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Equitable Edge Coloring of Strong Product of Cycle, Complete Graphs

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Abstract—An edge coloring of graph $\boldsymbol{G}$ is equitable if for each vertex $\boldsymbol{v}$ of $\boldsymbol{G}$, the number of edges of any one color incident with $\boldsymbol{v}$ differs from the number of edges of any other color incident with $\boldsymbol{v}$ by at most one. In this paper, we obtain the exact expressions for the equitable edge coloring of strong product of $\boldsymbol{C_n}\, \boldsymbol{K_m}$.

Keywords— *Equitable edge coloring, Product graph, Cycle, Complete graph.*

I. Introduction

Coloring problem is one among the most important research area in graph theory. As an extension of proper edge coloring [3,10,11] and conjectures on equitable edge coloring [1,4,6,9] is established. It is tough to find a result using equitable edge chromatic number.

In this paper, we consider a graph G as finite, simple and undirected. Let $G = (V(G), E(G))$ be an ordered pair of graph G with the vertices and the edges respectively. An equitable edge coloring of graph G is a mapping $f : E(G) \to N$, where N i s a set of colors satisfying the following conditions.

1. $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$.

2. $||E_i| - |E_j|| \leq 1;\ i, j = 1, 2, \dots, k.$

The minimum number of colors are required for an equitable edge coloring of graph G is called the equitable edge chromatic number of G and is denoted by $\chi_e'(G)$. The edge chromatic number of graph G is related to the maximum degree $\Delta(G)$, the greatest number of edges incident to any single vertex of G. it is clear that $\chi'(G) \geq \Delta(G)$, for if Δ various number of edges join at a single vertex v, then all of these edges to be received different colors from each other and that can be possible if there are at least Δ colors available to be received.

The edge chromatic number of graph G must be at least Δ, the greatest vertex degree of graph G given by Skiena [10]. However, Vizing [11] and Gupta [3] proved that any graph G can be edge colored with at most Δ+1 colors. Vizing's theorem states that, the tight bound of edge coloring for any simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. If a graph G with edge chromatic number equal to $\Delta(G)$, then the graph G is called Type-1 and if edge chromatic number is equal to $\Delta(G) + 1$, then it is called Type-2 graph. The number of colors for bipartite graph and high degree planar graphs is always Δ and for the multi graph may be as large as $3\Delta/2$. In 1964 Paul Erdős [1] conjectured that an equitable coloring is achievable with only one more color; for any graph G with greatest degree Δ has an equitable coloring with $\Delta + 1$ colors. This conjecture was proved in 1970 by Hajnal and Szemerédi [4] with lengthy and difficulted proof is called as the Hajnal Szemerédi Theorem. In the year 2008, Kierstead and Kostochka [6] was presented the same proof in a simple way. Seymour [9] introduced a good result in Hajnal Szemerédi theorem that conjecture is called Seymour's conjecture.

Theorem 1.1: [2] For any complete graph K_n,

$$\chi_e'(K_n) = \begin{cases} \Delta(G) + 1, & if\ n\ is\ odd \\ \Delta(G) & ,if\ n\ is\ even. \end{cases}$$

Theorem 1.2: [2] For any cycle graph C_n,

$$\chi_e'(C_n) = \begin{cases} \Delta(G) + 1, & if\ n\ is\ odd \\ \Delta(G) & ,if\ n\ is\ even. \end{cases}$$

Graph products were first defined by sabidussi [8] and vizing [12]. A lot of work was done on various topics related to graph product, but on the other hand there are still many open questions. Mohan et al. [7] proved the TCC for certain classes of product graphs.

In this paper, we obtain the exact expressions for the equitable edge coloring of $(C_n \boxtimes K_m)$ and $(K_n \boxtimes K_m)$.

II. RESULTS AND DISCUSSION

Definition 2.1:[13] Consider G and H be two graphs. The strong product $G \boxtimes H$, defined by $V(G \boxtimes H) = \{(g,h) | g \in V(G), h \in V(H)\}$ and $E(G \boxtimes H) = E(G \boxdot H) \cup E(G \times H)$.

Theorem 2.1: For any equitable edge coloring of $C_n \boxtimes K_m$ for all $n, m \geq 3$ and $n, m \in Z^+$ is

$$\chi'_e(C_n \boxtimes K_m) = \begin{cases} \Delta(C_n \boxtimes K_m), & if\ m\ is\ even \\ \Delta(C_n \boxtimes K_m) + 1, & if\ m\ is\ odd. \end{cases}$$

Proof: Let C_n be the cycle on n vertices $\{u_1, u_2, u_3, \dots, u_n\}$ and K_m be the complete graph on m vertices $\{v_1, v_2, v_3, \dots, v_m\}$ respectively. Then the strong product of $(C_n \boxtimes K_m)$ divide into two cases as follows:

Case (1): If m is even.

Here, $\Delta(C_n \boxtimes K_m)$ is $(3m-1)$ which is the maximum degree of $(C_n \boxtimes K_m)$. Then the strong product of $(C_n \boxtimes K_m)$ having $n(R_1, R_2, \dots, R_n)$ rows and $m\,(C_1, C_2, \dots, C_m)$ columns. We divide $\Delta(C_n \boxtimes K_m)$ into m partitions of color classes, say $X_1, X_2, \dots, X_m$. Each color classes $X_1, X_2, \dots, X_{m-1}$ contain 3 colors and X_m color class contains the remaining two colors of $\Delta(C_n \boxtimes K_m)$ respectively and case (1) divided into two subcases as follows:

Subcase (1.1): If both m and n are even.

If color the edges of $(C_n \boxtimes K_m)$, first color the edges of K_m using $(m-1)$ colors which are taken exactly one color from each color classes of $X_1, X_2, X_3, \dots, X_{m-1}$ and the vertex of K_m is $v_1, v_2, v_3, \dots, v_m$ respectively. In $(C_n \boxtimes K_m)$, Color all the edges in $C_1, C_2, C_3, \dots, C_m$ using X_m color. Then assign the colors to the remaining edges of $(C_n \boxtimes K_m)$ in the following way: The color which is assigned in the edges between v_1 and v_2 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_2 in $(C_n \boxtimes K_m)$. Then the color which is assigned in the edges between v_1 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_3 in $(C_n \boxtimes K_m)$. In the same way, the color which is assigned in the edges between v_2 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_2 and C_3 in $(C_n \boxtimes K_m)$. Proceeding like this manner for all the edges between $(C_3, C_4), (C_4, C_5) \dots, (C_n, C_1)$ which satisfies the condition of equitably edge colorable.

Subcase (1.2): If m is even and n is odd

To color the edges of $(C_n \boxtimes K_m)$, first color the edges of K_m using $(m-1)$ colors which are taken exactly one color from each color classes of $X_1, X_2, X_3, \dots, X_{m-1}$ and the vertex of K_m is $v_1, v_2, v_3, \dots, v_m$ respectively. In $(C_n \boxtimes K_m)$ assign X_m color to all the edges in $C_1, C_2, C_3, \dots, C_m$ from R_1 to R_n. Color the edges of R_n using one of the color of X_m and assign the remaining color of X_m in the R_1 edges as possible according to satisfying the equitable edge coloring conditions. Then assign the colors to the remaining edges of $(C_n \boxtimes K_m)$ in the following way: The color which is assigned in the edges between v_1 and v_2 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_2 in $(C_n \boxtimes K_m)$. Then the color which is assigned in the edges between v_2 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_3 in $(C_n \boxtimes K_m)$. In the same way, the color which is assigned in the edges between v_2 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_2 and C_3 in $(C_n \boxtimes K_m)$. Proceeding like this manner for all the edges between $(C_3, C_4), (C_4, C_5) \dots, (C_n, C_1)$. Finally, color the remaining edges between R_n and R_1 using the missing colors of $\Delta(C_n \boxtimes K_m)$ which satisfies the condition of equitably edge colorable.

Therefore, $\chi'_e(C_n \boxtimes K_m) = \Delta(C_n \boxtimes K_m)$.

Case (2): If m is odd and for any n.

Here, $\Delta(C_n \boxtimes K_m) + 1$ is $3m$ which is the equitable edge chromatic number of $C_n \boxtimes K_m$. Then the strong product of $(C_n \boxtimes K_m)$ having $n(R_1, R_2, \dots, R_n)$ rows and $m(C_1, C_2, \dots, C_m)$ columns. We divide $\Delta(C_n \boxtimes K_m) + 1$ into m equal partitions of color classes, say $X_1, X_2, X_3, \dots, X_m$ respectively.

To assign the color to the edges of $(C_n \boxtimes K_m)$, first color the edges of K_m using m colors which are taken exactly one color from each color classes of $X_1, X_2, X_3, \dots, X_m$ and the vertex of K_m is $v_1, v_2, v_3, \dots, v_m$ respectively.

To assign the colors to the edges of $(C_n \boxtimes K_m)$ in the following way: The color which is assigned in the edges between v_1 and v_2 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_2 in $(C_n \boxtimes K_m)$. Then the color which is assigned in the edges between v_1 and v_3 in K_m belongs to any one of the color classes. Using all the colors

in that color class, color all the edges between C_1 and C_3 in $(C_n \boxtimes K_m)$. In the same way, the color which is assigned in the edges between v_2 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_2 and C_3 in $(C_n \boxtimes K_m)$. Proceeding like this manner for all the edges between (C_3, C_4), (C_4, C_5), ...,
(C_n, C_1). Finally, color all the edges of $C_1, C_2, \dots, C_m$ using the missing color classes of $\Delta(C_n \boxtimes K_m) + 1$ according to satisfying the equitably edge colorable condition.

Therefore, $\chi'_e(C_n \boxtimes K_m) = \Delta\,(C_n \boxtimes K_m) + 1$.

Theorem 2.2: For any equitably edge colorable of $K_n \boxtimes K_m$ for all $n, m \geq 3$ and $n, m \in Z^+$ is

$$\chi'_e(K_n \boxtimes K_m) = \begin{cases} \Delta\,(K_n \boxtimes K_m), & \textit{if both n and m are even} \\ \Delta\,(K_n \boxtimes K_m) + 1, & \textit{otherwise.} \end{cases}$$

Proof: Let K_n be the complete graph on n vertices $\{u_1, u_2, u_3, \dots, u_n\}$ K_m be the complete graph on m vertices $\{v_1, v_2, v_3, \dots, v_m\}$ respectively. Then the strong product of $(K_n \boxtimes K_m)$ divide into two cases as follows:

Case (1): If both n and m are even.

Here, $\Delta(K_n \boxtimes K_m) = (\mathrm{nm}) - 1$ which is the maximum degree of $K_n \boxtimes K_m$. Then the strong product of $(K_n \boxtimes K_m)$ having $n\,(R_1, R_2, \dots, R_n)$ rows and $m\,(C_1, C_2, \dots, C_m)$ columns. We divide $\Delta(K_n \boxtimes K_m)$ into m partitions of color classes, say $X_1, X_2, \dots X_m$. Each color classes $X_1, X_2, \dots X_{m-1}$ contain n colors and X_m color class contains $n-1$ colors respectively and divided into two subcases as follows:

If color the edges of $(K_n \boxtimes K_m)$, first color the edges of K_m using $(m-1)$ colors which are taken exactly one color from each color classes of $X_1, X_2, \dots X_{m-1}$ and the vertex of K_m is $v_1, v_2, v_3, \dots, v_m$ respectively.

In $(K_n \boxtimes K_m)$, Color all the edges in $C_1, C_2, \dots, C_m$ using X_m color. Then assign the colors to the remaining edges of $(K_n \boxtimes K_m)$ in the following way: The color which is assigned in the edges between v_1 and v_2 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_2 in $(K_n \boxtimes K_m)$. Then the color which is assigned in the edges between v_1 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_3 in $(K_n \boxtimes K_m)$. In the same way, the color which is assigned in the edges between v_2 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_2 and C_3 in $(K_n \boxtimes K_m)$. Proceeding like this manner for all the edges between $(C_3, C_4), (C_3, C_5), \dots, (C_n, C_1)$ which satisfies the condition of equitably edge colorable.

Therefore, $\chi'_e(K_n \boxtimes K_m) = \Delta\,(K_n \boxtimes K_m)$.

Case (2): If both n and m are not even.

Here, $\Delta(K_n \boxtimes K_m) + 1 = \mathrm{nm}$ which is the equitable edge chromatic number of $(K_n \boxtimes K_m)$. Then the strong product of $(K_n \boxtimes K_m)$ having $n\,(R_1, R_2, \dots, R_n)$ rows and $m\,(C_1, C_2, \dots, C_m)$ columns. We divide $\Delta(K_n \boxtimes K_m) + 1$ into m equal partitions of color classes, say $X_1, X_2, \dots X_m$.

Subcase (2.1): If m is odd and for any n.

If color the edges of $(K_n \boxtimes K_m)$, first color the edges of K_m using m colors which are taken exactly one color from each color classes of $X_1, X_2, \dots X_m$ and the vertex of K_m is $v_1, v_2, v_3, \dots, v_m$ respectively.

To assign the colors to the edges of $(K_n \boxtimes K_m)$ in the following way: The color which is assigned in the edges between v_1 and v_2 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_2 in $(K_n \boxtimes K_m)$. Then the color which is assigned in the edges between v_1 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_3 in $(K_n \boxtimes K_m)$. In the same way, the color which is assigned in the edges between v_1 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_2 and C_3 in $(K_n \boxtimes K_m)$. Proceeding like this manner for all the edges between $(C_3, C_4), (C_3, C_5), \dots, (C_n, C_1)$. Finally, color all the edges of $C_1, C_2, \dots, C_m$ using the missing color classes of $\Delta(K_n \boxtimes K_m) + 1$ according to satisfying the equitably edge colorable condition.

Subcase (2.2): If n is odd and m is even.

If color the edges of $(K_n \boxtimes K_m)$, first color the edges of K_m using $(m-1)$ colors which are taken exactly one color from each color classes of $X_1, X_2, \dots X_{m-1}$ and the vertex of K_m is $v_1, v_2, v_3, \dots, v_m$ respectively.

In $(K_n \boxtimes K_m)$, Color all the edges in $C_1, C_2, \dots, C_m$ using X_m color in the same pattern and also color some of the edges in $R_1, R_2, \dots, R_n$ using the missing colors of X_m according to satisfying the equitably edge colorable condition. Then assign the colors to the remaining edges of $(K_n \boxtimes K_m)$ in the following way: The color which is assigned in the edges between v_1 and v_2 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_2 in $(K_n \boxtimes K_m)$. Then the color which is assigned in the edges between v_1 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_1 and C_3 in $(K_n \boxtimes K_m)$. In the same way, the color which is assigned in the edges between v_1 and v_3 in K_m belongs to any one of the color classes. Using all the colors in that color class, color all the edges between C_2 and C_3 in $(K_n \boxtimes K_m)$. Proceeding like this manner for all the edges between $(C_3, C_4), (C_3, C_5), \dots, (C_n, C_1)$ which satisfies the condition of equitably edge colorable.

Therefore, $\chi'_e(K_n \boxtimes K_m) = \Delta\,(K_n \boxtimes K_m) + 1.$

III. CONCLUSION

In this paper, we have mainly derived results about equitable edge coloring of strong product of cycle, complete graphs and we have to find tight bound of the chromatic number.

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