International Journal of Scientific Research in Mathematical and Statistical Sciences
Review Paper
Volume-5, Issue-6, pp.301-307, December (2018)
E-ISSN: 2348-4519

Application of Cauchy Characteristic in General Relativity

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Available online at: www.isroset.org

Received: 27/Nov/2018, Accepted: 16/Dec/2018, Online: 31/Dec/2018

Abstract- This paper reveals the development of numerical evolution codes for general relativity based upon the characteristic initial value problem. Progress is traced from the early stage of 1D feasibility studies to current 3D black hole codes that run forever. A prime application of characteristic evolution is Cauchy-characteristic matching, which is also reviewed.

Keywords**:** Cauchy-characteristic, general relativity, 3D codes, Hypersurfaces, etc.

I. INTRODUCTION

This paper is entering an era in which Einstein's equations can effectively be considered solved at the local level. Several groups, as reported in these Reviews, have developed 3D codes which are stable and accurate in some sufficiently local setting. Most work in numerical relativity is based upon the Cauchy "3 + 1" formalism, with the gravitational radiation extracted by perturbative Cauchy methods which introduce an artificial Schwarzschild background [1, 3, 2, 5]. These wave extraction methods have not been tested in a fully nonlinear 3D setting. Another approach specifically tailored to study radiation can be based upon the characteristic initial value problem. In the typical setting for a characteristic initial value problem, the domain of dependence of a single nonsingular null hypersurface is empty. In order to obtain a nontrivial evolution problem, the null hypersurface must either be completed to a caustic-crossover region where it pinches off, or an additional boundary must be introduced. So far, the only caustics that have been successfully evolved numerically in general relativity are pure point caustics (the complete null cone problem). When spherical symmetry is not present, it turns out that the stability conditions near the vertex of a nonsingular light cone place a strong restriction on the allowed time step. Point caustics in general relativity have been successfully handled this way for axisymmetric space-times, but the computational demands for 3D evolution would be prohibitive using current generation supercomputers. This is unfortunate because, away from the caustics, the characteristic evolution offers myriad computational and geometrical advantages.

As a result, at least in the near future, the computational application of characteristic evolution is likely to be restricted to some mixed form, in which boundary conditions are also set on a non-singular but incomplete initial null hypersurface and on a second nonsingular hypersurface (or perhaps several), which together with the initial null hypersurface present a nontrivial domain of dependence. This second hypersurface may itself be either (i) null, (ii) timelike or (iii) spacelike. These possibilities give rise to the (i) the double null problem, (ii) the nullcone-worldtube problem or (iii) the Cauchy-characteristic matching (CCM) problem.

In CCM, it is possible to choose the matching interface between the Cauchy and characteristic regions to be a null hypersurface, but it is more practical to match across a timelike worldtube. CCM combines the advantages of characteristic evolution in treating the outer radiation zone in spherical coordinates which are naturally adapted to the topology of the worldtube with the advantages of Cauchy evolution in Cartesian coordinates in the region where spherical coordinates would break down.

II. CHARACTERISTIC EVOLUTION SCHEMES

All characteristic evolution schemes share the same skeletal form. The fundamental ingredient is a foliation by null hypersurfaces $u = const$ which are generated by a 2-dimensional set of null rays, labeled by coordinates x^A, with a coordinate λ varying along the rays. In (u, λ, x^A) null coordinates, the main set of Einstein equations take the schematic form

$$F_{,\lambda} = H_F[F, G] \qquad 1$$

and

$$G_{,u\lambda} = H_G[F, G]. \qquad 2$$

Here F represents a set of hypersurface variables; G, a set of evolution variables; and H_F and H_G are nonlinear hypersurface operators, i.e. they operate locally on the values of F and G intrinsic to a single null hypersurface. In addition to these main equations, there is a subset of four Einstein equations which are satisfied by virtue of the

Bianchi identities, provided that they are satisfied on a hypersurface transverse to the characteristics. These equations have the physical interpretation as conservation laws. Mathematically they are analogous to the constraint equations of the canonical formalism. But they are not elliptic, since they are imposed upon null or timelike hypersurfaces, rather than spacelike.

III. EVOLUTION CODE

Computational implementation of characteristic evolution may be based upon different versions of the formalism (i.e. metric or tetrad) and different versions of the initial value problem (i.e. double null or worldtube-nullcone). The performance and computational requirements of the resulting evolution codes can vary drastically with the particular choice. However, all characteristic evolution codes have certain common advantages:

(i) There are no elliptic constraint equations. This eliminates the need for time consuming iterative methods to enforce constraints or to otherwise test the challenge of constraint free evolution.

(ii) No second time derivatives appear so that the number of basic variables is half the number for the corresponding version of the Cauchy problem.

(iii) The main Einstein equations form a system of coupled ordinary differential equations with respect to the parameter λ varying along the characteristics. This allows construction of an evolution algorithm in terms of a simple march along the characteristics.

(iv) In problems with isolated sources, the radiation zone can be compactified into a finite grid boundary using Penrose's conformal technique. Because the Penrose boundary is a null hypersurface, no extraneous outgoing radiation condition or other artificial boundary condition is required.

(v) The grid domain is exactly the region in which waves propagate, which is ideally efficient for radiation studies. Since each null hypersurface of the foliation extends to infinity, the radiation is calculated immediately (in retarded time) with no need to propagate it across the grid.

(vi) In black hole space-times, a large redshift at null infinity relative to internal sources is an indication of the formation of an event horizon and can be used to limit the evolution to the exterior region of space-time.

Characteristic schemes also share as a common disadvantage the necessity either to deal with caustics or to avoid them altogether. The scheme to tackle the caustics head on by including their development as part of the evolution is perhaps a great idea still ahead of its time, one that should not be forgotten. There are only a handful of structurally stable caustics, and they have well known algebraic properties. This makes it possible to model their singular structure in terms of Pade approximants. The structural stability of the singularities should in principle make this possible, and algorithms to evolve the elementary caustics have been proposed. In the axisymmetric case, cusps and folds are the only stable caustics, and they have already been identified in the horizon formation occurring in simulations of head-on collisions of black holes and in the temporarily toroidal horizons occurring in collapse of rotating matter.

One+One Dimensional Codes

It is now often said that the solution of the general ordinary differential equation is essentially known, in light of the success of computational algorithms and present day computing power. Perhaps this is an overstatement because investigating singular behavior is still an art. But, in the same vein, it is fair to say that the general system of hyperbolic partial differential equations in one spatial dimension is a solved problem. At least, it seems to be true in general relativity.

One of the earliest characteristic evolution codes was constructed by Corkill and Stewart to treat space-times with two Killing vectors. The grid was based upon a double null coordinate system, with the null hypersurfaces intersecting in the surfaces spanned by the Killing vectors. This allowed simulation of colliding plane waves (as well as the Schwarzschild solution). They were able to evolve the Khan-Penrose collision of impulsive (δ-function curvature) plane waves to within a few numerical zones from the singularity that forms after the collision, with extremely close agreement with the analytic results. Their simulations of collisions with more general waveforms, for which exact solutions are not known, have provided input to the theoretical understanding of singularity formation in this problem.

Most of the 1+1 dimensional applications of characteristic methods have been for spherically symmetric systems. Here matter must be included in order to make the system non-Schwarzschild. Except for some trivial applications to evolve dust cosmologies by characteristic evolution, matter has been represented by a massless Klein-Gordon field. This allows simulation of radiation effects in the simple context of spherical symmetry. Highly interesting results have been found this way.

On the analytic side, working in a characteristic initial value formulation based upon outgoing null cones, Christodoulou made a penetrating study of the existence and uniqueness of solutions to this problem. He showed that weak initial data evolve to Minkowski space asymptotically in time, but that sufficiently strong data form a horizon, with nonzero Bondi mass. In the latter case, he showed that the geometry is asymptotically Schwarzschild in the approach to I^+ (future time infinity) from outside the horizon, thus establishing a rigorous version of the no-hair theorem. What this analytic tour-de-force did not reveal was the remarkable critical behavior in the transition between these two regimes, which was discovered by Choptuik [26] using computational simulation.

A characteristic evolution algorithm for this problem centers about the evolution scheme for the scalar field, which constitutes the only dynamical field. Given the scalar field, all gravitational quantities can be determined by integration along the characteristics of the null foliation. But this is a coupled problem, since the scalar wave equation involves the curved space metric. It provides a good illustration of how null algorithms lead to a hierarchy of equations which can be integrated along the characteristics to effectively decouple the hypersurface and dynamical variables.

In a Bondi coordinate system based upon outgoing null hypersurfaces $u = const$, the metric is

ds2=−e2βdu(Vrdu+2dr)+r2(dθ2+sin2θdϕ2). 3

Smoothness at $r = 0$ allows the coordinate conditions

$V(u, r) = r + O(r^3)$ and $\beta(u, r) = O(r^2)$. 4

The field equations consist of the wave equation Φ = 0 for the scalar field and two hypersurface equations for the metric functions:

$\beta_{,r} = 2\pi r(\Phi_{,r})^2$ 5

$V_{,r} = e^{2\beta}$. 6

The wave equation can be expressed in the form

□(2)g−(Vr),re−2βgr=0, 7

where $g = r\Phi$ and $\Box^{(2)}$ is the D'Alembertian associated with the two dimensional submanifold spanned by the ingoing and outgoing null geodesics. Initial null data for evolution consists of $\Phi(u_0, r)$ at initial retarded time u_0.

Because any two dimensional geometry is conformally flat, the surface integral of $\Box^{(2)}g$ over a null parallelogram Σ gives exactly the same result as in a flat 2-space, and leads to an integral identity upon which a simple evolution algorithm can be based. Let the vertices of the null parallelogram be labeled by N, E, S and W corresponding, respectively, to their relative locations North, East, South and West in the 2-space. Upon integration of (7), curvature introduces an area integral correction to the flat space null parallelogram relation between the values of g at the vertices:

gN−gW−gE+gS=−12∫Σdudr(Vr),rgr.

This identity, in one form or another, lies behind all of the null evolution algorithms that have been applied to this system. The prime distinction between the different algorithms is whether they are based upon double null coordinates or Bondi coordinates as in Eq. (3). When a double null coordinate system is adopted, the points N, E, S and W can be located in each computational cell at grid points, so that evaluation of the left hand side of Eq. (8) requires no interpolation. As a result, in flat space, where the right hand side of Eq. (8) vanishes, it is possible to formulate an exact evolution algorithm. In curved space, of course, there is truncation error arising from the approximation of the integral by evaluating the integrand at the center of Σ.

The identity (8) gives rise to the following explicit marching algorithm. Let the null parallelogram lie at some fixed θ and ϕ and span adjacent retarded time levels u_0 and $u_0 + \Delta u$. Imagine for now that the points N, E, S and W lie on the spatial grid, with $r_N - r_W = r_E - r_S = \Delta r$. If g has been determined on the initial cone u_0, which contains the points E and S, and radially outward from the origin to the point W on the next cone $u_0 + \Delta u$, then Eq. (8) determines g at the next radial grid point N in terms of an integral over Σ. The integrand can be approximated to second order, i.e. to $O(\Delta r\Delta u)$, by evaluating it at the center of Σ. To this same accuracy, the value of g at the center equals its average between the points E and W, at which g has already been determined. Similarly, the value of $(V/r)_{,r}$ at the center of Σ can be approximated to second order in terms of values of V at points where it can be determined by integrating the hypersurface equations (5) and (6) radially outward from $r = 0$.

After carrying out this procedure to evaluate g at the point N, the procedure can then be iterated to determine g at the next radially outward grid point on the $u_0 + \Delta u$ level. Upon completing this radial march to null infinity, in terms of a compactified radial coordinate such as $x = r/(1 + r)$, the field g is then evaluated on the next null cone at $u_0 + 2\Delta u$, beginning at the vertex where smoothness gives the startup condition that $g(u, 0) = 0$.

In the compactified Bondi formalism, the vertices N, E, S and W of the null parallelogram Σ cannot be chosen to lie exactly on the grid because, even in Minkowski space, the velocity of light in terms of a compactified radial coordinate x is not constant. As a consequence, the fields g, β and V at the vertices of Σ are approximated to second order accuracy by interpolating between grid points. However, cancellations arise between these four interpolations so that Eq. (8) is satisfied to fourth order accuracy. The net result is that the finite difference version of (8) steps g radially outward one zone with an error of fourth order in grid size, $O((\Delta u)^2(\Delta x)^2)$. In addition, the smoothness conditions (4) can be incorporated into the startup for the numerical integrations for V and β to insure no loss of accuracy in starting up the march at $r = 0$. The resulting global error in g, after evolving a finite retarded time, is then $O(\Delta u\Delta x)$, after compounding errors from $1/(\Delta u\Delta x)$ number of zones.

Because of the explicit nature of this algorithm, its stability requires an analogue of the Courant-Friedrichs-

Lewy (CFL) condition that the physical domain of dependence be contained in the numerical domain of dependence. In the present spherically symmetric case, this condition requires that the ratio of the time step to radial step be limited by $(V/r)\Delta u \leq 2\Delta r$, where $\Delta r = \Delta[x/(1-x)]$. This condition can be built into the code using the value $V/r = e^{2H}$, corresponding to the maximum of V/r at $\mathcal{I}^+$. The strongest restriction on the time step then arises just before the formation of a horizon, where $V/r \to \infty$ at $\mathcal{I}^+$, This infinite redshift provides a mechanism for locating the true event horizon "on the fly" and restricting the evolution to the exterior space-time. Points near $\mathcal{I}^+$ must be dropped in order to evolve across the horizon in this gauge.

Such algorithms have been applied to many interesting problems. A characteristic algorithm based upon double null coordinates was used by Goldwirth and Piran in a study of cosmic censorship. Their early study lacked the sensitivity of adaptive mesh refinement which later enabled Choptuik to discover the critical phenomena appearing in this problem. The most accurate global treatment of this problem using a Cauchy code has been achieved by Marsa and Choptuik using ingoing Eddington-Finklestein coordinates. This use of a null based time slicing enabled them to avoid problems with the singularity by excising the black hole exterior and to construct a 1D code that runs forever.

The Southampton group has also constructed a 1+1 dimensional characteristic code for space-times with cylindrical symmetry. Their motivation was not to produce a stand-alone code for the study of cylindrically symmetric relativity but rather to use it as a test case for combining Cauchy and characteristic codes into a global scheme. Their work will be discussed later in this review under Cauchycharacteristic matching.

Two-D Codes

One dimensional characteristic codes enjoy a very special simplicity due to the two preferred sets (ingoing and outgoing) of characteristic null hypersurfaces. This eliminates a source of gauge freedom that otherwise exists in either two or three dimensional characteristic codes. However, the manner in which characteristics of a hyperbolic system determine domains of dependence and lead to propagation equations for shock waves is exactly the same as in the one dimensional case. This makes it desirable for the purpose of numerical evolution to enforce propagation along characteristics as extensively as possible. In basing a Cauchy algorithm upon shooting along characteristics, the infinity of characteristic rays (technically, *bicharacteristics*) at each point leads to an arbitrariness which, for a practical numerical scheme, makes it necessary either to average the propagation equations over the sphere of characteristic directions or to select out some preferred finite subset of propagation equations. The latter approach has been successfully applied by Butler [25] to the Cauchy evolution of 2-dimensional fluid flow but there seems to have been very little follow-up along these lines.

The formal ideas behind the construction of two or three dimensional characteristic codes are the same, although there are some additional technical complications in three dimensions associated with a nonsingular choice of angular coordinates for the null rays. Historically, most characteristic work graduated first from 1D to 2D because of the available computing power.

Three-D Characteristic Evolution

The Binary Black Hole Grand Challenge has fostered striking progress in developing a 3D characteristic code. At the outset of the Grand Challenge, the Pittsburgh group had just completed calibration of their axisymmetric characteristic code. Now, this has not only been extended to a full 3D code which calculates waveforms at infinity [20, 14], it has also been supplied with a horizon finder to successfully move distorted black holes on a computational grid. This has been accomplished with unlimited long term stability and demonstrated second order accuracy, in the harshest nonlinear physical regimes corresponding to radiation powers of a galactic rest mass per second, and with the harshest gauge conditions, corresponding to superluminal coordinate rotation.

The waveforms are initially calculated in arbitrary coordinates determined by the "3+1" gauge conditions on an inner worldtube. An important feature for the binary black hole problem is that these coordinates can be rigidly rotating, so that the evolution near infinity is highly superluminal, without affecting code performance. The waveforms are converted to the standard "plus and cross" inertial polarization modes by numerically carrying out the transformation to an inertial frame at infinity.

The eth Module

Spherical coordinates and spherical harmonics are standard analytic tools in the description of radiation, but, in computational work, spherical coordinates have mainly been used in axisymmetric systems, where polar singularities may be regularized by standard tricks. In the absence of symmetry, these techniques do not generalize and would be especially prohibitive to develop for tensor fields. A crucial ingredient of the 3D characteristic code is a module which allows use of spherical coordinates by implementing a computational version of the Newman-Penrose eth formalism. The eth module covers the sphere with two overlapping stereographic coordinate grids (North and South). It provides everywhere regular, second order accurate, finite difference expressions for tensor fields on the sphere and their covariant derivatives.

CCM for 3D Scalar Waves

CCM has been successfully implemented in the fully 3D problem of nonlinear scalar waves evolving in a flat space-time. [18, 17] The main purpose of the study was to demonstrate the feasibility of matching between Cartesian Cauchy coordinates and spherical null coordinates. This is the setup required to apply CCM to the binary black hole problem. Unlike the previous examples of matching, the Cauchy and characteristic patches do not now share a common coordinate which can be used to define the matching interface. This introduces a major complication into the matching procedure, resulting in extensive use of inter-grid interpolation. The accompanying short wavelength numerical noise presents a new challenge in obtaining a stable algorithm.

The nonlinear waves were modeled on the equation

$c^{-2}\partial_{tt}\Phi = \nabla^2\Phi + F(\Phi) + S(x, y, z, t),$ 9

with self-coupling $F(\Phi)$ and external source S. The initial Cauchy data $\Phi(x, y, z, t_0)$ and $\partial_t\Phi(x, y, z, t_0)$ are assigned in a spatial region bounded by a spherical matching surface of radius R_m.

The characteristic initial value problem (9) is expressed in standard spherical coordinates (r, θ, φ) and retarded time $u = t - r + R_m$:

2∂urg=∂rrg−L2gr2+r(F+S),

10

where $g = r\Phi$ and L^2 is the angular momentum operator

L2g=−∂θ(sinθ∂θg)sinθ−∂φφgsin2θ. 11

The initial null data is $g(r, \theta, \varphi, u_0)$, on the outgoing characteristic cone $u_0 = t_0$ emanating from the matching worldtube at the initial Cauchy time.

CCM was implemented so that, in the continuum limit, Φ and its normal derivatives would be continuous across the interface $r = R_m$ between the regions of Cauchy and characteristic evolution. The use of a Cartesian discretization in the interior and a spherical discretization in the exterior complicated the treatment of the interface. In particular, the stability of the matching algorithm required careful attention to the details of the inter-grid matching. Nevertheless, there was a reasonably broad range of discretization parameters for which CCM was stable.

Two different ways of handling the spherical coordinates were used. One was based upon two overlapping stereographic grid patches and the other upon a multiquadric approximation using a quasi-regular triangulation of the sphere. Both methods gave similar accuracy. The multiquadric method showed a slightly larger range of stability. Also, two separate tactics were used to implement matching, one based upon straightforward interpolations and the other upon maintaining continuity of derivatives in the outward null direction (a generalization of the Sommerfeld condition). Both methods were stable for a reasonable range of grid parameters. The solutions were second order accurate and the Richardson extrapolation technique could be used to accelerate convergence.

The performance of CCM was compared to traditional ABC's. As expected, the nonlocal ABC's yielded convergent results only in linear problems, and convergence was not observed for local ABC's, whose restrictive assumptions were violated in all of the numerical experiments. The computational cost of CCM was much lower than that of current nonlocal conditions. In strongly nonlinear problems, matching appears to be the only available method which is able to produce numerical solutions which converge to the exact solution with a fixed boundary.

The CCM Gravitational Module

The most important application of CCM is anticipated to be the binary black hole problem. The 3D Cauchy codes now being developed to solve this problem employ a single Cartesian coordinate patch [4]. A thoroughly tested and robust 3D characteristic code is now in place [14], ready to match to the boundaries of this Cauchy patch. Development of a stable implementation of CCM represents the last major step necessary to provide a global code for the binary problem.

A CCM module has been constructed and interfaced with Cauchy and characteristic evolution modules. It provides a model of how two of the best current codes to treat gravitation, the Grand Challenge ADM and characteristic codes, can be pieced together as modules to form a single global code. The documentation of the underlying geometrical algorithm is given in Ref. [19]. The main submodules of the CCM module are:

- The **outer boundary module** which sets the grid structures. This defines masks identifying which points in the Cauchy grid are to be evolved by the Cauchy module and which points are to be interpolated from the characteristic grid; and vice versa. The reference base for constructing the mask is the matching worldtube, which in Cartesian coordinates is the "Euclidean" sphere $x^2 + y^2 + z^2 = R^2$. The choice of lapse and shift for the Cauchy evolution govern the actual dynamical and geometrical properties of the matching worldtube.
- The **extraction module** whose input is Cauchy grid data in the neighborhood of the worldtube and whose output is the inner boundary data for the exterior characteristic evolution. This module numerically implements the transformation from Cartesian "3+1" coordinates to spherical null coordinates. The algorithm makes no perturbative assumptions and is based upon interpolations of the Cauchy data to a set of prescribed points on the worldtube. The metric information is then used to solve for the null geodesics normal to the slices of the worldtube. This provides the Jacobian for the transformation to null coordinates in the neighborhood of the worldtube. The characteristic evolution module is then used to propagate the data from the worldtube to null infinity, where the waveform is calculated.
- The **injection module** which completes the interface by using the exterior characteristic evolution to supply the

outer boundary condition for Cauchy evolution. This is the inverse of the extraction procedure but must be implemented outside the worldtube to allow for the necessary overlap between Cauchy and characteristic domains. (Without overlap, the domain of dependence of the initial value problem would be empty.) The overlap region is constructed so that it shrinks to zero in the continuum limit. As a result, the inverse Jacobian can be obtained to a prescribed accuracy in terms of an affine parameter expansion of the null geodesics about the worldtube.

The CCM module has been calibrated to give a second order accurate interface between Cauchy and characteristic evolution modules. When its long term stability has been established, it will provide an accurate outer boundary condition for an interior Cauchy evolution by joining it to an exterior characteristic evolution which extracts the waveform at infinity.

The Binary Black Hole Inner Boundary

It is clear that the 3-dimensional inspiral and coalescence of black holes challenges the limits of present computational know-how. CCM offers a new approach for excising an interior trapped region which might provide the enhanced flexibility to solve this problem. In a binary system, there are major computational advantages in posing the Cauchy evolution in a frame which is co-rotating with the orbiting black holes. Such a description seems necessary in order to keep the numerical grid from being intrinsically twisted and strangled. In this co-orbiting description, the Cauchy evolution requires an inner boundary condition inside the black holes and also an outer boundary condition on a worldtube outside of which the grid rotation is likely to be superluminal. An outgoing characteristic code can routinely handle such superluminal gauge flows in the exterior [14]. Thus, successful implementation of CCM would solve the exterior boundary problem for this co-orbiting description.

IV. CONCLUSION

This paper concludes that CCM also has the potential to handle the inner black hole boundaries of the Cauchy region. As described earlier, an ingoing characteristic code can evolve a moving black hole with long term stability. This means that CCM would also be able to provide the inner boundary condition for Cauchy evolution once stable matching has been accomplished. In this approach, the interior boundary of the Cauchy evolution is located *outside* the apparent horizon and matched to a characteristic evolution based upon ingoing null cones. The inner boundary for the characteristic evolution is a trapped or marginally trapped surface, whose interior is excised from the evolution.

This global strategy is tailor-made to treat two black holes in the co-orbiting gauge. Two disjoint characteristic evolutions based upon ingoing null cones would be matched across worldtubes to a central Cauchy region. The interior boundary of each of these interior characteristic regions would border a trapped surface. At the outer boundary of the Cauchy region, a matched characteristic evolution based upon outgoing null hypersurfaces would propagate the radiation to infinity.

Present characteristic and Cauchy codes can handle the individual pieces of this problem. Their unification appears to offer the best chance for simulating the inspiral and merger of two black holes. The CCM module is in place and calibrated for accuracy. The one missing ingredient is the long term stability of CCM, which would make future reviews of this subject very exciting.

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