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Gaussian Prime Labeling of Some Cycle Related Graphs

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Abstract—Gaussian Prime labeling is a bijection from the set of vertices of graph G of order n to the first 'n' Gaussian integers in the spiral ordering such that images of adjacent vertices are relatively prime. A graph G is called Gaussian prime graph if G admits Gaussian prime labeling. In this paper, we examine Gaussian prime labeling of some cycle related graphs and union of some cycle graphs.

Keywords—Gaussian integers, spiral order of Gaussian integers, Gaussian prime labeling.

I. INTRODUCTION

The concept of spiral ordering in Gaussian integers was introduced by Hunter Lehmann and Andrew Park[1]. By considering this ordering they defined Gaussian prime labeling inspired by prime labeling on natural numbers. So we can say that the Gaussian prime labeling is an extension of prime labeling in some sense.

Entringer conjectured that trees are prime graph. However, this conjecture has not been established up till now for all trees. A remarkable result that any tree with ≤ 72 vertices is Gaussian prime tree proved by Hunter Lehmann and Andrew Park[1]. S. Klee, H. Lehmann and A. Park[2] also proved that the path graph, star graph, n-centipede tree, $(n,2)-$ centipede tree, spider tree, (n,k,m) double star tree, $(n,3)$ firecracker tree are Gaussian prime graph. Let us start with the some definitions and the basic results of Gaussian integer before introducing the main results. We will use spiral ordering of Gaussian integers and its properties given by Steven Klee et al. [2].

The paper contains four sections in which Section I contains the introduction of Gaussian prime labeling and recent finding in Gaussian prime labeling. Section II contains introduction of spiral ordering of Gaussian integers. In Sections III Main results of this paper has been given. In Section IV concluding remarks indicates future directions and further scope of results.

II. SPIRAL ORDERING OF GAUSSIAN INTEGERS

Gaussian integers are complex numbers of the form $\gamma = x + iy$ where x and y are integers and $i^2 = -1$. The set of Gaussian integers is usually denoted by $Z[i]$. A Gaussian integer γ is called even if $1+i$ divides γ. Otherwise, it is an odd Gaussian integer. The **norm** on $Z[i]$ is defined as $d(x+iy) = x^2 + y^2$. $\pm 1, \pm i$ are the only units of $Z[i]$. The associates of γ are unit multiple of γ. In $Z[i]$, γ and γ' are relatively prime if the common divisors of γ and γ' are the only units of $Z[i]$. A γ is said to be prime Gaussian integer if and only if $\pm 1, \pm i, \pm\gamma, \pm i\gamma$ are the only divisors of γ.

In [2] Steven Klee et al. introduced the **Spiral ordering of the Gaussian integer**. The recursion relation of Spiral ordering of the Gaussian integers starting with $\gamma_1 = 1$ is defined as follows:

$$\gamma_{n+1} = \begin{cases} \gamma_n + i \ ; \text{if } \mathrm{Re}(\gamma_n) \equiv 1 \,(\text{mod } 2),\ \mathrm{Re}(\gamma_n) > \mathrm{Im}(\gamma_n) + 1 \\ \gamma_n - 1 \,; \text{if } \mathrm{Im}(\gamma_n) \equiv 0 \,(\text{mod } 2),\ \mathrm{Re}(\gamma_n) \le \mathrm{Im}(\gamma_n) + 1,\ \mathrm{Re}(\gamma_n) > 1 \\ \gamma_n + 1 \,; \text{if } \mathrm{Im}(\gamma_n) \equiv 1 \,(\text{mod } 2),\ \mathrm{Re}(\gamma_n) < \mathrm{Im}(\gamma_n) + 1 \\ \gamma_n + i \,; \text{if } \mathrm{Im}(\gamma_n) \equiv 0 \,(\text{mod } 2),\ \mathrm{Re}(\gamma_n) = 1 \\ \gamma_n - i \,; \text{if } \mathrm{Re}(\gamma_n) \equiv 0 \,(\text{mod } 2),\ \mathrm{Re}(\gamma_n) \ge \mathrm{Im}(\gamma_n) + 1,\ \mathrm{Im}(\gamma_n) > 0 \\ \gamma_n - i \,; \text{if } \mathrm{Re}(\gamma_n) \equiv 0 \,(\text{mod } 2),\ \mathrm{Im}(\gamma_n) = 0 \end{cases}$$

The notation γ_n is used to denote the n^{th} Gaussian integer under the above ordering. We symbolically write first $'n'$ Gaussian integers by $[\gamma_n]$. Index of Gaussian integer $x+iy$ in the spiral ordering is denoted by $I(x+iy)$.

In [1] Steven Klee et al. had already established some interesting basic properties about Gaussian integers with the above ordering like :

- Any two consecutive Gaussian integer are relatively prime.
- Any two consecutive odd Gaussian integer are relatively prime.
- γ and $\gamma+\mu$ are relatively prime, if γ is a Gaussian integer and μ is a unit.
- γ and $\gamma+\mu(1+i)^k$ are relatively prime, if γ is an odd Gaussian integer and μ is a unit. where k is a positive integer.
- γ and $\gamma+\pi$ are relatively prime if and only if π does not divides γ where γ is Gaussian integer and π is a prime Gaussian integer.

Now let us prove one simple lemma which we will use to prove some of our results.

Lemma 2.1: $\gamma_{2^{2n}}$ and $\gamma_{2^n(2^n+1)+2}$ are relatively prime to any odd Gaussian integers.

Proof: We observe that prime factorizations of the Gaussian integers of the form 2^n and $2^n+2^n i$ are $(1+i)^{2n}$ and $(1+i)^{2n+1}$ respectively.

By the definition of even and odd Gaussian integers, any Gaussian integer whose prime factorization is of the form $(1+i)^m$ for some positive integer m is relatively prime to any odd Gaussian integer.

Now from Theorem 2.7 in [6], we have

$I(2^n+0i)=(2^n)^2=2^{2n}$ and

$I(2^n+2^n i)=(2^n+1)^2-2^n+1=2^n(2^n+1)+2$

Thus, $\gamma_{2^{2n}}=2^n$ and $\gamma_{2^n(2^n+1)+2}=2^n+2^n i$ .Hence $\gamma_{2^{2n}}$ and $\gamma_{2^n(2^n+1)+2}$ are relatively prime to odd Gaussian integers.

Definition 2.2: Let G be a graph having n vertices. A bijective function $g:V(G)\to[\gamma_n]$ is called *Gaussian prime labeling*, if the images of adjacent vertices are relatively prime. A graph which admits Gaussian prime labeling is known as *Gaussian prime graph*.

Definition 2.3: The n *-sunlet graph* of $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n .

Definition 2.4: An *Independent set* of vertices in a graph G is a set of mutually non-adjacent vertices.

Definition 2.5: The *Independence number* of a graph G is the maximum cardinality of an independent set of vertices. It is denoted by $\alpha(G)$.

Let G be a Gaussian prime graph of order n . It is easy to see that the set of vertices which are labeled with even Gaussian integers less or equal to γ_n is an independent set. Hence we have the following similar lemma as lemma 2.1 in [3].

Lemma 2.6: If G is a Gaussian prime graph of order n then the independence number $\alpha(G)\ge\left\lfloor\frac{n}{2}\right\rfloor$.

Definition 2.7: The graph $C_n^{(k)}$ (where $k\ge 2$) is known as the *one point union of k copies of the cycle C_n* obtained from the k copies of the cycle C_n by identifying exactly one vertex of each of these k copies of C_n .

We can easily see that the graph C_n^k contains $k(n-1)+1$ vertices and kn edges. From past many years, the various types of labeling techniques are used to study such types of graphs, but in this paper we prove some results for union of such graphs for Gaussian prime labeling. Just like C_n , it is very simple to prove that C_n^k is Gaussian prime graph for all n and k . But thinking about union of C_n and C_n^k is little bit tricky.

In this paper, we considered all graphs are undirected, finite and simple. We follow Gross and Yellen[4] for notations and graph theoretic terminology and D. M. Burton[5] for number theoretic results. Throughout this paper, we will understand that the graph is Gaussian prime graph means to be a

Gaussian prime graph with respect to the spiral ordering of Gaussian integers.

III. Results and Discussion

Theorem 3.1: $C_n \cup C_m$ is a Gaussian prime graph if and only if either m or n is even.

Proof: Let $G = C_n \cup C_m$. If n and m both are odd, then $\alpha(G) = \frac{n+m}{2} - 1 < \frac{n+m}{2} = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ and so from Lemma 2.6, G is not a Gaussian prime graph. However, if n is even and $\{v_1, v_2, ..., v_n\}$ and $\{u_1, u_2, ..., u_m\}$ are the vertex sets of C_n and C_m respectively then it is easy to see that

$h: V(G) \to [\gamma_{n+m}]$ defined by

$$h(u_1) = \gamma_1$$

$$h(v_i) = \gamma_{i+1} \;;\quad 1 \le i \le n,$$

$$h(u_i) = \gamma_{i+n} \;;\quad 2 \le i \le m,$$

is a Gaussian prime labeling of G. Thus, G is a Gaussian prime graph if and only if either n or m is even.

Theorem 3.2: $\bigcup_{i=1}^{m} C_{n_i}$ is not a Gaussian prime graph if C_{n_i}'s are odd cycles, where $m > 1$.

Proof: Let $G = \bigcup_{i=1}^{m} C_{n_i}$. We know that if n is odd then independence number $\alpha(C_n) = \frac{n-1}{2}$

Case 1: m is even

$$\begin{aligned}
\alpha(G) &= \frac{n_1 - 1}{2} + \frac{n_2 - 1}{2} + ... + \frac{n_m - 1}{2} \\
&= \frac{n_1}{2} - \frac{1}{2} + \frac{n_2}{2} - \frac{1}{2} + ... + \frac{n_m}{2} - \frac{1}{2} \\
&= \frac{n_1}{2} + \frac{n_2}{2} + ... + \frac{n_m}{2} - \frac{m}{2} \\
&< \frac{n_1}{2} + \frac{n_2}{2} + ... + \frac{n_m}{2} \\
&= \frac{n_1 + n_2 + ... + n_m}{2} = \frac{|V(G)|}{2} = \left\lfloor \frac{|V(G)|}{2} \right\rfloor
\end{aligned}$$

Case 2: m is odd

$$\begin{aligned}
\alpha(G) &= \frac{n_1 - 1}{2} + \frac{n_2 - 1}{2} + ... + \frac{n_m - 1}{2} \\
&= \frac{n_1}{2} - \frac{1}{2} + \frac{n_2}{2} - \frac{1}{2} + ... + \frac{n_m}{2} - \frac{1}{2} \\
&= \frac{n_1}{2} + \frac{n_2}{2} + ... + \frac{n_m}{2} - \frac{m}{2} \\
&< \frac{n_1}{2} + \frac{n_2}{2} + ... + \frac{n_m}{2} - \frac{1}{2} \\
&= \frac{n_1 + n_2 + ... + n_m}{2} - \frac{1}{2} = \frac{|V(G)|}{2} - \frac{1}{2} = \left\lfloor \frac{|V(G)|}{2} \right\rfloor
\end{aligned}$$

Hence by Lemma 2.6, G is not a Gaussian prime graph.

Theorem3.3:
$C_{2^{2n}-2} \cup C_{3\cdot 2^{2n}} \cup C_{3\cdot 2^{2(n+1)}} \cup C_{3\cdot 2^{2(n+2)}} \cup ... \cup C_{3\cdot 2^{2(n+k)}} \cup C_{2m} \cup C_r$ is a Gaussian prime graph for any positive integers n, m, k and r.

Proof: Let
$G = C_{2^{2n}-2} \cup C_{3\cdot 2^{2n}} \cup C_{3\cdot 2^{2(n+1)}} \cup C_{3\cdot 2^{2(n+2)}} \cup ... \cup C_{3\cdot 2^{2(n+k)}} \cup C_{2m} \cup C_r$ and $|V(G)| = 2^{2(n+k+1)} + 2m + r - 2$. Let

$\{v_1, v_2, ..., v_{2^{2n}-2}, v_{2^{2n}-1}, v_{2^{2n}}, ..., v_{2^{2(n+1)}-2}, v_{2^{2(n+1)}-1}, v_{2^{2(n+1)}}, ...,$
$v_{2^{2(n+2)}-2}, ..., v_{2^{2(n+k)}-1}, v_{2^{2(n+k)}}, ..., v_{2^{2(n+k+1)}-2}, v_{2^{2(n+k+1)}-1}, v_{2^{2(n+k+1)}}, ...$
$v_{2^{2(n+k+1)}+2m-2}, v_{2^{2(n+k+1)}+2m-1}, v_{2^{2(n+k+1)}+2m}, ..., v_{2^{2(n+k+1)}+2m+r-2}\}$

be the set of vertices of graph G consisted by consecutive vertices of cycle graphs $C_{2^{2n}-2}, C_{3*2^{2n}}, C_{3*2^{2(n+1)}} C_{3*2^{2(n+2)}}$, $..., C_{3*2^{2(n+k)}}, C_{2m}, C_r$.

Now define a bijection $f: V(G) \to [\gamma_{2^{2(n+k+1)}+2m+r-2}]$ as

$$f(x) = \begin{cases} \gamma_{2^{2n}-1} & ;\; x = v_1 \\ \gamma_{2^{2(n+i)}-1} & ;\; x = v_{2^{2(n+i-1)}-1},\; 1 \le i \le k+1 \\ \gamma_{2^{2(n+k+1)}+2m-1} & ;\; x = v_{2^{2(n+k+1)}-1} \\ \gamma_1 & ;\; x = v_{2^{2(n+k+1)}+2m-1} \\ \gamma_j & ;\; \text{for remaining } x = v_j \end{cases}$$

The labels of all the adjacent vertices of G are consecutive Gaussian integers except the pairs v_1, v_2 & $v_{2^{2(n+i-1)}-1}, v_{2^{2(n+i-1)}}$ for $1 \le i \le k+1$ and $v_{2^{2(n+k)}+2m-1}$, $v_{2^{2(n+k)}+2m}$. we can easily see that $gcd(f(v_1), f(v_2)) =$

$gcd(\gamma_{2^{2n}-1},\gamma_2)=1$ and $gcd(f(v_{2^{2(n+k)}+2m-1}), f(v_{2^{2(n+k)}+2m}))=gcd(\gamma_1,\gamma_{2^{2(n+k)}+2m})=1$.

Now in view of Lemma 2.1, $gcd(f(v_{2^{2(n+i)}-1}), f(v_{2^{2(n+i)}}))=gcd(\gamma_1,\gamma_{2^{2(n+k)}+2m})=1$ for $0\le i\le k$. Hence G is a Gaussian prime graph.

By using Lemma 2.1 and the similar argument as in theorem 3.3 we can easily prove the following theorem.

Theorem3.4: $C_{2^n(2^n+1)}\cup C_{3\cdot 2^{2n}+2^n}\cup C_{3\cdot 2^{2(n+1)}+2^{(n+1)}}\cup\ldots\cup C_{3\cdot 2^{2(n+k)}+2^{(n+k)}}\cup C_{2m}\cup C_r$ is a Gaussian prime graph for any positive integers n,m,k and r.

Theorem 3.5: Two cycles C_n and C_m joined by path P_k is a Gaussian Prime Graph if and only if either m or n is even.

Proof: Let G be the graph obtained by joining cycles C_n and C_m by path a P_k.

We note that $|V(G)|=n+m+k-2$

Let $x_1,x_2,...,x_n$ be the consecutive vertices of cycle C_n and $y_1,y_2,...,y_m$ be the consecutive vertices of cycle C_m and $z_1,z_2,...,z_k$ be the consecutive vertices of path P_k with $x_1=z_1$ and $y_1=z_k$.

<u>Case 1:</u> n and k are of opposite parity and m is even

Define a bijection $f:V(G)\to[\gamma_{n+m+k-2}]$ as

$$f(x)=\begin{cases}\gamma_1 & ;\ x=x_1\\ \gamma_2 & ;\ x=y_1\\ \gamma_{k+2-i} & ;\ x=z_i,\ 2\le i\le k\\ \gamma_{k+i-1} & ;\ x=x_i,\ 2\le i\le n\\ \gamma_{n+k+i-2} & ;\ x=y_i,\ 2\le i\le m\end{cases}$$

Observe that $N(x_1)=\{z_2,x_2,x_n\}$ and $N(y_1)=\{z_{k-1},y_2,y_m\}$

Since $f(x_1)=\gamma_1$, $gcd(f(x_1),f(z_2))=gcd(f(x_1),f(x_2))=gcd(f(x_1),f(x_n))=1$ and since $f(y_1)=\gamma_2$ and $\gamma_3,\gamma_{n+k},\gamma_{n+k+m-2}$ are odd Gaussian integers, $gcd(f(y_1),f(z_{k-1}))=gcd(f(y_1),f(y_2))=gcd(f(y_1),f(y_m))=1$

The labels of all the remaining vertices are consecutive Gaussian integers. So we are through.

<u>Case 2:</u> n and k are even and m is odd

In this case we interchange the labels of x_i and y_i for each i and reverse the labels of all z_i except z_1 and z_k in the labeling of case 1. This new bijection is a Gaussian prime labeling of G and hence G is a Gaussian prime graph.

<u>Case 3:</u> n, m and k are even

Define a bijection $f:V(G)\to[\gamma_{n+m+k-2}]$ as

$$f(x)=\begin{cases}\gamma_1 & ;\ x=x_1\\ \gamma_{m+i} & ;\ x=z_i,\ 2\le i\le k\\ \gamma_{m+k+i-2} & ;\ x=x_i,\ 2\le i\le n\\ \gamma_{i+1} & ;\ x=y_i,\ 1\le i\le m\end{cases}$$

Observe that $N(x_1)=\{z_2,x_2,x_n\}$ and $N(y_1)=\{z_{k-1},y_2,y_m\}$

Since $f(x_1)=\gamma_1$, $gcd(f(x_1),f(z_2))=gcd(f(x_1),f(x_2))=gcd(f(x_1),f(x_n))=1$ and since $f(y_1)=\gamma_2$ and $\gamma_3,\gamma_{n+k},\gamma_{n+k+m-2}$ are odd Gaussian integers,

$gcd(f(y_1),f(z_{k-1}))=gcd(f(y_1),f(y_2))=gcd(f(y_1),f(y_m))=1$

The labels of all the remaining vertices are consecutive Gaussian integers. So we are done.

<u>Case-4:</u> m and k are odd and n is even

Define a bijection $f:V(G)\to[\gamma_{n+m+k-2}]$ as

$$f(x)=\begin{cases}\gamma_1 & ;\ x=y_m\\ \gamma_{n+i} & ;\ x=z_i,\ 1\le i\le k\\ \gamma_i & ;\ x=x_i,\ 2\le i\le n\\ \gamma_{n+k+i} & ;\ x=y_i,\ 2\le i\le m-1\end{cases}$$

Observe that $N(y_m)=\{y_1,y_{m-1}\}$

Since $f(y_m)=\gamma_1$ and γ_{n+1} is an odd Gaussian integer, $gcd(f(y_m),f(y_{m-1}))=gcd(f(y_m),f(y_1))=1$ and $gcd(f(x_1),f(x_2))=gcd(\gamma_{n+1},\gamma_2)=1$.

The labels of all the remaining vertices are consecutive Gaussian integers. So we are through.

<u>Case 5:</u> n and k are odd and m is even

In the labeling of case 5, we interchange the labels of x_i and y_i for each i and reverse the labels of all z_i except z_1 and z_k. This new bijection is a Gaussian prime labeling of G.

Hence G is a Gaussian prime graph if either m or n is even.

Now we prove that G is not a Gaussian prime graph if both m and n are odd. This leads to two cases

<u>Case 1:</u> n, k and m are odd

In this case, the independence number $\alpha(G)=\frac{m-1}{2}+\frac{n-1}{2}+\frac{k-2}{2}=\frac{m+n+k}{2}-2$

and since $\frac{m+n+k}{2}-2<\frac{m+n+k}{2}-1$, $\alpha(G)<\left\lfloor\frac{|V(G)|}{2}\right\rfloor$

Case 2: n and m are odd and k is even

In this case, the independence number $\alpha(G)=\frac{m-1}{2}+\frac{n-1}{2}+\frac{k-1}{2}=\frac{m+n+k}{2}-\frac{3}{2}$

and since $\frac{m+n+k}{2}-\frac{3}{2}<\frac{m+n+k}{2}-1$, $\alpha(G)<\left\lfloor\frac{|V(G)|}{2}\right\rfloor$

Thus by Lemma 2.6, G is not a Gaussian prime graph if both m and n are odd.

Theorem 3.6: If n and m both are odd, then $C_n^{(j)}\cup C_m^{(k)}$ is not a Gaussian prime graph.

Proof: Let $G=C_n^{(j)}\cup C_m^{(k)}$. Since n and m are odd, the independence numbers $\alpha(C_n)=\frac{n-1}{2}$ and $\alpha(C_m)=\frac{m-1}{2}$.

Therefore, $\alpha(G)=j(\frac{n-1}{2})+k(\frac{m-1}{2})$.

But $|V(G)|=j(n-1)+k(m-1)+2$ and so

$\left\lfloor\frac{|V(G)|}{2}\right\rfloor=\frac{j(n-1)+k(m-1)}{2}+1$.

Thus, $\alpha(G)<\left\lfloor\frac{|V(G)|}{2}\right\rfloor$.

Hence by Lemma 2.6, we can conclude that G is not a Gaussian Prime Graph.

Theorem 3.7: Let $G=\left(\bigcup_{k=1}^{N}C_{n_k}^{(2)}\right)\cup\left(\bigcup_{j=1}^{M}C_{m_j}^{(2)}\right)$, where each n_k is an odd integer and each m_j is an even integer. Then G is not a Gaussian prime graph if $M\le N-2$.

Proof: We know that independence number of each $C_{n_k}^{(2)}$ and each $C_{m_j}^{(2)}$ is n_k-1 and m_j respectively, therefore we have

$$\alpha(G)=\sum_{k=1}^{N}(n_k-1)+\sum_{j=1}^{M}m_j=\sum_{k=1}^{N}n_k+\sum_{j=1}^{M}m_j-N \qquad (1)$$

Also,

$$|V(G)|=(2n_1-1)+(2n_2-1)+\ldots+(2n_N-1)+(2m_1-1)+(2m_2-1)+\ldots+(2m_M-1)$$

$$=\sum_{k=1}^{N}2n_k+\sum_{j=1}^{M}2m_j-(N+M).$$

So, $$\left\lfloor\frac{|V(G)|}{2}\right\rfloor=\sum_{k=1}^{N}n_k+\sum_{j=1}^{M}m_j-\left\lceil\frac{N+M}{2}\right\rceil. \qquad (2)$$

Since $M\le N-2$, it follows from (1) and (2) that, $\alpha(G)<\left\lfloor\frac{|V(G)|}{2}\right\rfloor$.

Therefore, G is not a Gaussian prime graph due to Lemma 2.6.

Theorem 3.8: W_n is a Gaussian Prime Graph if n is even.

Proof: Let $V(W_n)=\{v_0,v_1,v_2,,v_n\}$ where v_0 is the apex vertex and remaining v_i's are consecutive rim vertices of W_n.

Now define a bijection $f:V(W_n)\to[\gamma_{n+1}]$ as $f(v_i)=v_{i+1},\ 0\le i\le n.$

Here labels of all the adjacent pairs of vertices are either consecutive Gaussian integers or one of the label is γ_1 except the pair v_1,v_n. But $gcd(f(v_1),f(v_n))=gcd(\gamma_2,\gamma_{n+1})=1$ because γ_{n+1} is an odd Gaussian integer.

Hence W_n is a Gaussian prime graph if n is even.

Theorem 3.9: $\bigcup_{i=1}^{k}W_{n_i}$ is not a Gaussian prime graph if n_i's are odd.

Proof: Let $G=\bigcup_{i=1}^{k}W_{n_i}$. We know that,

$$|V(G)|=n_1+1+n_2+1+\ldots+n_k+1$$

$$=n_1+n_2+\ldots+n_k+k=\sum_{i=1}^{k}n_i+k$$

Since $\alpha(W_n)=\left\lfloor\frac{n}{2}\right\rfloor$,

$$\alpha(G)=\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lfloor\frac{n_2}{2}\right\rfloor+\ldots+\left\lfloor\frac{n_k}{2}\right\rfloor$$

$$\le\frac{n_1}{2}+\frac{n_2}{2}+\ldots+\frac{n_k}{2}$$

$$=\frac{n_1+n_2+\ldots+n_k}{2}$$

$$<\frac{\sum_{i=1}^{k}n_i+k}{2}=\frac{|V(G)|}{2}=\left\lfloor\frac{|V(G)|}{2}\right\rfloor$$

Therefore, G is not a Gaussian Prime Graph by Lemma 2.6.

By Theorem 3.8 and Theorem 3.9 we can conclude that W_n is a Gaussian Prime Graph if and only if n is even.

Theorem 3.10: The helm graph H_n is a Gaussian prime graph.

Proof: Let $V(H_n)=\{v_0,v_1,v_2,...,v_n,v_1',v_2',...,v_n'\}$ where v_0 is the apex vertex and

$$E(H_n)=\{v_0v_i;1\le i\le n\}\cup\{v_iv_i';1\le i\le n-1\}\cup\{v_iv_{i+1};1\le i\le n-1\}\cup\{v_1v_n\}$$

Define a bijection $f:V(H_n)\to[\gamma_{2n+1}]$ as

$$f(x)=\begin{cases}\gamma_1 & ;\ x=v_0\\ \gamma_2 & ;\ x=v_1\\ \gamma_3 & ;\ x=v_1'\\ \gamma_{2i+1} & ;\ x=v_i,\ 2\le i\le n\\ \gamma_{2i} & ;\ x=v_i',\ 2\le i\le n\end{cases}$$

The labels of all the adjacent pairs of vertices are either consecutive Gaussian integers or consecutive odd Gaussian integers or one of the label is γ_1 except the pair v_1,v_n. But $gcd(f(v_1),f(v_n))=gcd(\gamma_2,\gamma_{2n+1})=1$ because γ_{2n+1} is odd gaussian integer.

Then f is a Gaussian prime labeling and hence H_n is a Gaussian prime graph.

Theorem 3.11: The sunlet graph S_n is a Gaussian prime graph.

Proof: Let $V(S_n)=\{v_1,v_2,...,v_n,v_1',v_2',...,v_n'\}$ and

$$E(S_n)=\{v_iv_i';1\le i\le n\}\cup\{v_iv_{i+1};1\le i\le n-1\}\cup\{v_1v_n\}$$

Now define a bijection $f:V(G)\to[\gamma_{2n}]$ as

$$f(x)=\begin{cases}\gamma_{2i-1} & ;\ x=v_i,\ 1\le i\le n\\ \gamma_{2i} & ;\ x=v_i',\ 1\le i\le n\end{cases}$$

The labels of all the adjacent pair of vertices are either consecutive Gaussian integers or consecutive odd Gaussian integers or one of the label is γ_1 Then f is a Gaussian prime labeling and hence S_n is a Gaussian prime graph.

Theorem 3.12: The union of two sunlet graphs is a Gaussian prime graph.

Proof: Let $G=S_n\cup S_m$ and

$$V(G)=\{v_1,v_2,...,v_n,v_1',v_2',...,v_n',u_1,u_2,...,u_n,u_1',u_2',...,u_n'\}$$

and

$$E(G)=\{v_iv_i';1\le i\le n\}\cup\{v_iv_{i+1};1\le i\le n-1\}\cup\{u_iu_i';1\le i\le n\}\cup\{u_iu_{i+1};1\le i\le n-1\}\cup\{v_1v_n,u_1u_n\}$$

Here $|V(G)|=4n$

Now Define a bijection $f:V(G)\to[\gamma_{4n}]$ as

$$f(x)=\begin{cases}\gamma_1 & ;\ x=u_1\\ \gamma_2 & ;\ x=v_1\\ \gamma_3 & ;\ x=v_1'\\ \gamma_{2n} & ;\ x=u_1'\\ \gamma_{2i+1} & ;\ x=v_i,\ 2\le i\le n\\ \gamma_{2i} & ;\ x=v_i',\ 2\le i\le n\\ \gamma_{2n+2i-1} & ;\ x=u_i,\ 2\le i\le m\\ \gamma_{2n+2i-2} & ;\ x=u_i',\ 2\le i\le m\end{cases}$$

The labels of all the adjacent pairs of vertices are either consecutive Gaussian integers or consecutive odd Gaussian integers or one of the label is γ_1 except the pair v_1,v_n. But $gcd(f(v_1),f(v_n))=gcd(\gamma_2,\gamma_{2n+1})=1$ because γ_{2n+1} is odd gaussian integer.

Then f is a Gaussian prime labeling and hence G is aa Gaussian prime graph.

Theorem 3.13: Two sunlet graph joined by path P_k is a Gaussian prime graph.

Proof: Let G be a graph obtained by joining sunlet graphs S_n and S_m by a path P_k.

We note that $|V(G)|=2n+2m+k-2$.

Let $\{u_1,u_2,...,u_n,u_1',u_2',...,u_n'\}$ be the vertex set of S_n and $\{v_1,v_2,...,v_m,v_1',v_2',...,v_m'\}$ be the vertex set of S_m and $\{w_1,w_2,...,w_k\}$ be the vertex set of path P_k with $u_1'=w_1$ and $v_1'=w_k$.

$$E(G)=\{v_iv_i';1\le i\le m\}\cup\{v_iv_{i+1};1\le i\le m-1\}\cup\{u_iu_i';1\le i\le n\}\cup\{u_iu_{i+1};1\le i\le n-1\}\cup\{w_iw_{i+1};1\le i\le k-1\}\cup\{v_1v_m,u_1u_n\}$$

Now if k is even, then we define a bijection $f:V(G)\to[\gamma_{2n+2m+k-2}]$ as

$$f(x)=\begin{cases}\gamma_1 & ;\ x=v_1\\ \gamma_2 & ;\ x=u_1\\ \gamma_{2n+k} & ;\ x=v_1^{'}\\ \gamma_{2n+1} & ;\ x=u_1^{'}\\ \gamma_{2n+k+2i-3} & ;\ x=v_i\,, 2\le i\le m\\ \gamma_{2n+k+2i-2} & ;\ x=v_i^{'}\,, 2\le i\le m\\ \gamma_{2i-1} & ;\ x=u_i\,, 2\le i\le n\\ \gamma_{2i} & ;\ x=u_i^{'}\,, 2\le i\le n\\ \gamma_{2n+i} & ;\ x=w_i\,, 2\le i\le k-1\end{cases}$$

and if k is odd integer, then we have to do only one change in the above labeling that is to interchange labels of v_i and $v_i^{'}$. So in both the cases, labels of all the adjacent pairs of vertices are either consecutive Gaussian integers or consecutive odd Gaussian integers or one of the label is γ_1 except the pairs u_1, u_n and $u_1, u_1^{'}$. But $gcd(f(u_1), f(u_n)) = gcd(\gamma_2, \gamma_{2n-1}) = 1$ and $gcd(f(u_1), f(u_1^{'})) = gcd(\gamma_2, \gamma_{2n+1}) = 1$ because γ_{2n-1} and γ_{2n+1} are odd gaussian integers.

Then f is a Gaussian prime labeling and hence G is a Gaussian prime graph.

IV. CONCLUDING REMARKS

It may be interesting to find necessary and sufficient conditions for which finite union of disjoint cycles of arbitrary length are Gaussian prime graph. It may be somewhat tricky but interesting to find solution of above stated problem for infinite union of disjoint cycles. The same problem can also be discussed for disjoint union of one point union of cycles.

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