality

$$[d(Sx,Ty)]^3 \precsim \alpha [d(Ix,Iy)\, d(Ix,Sx)\, d(Iy,Ty)] + \beta \frac{[d(Sx,Iy)\, d(Ix,Iy)\, d(Iy,Ty)\, d(Ix,Sx)]}{d(Ix,Ty+d(Sx,Iy)}$$

For all x, y in X, where $\alpha, \beta \in (0,1)$. Suppose that the following conditions hold:

(i) $I(X) \supseteq S(X) \cup T(X)$, ,

(ii) both the pairs {I, S} and {I, T} are weakly compatible,

(iii) one of the pairs {I, S} and {I, T} satisfies the property (E.A).

If I(X) is complete subspace of X, then S, T and I have a unique common fixed point in X.

In theorem (3.1), if we put S=T and I = J, we have the following.

Corollary (3.2) Let (X, d) be a complex valued metric space and let S and T be two self-mappings satisfying the following:

(i) $S(X) \subseteq I(X);$

(ii) $[d(Sx, Sy)]^3 \precsim \alpha(d(Ix, Iy)\,d(Ix, Sx)\,d(Iy, Sy)] + \beta \dfrac{[d(Sx, Iy)\,d(Ix, Iy)\,d(Iy, Sy)\,d(Ix, Sx)]}{d(Ix, Sy) + d(Sx, Iy)}$

(iii) {I, S} is a weakly compatible pair;

(iv) the pair {I, S} satisfies property (E.A).

If I(X) is complete subspace of X, then I and S have the unique common fixed point in X.

Theorem 3.2 Let (X, d) be a complex valued metric space and let S, T, I, J: X→X be four self-mappings satisfying the following:

(i) $T(X) \subseteq I(X),\ S(X) \subseteq J(X);$

(ii) $[d(Sx, Ty)]^3 \precsim \alpha[d(Ix, Jy)\,d(Ix, Sx)\,d(Jy, Ty)] + \beta \dfrac{[d(Sx, Jy)\,d(Ix, Jy)\,d(Jy, Ty)\,d(Ix, Sx)]}{d(Ix, Ty) + d(Sx, Jy)}$ .....(3)

for all x, y, $\in$ X, where $\alpha, \beta \in (0, 1)$;

(iii) the pairs {S, I} and {T, J} are weakly compatible.

If the pairs {S, I} satisfies (CLR_S) - property or the pair {T, J} satisfies (CLR_T) - property, then the mappings I, J, S and T have a unique common fixed point in X.

Proof: Let us suppose that the pair {T, J} satisfies (CLR_T) - property; then by definition (2.5) there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n = Tx \quad \text{..... (4)}$$

For some x $\in X$ and also, since $T(x) \subseteq I(x)$, we have $Tx = Iz$ for some $z \in X$. We claim that $Sz = Iz = u$. Then putting x = z and $y = x_n$ in inequality (3) we have

$$d(Sz, Tx_n) \precsim \alpha[d(Iz, Jx_n)\,d(Iz, Sz)\,d(Jx_n, Tx_n)] + \beta \frac{[d(Sz, Jx_n)\,d(Iz, Jx_n)\,d(Jx_n, Tx_n)\,d(Tz, Sz)]}{d(Iz, Tx_n) + d(Sz, Jx_n)}$$

If n tends to infinity and with equality (4), then

$$|d(Sz, Iz)| \le 0 ,$$

which is possible for Iz = Sz since $\alpha, \beta \in (0, 1)$. Therefore, Sz= Iz= u, that is, z is a coincidence point of the pair {S, I}.

Also weak compatibility of the mappings S and I implies the following equality:

ISz =SIz=Iu=Su.

Besides, since S(x) $\subseteq$ J(x), there exist some $\omega \in X$ such that Sz = J ω. We claim that T ω = u. Then from (3), we have

$$d(Sz, T\omega) \precsim \alpha[(d(Iz, J\omega)\,d(Iz, Sz)\,d(J\omega, T\omega)] + \beta \frac{[d(Sz, J\omega)\,d(Iz, J\omega)\,d(J\omega, T\omega)\,d(Iz, Sz)]}{d(Iz, T\omega) + d(Sz, J\omega)}$$

$$d(u, T\omega) \precsim \alpha[d(u, Sz)\,d(u, u)\,d(u, T\omega)] + \beta \frac{[d(Sz, Sz)\,d(u, u)\,d(u, T\omega)\,d(u, u)]}{d(u, T\omega) + d(Sz, Sz)}$$

Thus, $|d(u, T\omega)| \le 0$,

which implies that $T\omega$ = u, since $\alpha, \beta \in (0,1)$

Hence $Iz = Sz = u = T\omega = J\omega$

and this shows that ω is a coincidence point of the pair {T, J}, weak compatibility of the pair {T, J} yields that TJω =JTω = Tu = Ju. In conclusion we show that u is a common fixed point of I, J, S and T. Using (3) we get

$$d(u, Tu) = d(Sz, Tu) \precsim \alpha[d(Iz, Ju)\,d(Iz, Sz)\,d(Ju, Tu)] + \beta \frac{[d(Sz, Ju)\,d(Iz, Ju)\,d(Ju, Ju)\,d(Iz, Sz)]}{d(Iz, Tu) + d(Sz, Ju)}$$

$$d(u,Tu) = d(Sz,Tu) \precsim \alpha[d(u,Tu)\,d(u,u)\,d(Tu,Tu)] + \beta\,\frac{[d(u,Tu)\,d(u,Tu)\,d(Tu,Tu)\,d(u,u)]}{d(u,Tu)+d(u,Tu)}$$

Thus, $|d(u,Tu)| \le 0$,

and hence Tu = u which is the desired result. The uniqueness of common fixed point u follows easily. The details of the proof of this theorem can be obtained by using the argument that the pair {S, I} satisfies (CLR_S) -property with suitable modifications. This completes the proof.

Theorem 3.3 Let (X,d) be a complex valued metric space and let $S,T,I,J : X \to X$ be four self mappings satisfying the following:

(i) $T(X) \subseteq I(X),\ S(X) \subseteq J(X)$;

(ii) $$d(Sx,Ty) \precsim \alpha \max\left\{\frac{d(Ix,Ty)[1+d(Ix,Sx)\,d(Jy,Ty)+d(Jy,Sx)\,d(Ix,Ty)]}{2[1+d(Ix,Jy)]},\ \frac{d(Jy,Sx)[1+d(Ix,Sx)\,d(Ix,Ty)+d(Jy,Sx)\,d(Jy,Ty)]}{2[1+d(Ix,Jy)]}\right\} \quad ....(5)$$

for all $x,y \in X$, where $\alpha \in (0,1)$;

(iii) the pairs $\{S,I\}$ and $\{T,J\}$ are weakly compatible;

(iv) one of the pairs $\{S,I\}$ or $\{T,J\}$ satisfies property (E.A)

If the range of one of the mappings J(X) or I(X) is a complete subspace of X, then the mappings I, J, S and T have a unique common fixed point in X.

Proof: Suppose that the pair {T, J} satisfies property (E.A) then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n = z, \quad \text{for some } z \in X$$

Further, since $T(X) \subseteq I(X)$, there exists a sequence $\{y_n\}$ in X such that

$$Tx_n = Iy_n, \text{ Hence } \lim_{n\to\infty} Iy_n = z.$$

Our claim is $\lim_{n\to\infty} Sy_n = z$, using condition (5), we have

$$d(Sy_n,Tx_n) \precsim \alpha \max\left\{\frac{d(Iy_n,Tx_n)[1+d(Iy_n,Sy_n)\,d(Jx_n,Tx_n)+d(Jx_n,Sy_n)\,d(Iy_n,Tx_n)]}{2[1+d(Iy_n,Jx_n)]},\ \frac{d(Jx_n,Sy_n)[1+d(Iy_n,Sy_n)\,d(Iy_n,Tx_n)+d(Jx_n,Sy_n)\,d(Jx_n,Tx_n)]}{2[1+d(Iy_n,Jx_n)]}\right\}$$

$$\precsim \alpha \max\left\{\frac{d(Tx_n,Tx_n)[1+d(Tx_n,Sy_n)\,d(Tx_n,Tx_n)+d(Tx_n,Sy_n)\,d(Tx_n,Tx_n)]}{2[1+d(Tx_n,Tx_n)]},\ \frac{d(Tx_n,Sy_n)[1+d(Tx_n,Sy_n)\,d(Tx_n,Tx_n)+d(Tx_n,Sy_n)\,d(Tx_n,Tx_n)]}{2[1+d(Tx_n,Tx_n)]}\right\}$$

$$\precsim \alpha \max\left\{0, \frac{d(Tx_n,Sy_n}{2}\right\},$$

and letting $n \to \infty$ we have,

$$\left(1-\frac{\alpha}{2}\right)|d(Sy_n,z)| \le 0.$$

Which is a contradiction, since $\alpha \in (0,1)$.

Therefore, $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = z$.

Now assuming $I(X)$ is complete subspace of X, then z = Iu for some $u \in X$

Right after, we obtain

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n == \lim_{n\to\infty} Iy_n = z = Iu \quad ....(6)$$

Our aim is to prove Su = Iu and for this putting x = u and y = x_n in (5), we get

$$d(Su, Tx_n) \precsim \alpha \max\left\{\frac{d(Iu, Tx_n)[1 + d(Iu, Su)\, d(Jx_n, Tx_n) + d(Jx_n, Su)\, d(Iu, Tx_n)]}{2[1 + d(Iu, Jx_n)]}, \frac{d(Jx_n, Su)[1 + d(Iu, Su)\, d(Iu, Tx_n) + d(Jx_n, Su)\, d(Jx_n, Tx_n)]}{2[1 + d(Iu, Jx_n)]}\right\}$$

and letting $n \to \infty$ and using (6), we have

$$|d(Su, z)| \precsim \alpha \left|\max\left\{0, \frac{d(z, Su)}{2}\right\}\right|$$

and hence $\left(1 - \frac{\alpha}{2}\right)|d(Su, z)| \le 0$

and consequently Su = z = Iu. Since $\alpha \in (0,1)$.

Thus z is a coincidence point of $\{S, I\}$ weak compatibility of the pair $\{S, I\}$ implies that

$$SIu = ISu = Sz = Iz.$$

Conversely, since $S(X) \subseteq J(X)$, there exists $v \in X$ such that $Su = Jv$.

Hence Su = Iu = Jv = z.

Now we show that v is a coincidence point of $\{T, J\}$; that is $Tv = Jv = z$ putting $x = u, y = v$ in (5) we get

$$d(Su, Tv) \precsim \alpha \max\left\{\frac{d(Iu, Tv)[1 + d(Iu, Su)\, d(Jv, Tv) + d(Jv, Su)\, d(Iu, Tv)]}{2[1 + d(Iu, Jv)]}, \frac{d(Jv, Su)[1 + d(Iu, Su)\, d(Iu, Tv) + d(Jv, Su)\, d(Jv, Tv)]}{2[1 + d(Iu, Jv)]}\right\}$$

$$|d(z, Tv)| \precsim \alpha \left|\max\left\{\frac{d(z, Tv)}{2}, 0\right\}\right|$$

$$\left(1 - \frac{\alpha}{2}\right)|d(z, Tv)| \le 0$$

Thus Tv = z. Hence Tv = Jv = z and v is a coincidence point of T and J.

Weak compatibility of the pair {T, J} implies that

$$TJv = JTv = Tz = Jz.$$

Therefore, z is a common coincidence point of S, T, I and J.

In order to show that z is a common fixed point of these mappings, we write in (5)

$$d(z, Tz) = d(Su, Tz) \precsim \alpha \max\left\{\frac{d(Iu, Tz)[1 + d(Iu, Su)\, d(Jz, Tz) + d(Jz, Su)\, d(Iu, Tz)]}{2[1 + d(Iu, Jz)]}, \frac{d(Jz, Su)[1 + d(Iu, Su)\, d(Iu, Tz) + d(Jz, Su)\, d(Jz, Tz)]}{2[1 + d(Iu, Jz)]}\right\}$$

$$|d(z, Tz)| \precsim \alpha \left|\max\left\{\frac{d(z, Tz)}{2}, \frac{d(z, Tz)}{2}\right\}\right|$$

$$\left(1-\frac{\alpha}{2}\right)|d(z,Tz)| \leq 0.$$

Thus $Sz = Iz = Jz = Tz = z$.

A similar argument derives if we assume that J(X) is a complete subspace of X and also using the property (E.A) of the pair {S, I} gives us the same result.

Uniqueness: To prove that z is a unique common fixed point, let us suppose that z* is another common fixed point of I, J, S and T. In (5) take x = z^* and y = z then

$$d(z^*,z) = d(Sz^*,Tz) \precsim \alpha \max\left\{\frac{d(Iz^*,Tz)[1+d(Iz^*,Sz^*)\,d(Jz,Tz)+d(Jz,Sz^*)\,d(Iz^*,Tz)]}{2[1+d(Iz^*,Jz)]}\right.$$
$$\left.,\frac{d(Jz,Sz^*)[1+d(Iz^*,Sz^*)\,d(Iz^*,Tz)+d(Jz,Sz^*)\,d(Jz,Tz)}{2[1+d(Iz^*,Jz)]}\right\}$$

$$\precsim \alpha \max\left\{\frac{d(z^*,z)[1+d(z^*,z^*)\,d(z,z)+d(z,z^*)\,d(z^*,z)]}{2[1+d(z^*,z)]}\right.$$
$$\left.,\frac{d(z,z^*)[1+d(z^*,z^*)\,d(z^*,z)+d(z,z^*)\,d(z,z)}{2[1+d(z^*,z)]}\right\}$$

$\left(1-\frac{\alpha}{2}\right)|d(z^*,z)| \leq 0$ is a contradiction. Thus $z = z^*$.

Consequently, $Sz = Iz = Tz = Jz = z$ and z is the unique common fixed point of I, J, S and T.

Putting J = I in theorem (3.3) we have the following corollary.

Corollary (3.3): Let S, T, and I be three self -mappings of a complex valued metric space (X, d) satisfying the inequality

$$d(Sx,Ty) \precsim \alpha \max\left\{\frac{d(Ix,Ty)[1+d(Ix,Sx)\,d(Iy,Ty)+d(Iy,Sx)\,d(Ix,Ty)]}{2[1+d(Ix,Iy)]}\right.$$
$$\left.,\frac{d(Iy,Sx)[1+d(Ix,Sx)\,d(Ix,Ty)+d(Iy,Sx)\,d(Iy,Ty)}{2[1+d(Ix,Iy)]}\right\}$$

For all x, y in X, where $\alpha \in (0,1)$. Suppose that the following conditions hold:

(i) $I(X) \supseteq S(X) \cup T(X)$,

(ii) both the pairs {I, S} and {I, T} are weakly compatible,

(iii) One of the pairs {I, S} and {I, T} satisfies the property (E.A).

If I(X) is complete subspace of X, then S, T and I have a unique common fixed point in X.

In theorem (3.3), if we put S=T and I = J, we have the following.

Corollary (3.4) Let (X, d) be a complex valued metric space and let S and T be two self-mappings satisfying the following:

(i) $S(X) \subseteq I(X)$;

(ii) $$d(Sx,Sy) \precsim \alpha \max\left\{\frac{d(Ix,Sy)[1+d(Ix,Sx)\,d(Iy,Ty)+d(Iy,Sx)\,d(Ix,Sy)]}{2[1+d(Ix,Iy)]}\right.$$
$$\left.,\frac{d(Iy,Sx)[1+d(Ix,Sx)\,d(Ix,Sy)+d(Iy,Sx)\,d(Iy,Sy)}{2[1+d(Ix,Iy)]}\right\}$$

(iii) {I, S} is a weakly compatible pair;

(iv) the pair {I, S} satisfies property (E.A).

If I(X) is complete subspace of X, then I and S have the unique common fixed point in X.

Theorem 3.4 Let (X, d) be a complex valued metric space and let S, T, I, J: X→X be four self-mappings satisfying the following:

(i) $T(X) \subseteq I(X), S(X) \subseteq J(X);$

(ii) $$d(Sx,Ty) \precsim \alpha \max\left\{ \frac{d(Ix,Ty)[1+d(Ix,Sx)\,d(Jy,Ty)+d(Jy,Sx)\,d(Ix,Ty)]}{2[1+d(Ix,Jy)]}, \frac{d(Jy,Sx)[1+d(Ix,Sx)\,d(Ix,Ty)+d(Jy,Sx)\,d(Jy,Ty)]}{2[1+d(Ix,Jy)]} \right\} \quad ....(7)$$

for all x, y $\in$ X , where $\alpha \in (0, 1)$;

(iii) the pairs {S, I} and {T, J} are weakly compatible.

If the pair {S, I} satisfies (CLR_S) - property or the pair {T, J} satisfies (CLR_T) - property, then the mappings I, J, S and T have a unique common fixed point in X.

Proof: Let us suppose that the pair {T, J} satisfies (CLR_T) - property; then by definition (2.6) there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Jx_n = Tx \quad .....(8)$$

For some x $\in X$ and also, since $T(X) \subseteq I(X)$, we have $Tx = Iz$ for some $z \in X$. We claim that $Sz = Iz = u$. Then putting x = z and $y = x_n$ in inequality (7) we have

$$d(Sz,Tx_n) \precsim \alpha \max\left\{ \frac{d(Iz,Tx_n)[1+d(Iz,Sz)\,d(Jx_n,Tx_n)+d(Jx_n,Sz)\,d(Iz,Tx_n)]}{2[1+d(Iz,Jx_n)]}, \frac{d(Jx_n,Sz)[1+d(Iz,Sz)\,d(Iz,Tx_n)+d(Jx_n,Sz)\,d(Jx_n,Tx_n)]}{2[1+d(Iz,Jx_n)]} \right\}$$

If n tends to infinity and with equality (8), then

$$|d(Sz,Iz)| \precsim \alpha \left| \max\left\{ 0, \frac{d(Iz,Sz)}{2} \right\} \right|$$

$$\left(1-\frac{\alpha}{2}\right)|d(Sz,Iz)| \le 0,$$

which is possible for Iz = Sz since $\alpha \in (0, 1)$. Therefore, Sz= Iz= u, that is, z is a coincidence point of the pair {S, I}.

Also weak compatibility of the mappings S and I implies the following equation:

$$ISz = SIz = Iu = Su$$

Besides, since S(X) $\subseteq$ J(X), there exist some $\omega \in X$ such that Sz = Jω, we claim that Tω = u. Then from (7), we have

$$d(Sz,T\omega) \precsim \alpha \max\left\{ \frac{d(Iz,T\omega)[1+d(Iz,Sz)\,d(J\omega,T\omega)+d(J\omega,Sz)\,d(Iz,T\omega)]}{2[1+d(Iz,J\omega)]}, \frac{d(J\omega,Sz)[1+d(Iz,Sz)\,d(Iz,T\omega)+d(J\omega,Sz)\,d(J\omega,T\omega)]}{2[1+d(Iz,J\omega)]} \right\}$$

$$d(u,T\omega) \precsim \alpha \max\left\{ \frac{d(u,T\omega)[1+d(u,u)\,d(u,T\omega)+d(Sz,Sz)\,d(u,T\omega)]}{2[1+d(u,u)]}, \frac{d(Sz,Sz)[1+d(u,u)\,d(u,T\omega)+d(Sz,Sz)\,d(u,T\omega)]}{2[1+d(u,u)]} \right\}$$

$$|d(u,T\omega)| \precsim \alpha \left| \max\left\{ \frac{d(u,T\omega)}{2}, 0 \right\} \right|$$

$$\left(1-\frac{\alpha}{2}\right)|d(u,T\omega)| \le 0,$$

which implies that $T\omega$ = u, since $\alpha \in (0, 1)$.

Hence $Iz = Sz = u = T\omega = J\omega.$

and this shows that ω is a coincidence point of the pair {T, J}. Weak compatibility of the pair {T, J} yields that TJ ω =JT ω = Tu = Ju. In conclusion we show that u is a common fixed point of I, J, S and T. Using (7) we get

$$d(u,Tu) = d(Sz,Tu) \precsim \alpha \max \left\{ \frac{d(Iz,Tu)[1+d(Iz,Sz)\,d(Ju,Tu)+d(Ju,Sz)\,d(Iz,Tu)]}{2[1+d(Iz,Ju)]}, \frac{d(Ju,Sz)[1+d(Iz,Sz)\,d(Iz,Tu)+d(Ju,Sz)\,d(Ju,Tu)]}{2[1+d(Iz,Ju)]} \right\}$$

$$d(u,Tu) = d(Sz,Tu) \precsim \alpha \max \left\{ \frac{d(u,Tu)[1+d(u,u)\,d(u,Tu)+d(u,u)\,d(u,Tu)]}{2[1+d(u,u)]}, \frac{d(u,u)[1+d(u,u)\,d(u,Tu)+d(u,u)\,d(u,Tu)]}{2[1+d(u,u)]} \right\}$$

Thus, $\left(1-\frac{\alpha}{2}\right)|d(u,Tu)| \le 0,$

and hence Tu = u which is the desired result. The uniqueness of common fixed point u follows easily. The details of the proof of this theorem can be obtained by using the argument that the pair {S, I} satisfies (CLR_S) -property with suitable modifications. This completes the proof.

Example 3.5 Let $X = (0,5]$ be equipped with the complex valued metric space

$$d(x,y) = |x-y|i$$

Let S, T, I and J be self-maps of X, defined by

$$S(x) = \begin{cases} 1 & ; \text{ if } x \in \{1\} \cup (1,5] \\ \frac{1}{5} & ; \text{ if } x \in (0,1) \end{cases}$$

$$T(x) = \begin{cases} 1 & ; \text{ if } x \in \{1\} \cup (1,5] \\ \frac{1}{4} & ; \text{ if } x \in (0,1) \end{cases}$$

$$I(x) = \begin{cases} 1 & ; \text{ if } x \in (0,5) \\ \frac{1}{5} & ; \text{ if } x \in (0,1) \\ \frac{2x+6}{4} & ; \text{ if } x \in (1,5] \end{cases}$$

$$J(x) = \begin{cases} 1 & ; \text{ if } x = 1 \\ \frac{1}{4} & ; \text{ if } x \in (0,1) \cup \{5\} \\ 2x-1 & ; \text{ if } x \in [1,5) \end{cases}$$

Clearly,

$$S(x) = \left\{1,\ \frac{1}{5}\right\} \subseteq [1, 4] \cup \left\{\frac{1}{5}\right\} = I(x),$$

$$T(x) = \left\{1,\ \frac{1}{4}\right\} \subseteq (0, 1] \cup \left\{\frac{1}{4}\right\} = J(x),$$

Let $x_n = \left\{5 - \frac{1}{n^2+1}\right\}$ & $y_n = \left\{1 + \frac{1}{n}\right\}$ be two sequences in X.

The pairs (S, I) and (T, J) satisfy (E.A)-property. Also the pair (S, I) or (T, J) satisfy (CLR_S) or (CLR_T)-property. Clearly, the pairs (S, I) and (T, J) are weakly compatible. "1" is the unique common fixed point of S, T, I and J.

IV. CONCLUSIONS

In this paper, we discussed some common fixed point theorems satisfying contractive conditions involving rational expression for two pairs of weakly compatible self mappings in complex valued metric spaces. Theorems established in this article will be helpful for researchers to work on rational contractions and derived some more common fixed point.

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