International Journal of Scientific Research in Mathematical and Statistical Sciences
Research Paper
Vol.6, Issue.2, pp.258-261, April (2019)
E-ISSN: 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i2.258261

Common Fixed Point Theorems in Complex Valued $b-$ Metric Spaces

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Available online at: www.isroset.org

Received: 20/Apr/2019, Accepted: 30/Apr/2019, Online: 30/Apr/2019

Abstract- Many authors prove several fixed point results for mappings satisfying certain contraction conditions. b- metric spaces in fixed point has attracted much attention in recent times. In this paper, we prove the existence and uniqueness of fixed point for two self mappings satisfying rational expressions in Complex Valued b –metric spaces. These theorems generalize many previously obtained fixed point results. An example is given to illustrate the main result.

Keywords: Fixed point theorem, Complex valued metric spaces,$b-$ metric spaces

2010 *mathematics subject classification*:47H10, 54H25, 30L99

I. Introduction

The Banach contraction principle [3] is a very popular and effective tool in solving existence problems in many branches of mathematical analysis. Due to simplicity and usefulness of this classic and celebrated theorem, it has become a very popular source of existence and uniqueness theorems in different branches of mathematical analysis. In 1989 Bakhtin [2] introduced the concept of b-metric space as a generalization of metric spaces. In 2011 Azam et al .[1] introduce the notion of complex valued metric spaces and also proved common fixed point theorems for mapping satisfying rational expression .The concept of complex valued b- metric spaces was introduced in 2013 by Rao et al.[5]. In the sequel, Mukheimer [4] proved some common fixed point theorems in complex valued b- metric spaces.
In this paper we present a class of mappings satisfying a rational expression in the setting of complex valued b-metric spaces.

II. Preliminaries

Consistent with Azam, Fisher and Khan [1], *the following definitions and results will be needed in the sequel.*
Let C be the set of complex numbers and let $z_1, z_2 \in C$.
Define a partial order $\precsim$ *on C as follows*: $z_1 \precsim z_2$
if and only if $Re(z_1) \le Re(z_2)$, $\quad Im(z_1) \le Im(z_2)$.
It follows that: $z_1 \precsim z_2$ *if one of the following conditions is satisfied*:
(1) $Re(z_1) = Re(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(2) $Re(z_1) < Re(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(3) $Re(z_1) = Re(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.
In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ *and one of* (1), (2) *and* (3) *is satisfied and we will write* $z_1 \prec z_2$ *if only* (3) *is satisfied.*
Definition 2.1: [**1**] *Let X be a nonempty set. Suppose that the mapping* $d: X \times X \to C$ *Satisfies*:
(a) $0 \precsim d(x,y)$ *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$

$(b)\ d(x,y) = d(y,x)\ for\ all\ x,y \in X$
$(c)\ d(x,y) \precsim d(x,z) + d(z,y)\ for\ all\ x,y,z \in X.$
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 2.2: [1] *Let* $\{x_n\}$ *be a sequence in X and* $x \in X$. *If for every* $c \in C$, *with* $0 < c$ *there is* $n_0 \in N$ *such that for all* $n > n_0$, $d(x_n, x) < c$, then x *is called the limit of* $\{x_n\}$ *and we write* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

Definition 2.3: [1] *If every Cauchy sequence is convergent in* (X,d), *then* (X,d) *is called a Complete Complex valued metric space.*

Lemma 2.4: [1] *Let* (X,d) *be a complex valued metric space and* $\{x_n\}$ *a sequence in X. Then* $\{x_n\}$ *converges to x if and only if* $|d(x_n, x)| \to 0$ *as* $n \to \infty$.

Lemma 2.5: [1] *Let* (X,d) *be a complex valued metric space and* $\{x_n\}$ *a sequence in X. Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|d(x_n, x_{n+m})| \to 0$ *as* $n \to \infty$ *where* $m, n \in N$.

Definition 2.6: [5] *Let X be a nonempty set and let* $s \geq 1$ *be a given real number. A function* $d: X \times X \to C$ *is called a complex valued b − metric if for all* $x, y, z \in X$ *the following conditions are satisfied:*
$(a)\ d(x,y) = 0\ if\ and\ only\ if\ x = y$
$(b)\ d(x,y) = d(y,x)\ for\ all\ x,y \in X$
$(c)\ d(x,y) \precsim s[\,d(x,z) + d(z,y)]\ for\ all\ x,y,z \in X.$
Then the pair (X,d) *is called Complex valued b − metric space. The number* $s \geq 1$ *is called the coefficient of* (X,d).

Definition 2.7: [5] *Let* (X,d) *be a complex valued b − metric space let* $\{x_n\}$ *be a sequence in X and* $x \in X$. *If for every* $c \in C$, *with* $0 < c$ *there is* $n_0 \in N$ *such that for all* $n > n_0$, $d(x_n, x) < c$, *then x is called the limit of* $\{x_n\}$ *and we write* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.
i. If for every $c \in C$, *with* $0 < c$, *there is an* $n_0 \in N$ *such that for all* $n > n_0$, $d(x_n, x_{n+m}) < c$, *then* $\{x_n\}$ *is called a Cauchy sequence in* (X,d).
ii. If every Cauchy sequence is convergent in (X,d) *then* (X,d) *is called a complete complex valued b − metric space.*

Lemma 2.8: [5] *Let* (X,d) *be a complex valued b − metric space and* $\{x_n\}$ *a sequence in X*. Then $\{x_n\}$ *converges to x if and only if* $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.9: [5] *Let* (X,d) *be a complex valued b − metric space and* $\{x_n\}$ *a sequence in X. Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|d(x_n, x_{n+m})| \to 0$ *as* $n \to \infty$.

3. *Main Result*:

Theorem 3.1: *Let* (X,d) *be a complete complex valued b metric space with the coefficient* $s \geq 1$ *and let* $S, T: X \to X$ *be mapping satisfying*

$$d(Sx,Ty) \precsim A\, d(x,y) + B\, \frac{d(x,Sx)\sqrt{d(x,Ty)} + d(y,Ty)\sqrt{d(y,Sx)}}{1+\sqrt{d(x,Ty)} + \sqrt{d(y,Sx)}} + C\, max\left\{d(x,y), \frac{d(x,Ty)d(y,Sx)}{1+d(Sx,Ty)}\right\}$$

for all $x, y \in X$ *and A, B are non negative real with* $s(A + B + C) < 1$ *then S and T have a unique common fixed point in X.*

Proof: for any arbitrary point $x_0 \in X$ *define sequence* $\{x_n\}$ *in X such that* $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = T\,x_{2k+1}$ *for* $n = 0,1,2,3\ldots$
Now we show that the sequence $\{x_n\}$ *is Cauchy sequence let* $x = x_{2k}$ *and* $y = x_{2k+1}$. *We have*

$$d(x_{2k+1}, x_{2k+2}) = d(S\,x_{2k}, Tx_{2k+1}) \precsim Ad(x_{2k}, x_{2k+1})$$

$$+ B \frac{d(x_{2k},Sx_{2k})\sqrt{d(x_{2k},Tx_{2k+1})} + d(x_{2k+1},Tx_{2k+1})\sqrt{d(x_{2k+1},Sx_{2k})}}{1+\sqrt{d(x_{2k},Tx_{2k+1})} + \sqrt{d(x_{2k+1},Sx_{2k})}}$$

$$+ C\, max\left\{d(x_{2k}, x_{2k+1}), \frac{d(x_{2k},Tx_{2k+1})d(x_{2k+1},Sx_{2k})}{1+d(Sx_{2k},Tx_{2k+1})}\right\}$$

$$\lesssim Ad(x_{2k}, x_{2k+1})$$

$$+B \frac{d(x_{2k},x_{2k+1})\sqrt{d(x_{2k},x_{2k+2})} + d(x_{2k+1},x_{2k+2})\sqrt{d(x_{2k+1},x_{2k+1})}}{1+\sqrt{d(x_{2k},x_{2k+2})} + \sqrt{d(x_{2k+1},x_{2k+1})}}$$

$$+ C\, max\left\{d(x_{2k}, x_{2k+1}), \frac{d(x_{2k},x_{2k+2})d(x_{2k+1},x_{2k+1})}{1+d(x_{2k+1},x_{2k+2})}\right\}$$

$$\lesssim Ad(x_{2k}, x_{2k+1}) + B\, d(x_{2k}, x_{2k+1}) + C\, d(x_{2k}, x_{2k+1})$$

$$\lesssim (A + B + C)\, d(x_{2k}, x_{2k+1})$$

$$|\, d(x_{2k+1}, x_{2k+2})\, | \leq (A + B + C)\ |\ d(x_{2k}, x_{2k+1})\ |$$

$If\ \ \delta = (A + B + C) < 1\ then$

$$|\, d(x_{2k+1}, x_{2k+2})\, | \leq \delta\ |\, d(x_{2k}, x_{2k+1})\, | \leq \delta^2\, |\, d(x_{2k-1}, x_{2k+1})\, | \ldots$$

$$\leq \delta^{2k+1} |d(x_0, x_1)| \qquad (3.1.1)$$

$Thus\ for\ any\ m > n, m, n \in N\ and\ since\ s\delta = s(A + B + C) < 1\ \ we\ get$

$$|\, d(x_n, x_m)\, | \leq s\, |\, d(x_n, x_{n+1})\, | + s\ |\, d(x_{n+1}, x_m)\, |$$

$$\leq s\ |\, d(x_n, x_{n+1})\, | + s^2\ |\, d(x_{n+1}, x_{n+2}) + s^2\ |\, d(x_{n+2}, x_m)\, |$$

$$\leq s\ |\, d(x_n, x_{n+1})\, | + s^2\ |\, d(x_{n+1}, x_{n+2}) + s^3\ |\, d(x_{n+2}, x_{n+3})\, |$$

$$+ s^3\ |\, d(x_{n+3}, x_m)\, |$$

$$\leq s\ |\, d(x_n, x_{n+1})\, | + s^2\ |\, d(x_{n+1}, x_{n+2}) + s^3\, |\, d(x_{n+2}, x_m)\, | + \ldots$$

$$+ s^{m-n-2}\ |\, d(x_{m-3}, x_{m-2})\, | + \ldots + s^{m-n-1}\, |\, d(x_{m-2}, x_{m-1})\, |$$

$$+ s^3\ |\, d(x_{m-1}, x_m)\, |$$

$By\ using\ ((3.1.1)\ we\ get$

$$|\, d(x_n, x_m)\, | \leq s\delta^n\ |\, d(x_0, x_1)\, | + \delta^{n+1}\ s^2\ |\, d(x_0, x_1) + s^3\, \delta^{n+2}\ |\, d(x_0, x_1)\, | + \ldots + s^{m-n-2}\ \delta^{m-3}\ |\, d(x_0, x_1)\, | + \ldots + s^{m-n-1}\, \delta^{m-2}\ |\, d(x_0, x_1)\, | + s^{m-n}\, \delta^{m-1}\ |\, d(x_0, x_1)\, |$$

$$|\, d(x_n, x_m)\, | \leq \sum_{i=1}^{m-n} s^i\ \delta^{i+n-1}\ |\, d(x_0, x_1)\, |$$

Therefore

$$|\, d(x_n, x_m)\, | \leq \sum_{i=1}^{m-n} s^{i+n-1}\ \delta^{i+n-1}\ |\, d(x_0, x_1)\, |$$

$$= \sum_{t=n}^{m-1} s^t\ \delta^t\ |\, d(x_0, x_1)\, |$$

$$\leq \sum_{t=n}^{\infty} (s\delta)^t\ |\, d(x_0, x_1)\, |$$

$$\leq \frac{(s\delta)^n}{1-s\delta}\ |\, d(x_0, x_1)\, | \quad and\ hence$$

$$|\, d(x_n, x_m)\, | \leq \frac{(s\delta)^n}{1-s\delta}\ |\, d(x_0, x_1)\, | \to 0\ as\ m, n \to \infty.$$

$Thus\ \{x_n\}\ is\ a\ Cauchy\ sequence\ in\ X.$

$Since\ X\ is\ complete\ there\ exists\ some\ u \in X\ such\ that\ x_n \to u\ \ as$
$n \to \infty.$

$Now\ we\ show\ that\ Su = u$

$$d(u, Su) \lesssim s\, d(u, x_{2k+2}) + s\, d(x_{2k+2}, Su)$$

$$d(u, Su) \lesssim s\, d(u, x_{2k+2}) + s\, d(Su, Tx_{2k+1})$$

$$d(u, Su) \lesssim s\, d(u, x_{2k+2}) + As\, d(u, x_{2k+1})$$

$$+ B \frac{d(u,Su)\sqrt{d(u,Tx_{2k+1})} + d(x_{2k+1},Tx_{2k+1})\sqrt{d(x_{2k+1},Su)}}{1+\sqrt{d(u,Tx_{2k+1})} + \sqrt{d(x_{2k+1},Su)}}$$

$$+ C\, max\left\{d(u, x_{2k+1}), \frac{d(u,Tx_{2k+1})d(x_{2k+1},Su)}{1+d(Su,Tx_{2k+1})}\right\}$$

$$d(u, Su) \lesssim s\, d(u, x_{2k+2}) + A\, s\, d(u, x_{2k+1})$$

$$+ B \frac{d(u,Su)\sqrt{d(u,x_{2k+2})} + d(x_{2k+1}x_{2k+2})\sqrt{d(x_{2k+1},Su)}}{1+\sqrt{d(u,x_{2k+2})} + \sqrt{d(x_{2k+1},Su)}}$$

$$+ C\, max\left\{d(u, x_{2k+1}), \frac{d(u,x_{2k+2})d(x_{2k+1},Su)}{1+d(Su,x_{2k+2})}\right\}$$

as $n \to \infty, x_n \to u\ so\ d(u, Su) = 0$

$so\ that\ u = Su$

$similarly\ u = Tu.$

$Therefore\ u\ is\ common\ fixed\ point\ of\ S\ and\ T.$

III. *Uniqueness*

$let\ v \in X\ be\ another\ common\ fixed\ point\ of\ S\ and\ T$

$such\ that\ \ Sv = Tv = v$

$$Now\ consider\ \ d(u,v) = d(Su, Tv) \lesssim Ad(u,v) + B\frac{d(u,Su)\sqrt{d(u,Tv)} + d(v,Tv)\sqrt{d(v,Su)}}{1+\sqrt{d(u,Tv)} + \sqrt{d(v,Su)}} + C\max\left\{d(u,v), \frac{d(u,Tv)d(v,Su)}{1+d(Su,Tv)}\right\}$$

$d(u,v) \ \lesssim\ \mathrm{Ad(u,v)} + \mathrm{C\,d(u,v)}$

$(1 - \mathrm{A} - \mathrm{C})\, d(u,v) \lesssim 0$

$|\, d(u,v))| = 0$

$So\ that\ u = v.$

$Which\ prove\ the\ uniqueness\ of\ common\ fixed\ point.$

$By\ putting\ S = T\ in\ theorem\ 3.1, we\ get\ the\ following\ corollary$

***Corollary* 3.2**: $Let\ (X,d)\ be\ a\ complete\ complex\ valued\ b\ \ metric\ space$ $with\ the\ coefficient\ s\ \geq\ 1\ and\ let\ T: X \to X\ be\ mapping\ satisfying$

$$d(Tx,Ty) \lesssim A\,d(x,y) + B\frac{\mathrm{d(x,Tx)}\sqrt{d(x,Ty)} + \mathrm{d(y,Ty)}\sqrt{d(y,Tx)}}{1+\sqrt{d(x,Ty)} + \sqrt{d(y,Tx)}} + C\max\left\{d(x,y), \frac{d(x,Ty)d(y,Tx)}{1+d(Tx,Ty)}\right\}$$

$for\ all\ x, y \in X\ and\ A, B\ \ are\ non\ negative\ real\ with$ $\mathrm{s(A + B + C)} < 1\ \ then\ \ T\ has\ a\ unique\ common\ fixed\ point\ \ \ in\ X$.

***Example* 3.3**: $Let\ (X,d)\ \ be\ a\ Complex\ valued\ b\ metric\ space\ with\ a$ $metric\ \ d(\mathrm{x,y}) = |\ x_1 - x_2\ | + \mathrm{i}\ |\ y_1 - y_2\ |\ \ Consider\ the\ mapping$

$S, T: X \to X$ defined by $Sz = \dfrac{z}{5}\ and\ Tz = \dfrac{z+1}{5}\ \ where\ \ z = x + iy$

$Here\ S\ and\ T\ satisfy\ all\ the\ \ conditions\ of\ Theorem\ 3.1, S\ and\ T\ have\ a\ unique\ common\ fixed\ point\ 0 \in X$.

IV. *Conclusion*

In this paper, we presented a fixed point theorem for a pair of mapping with rational expressions in complex valued b- metric spaces with example and other corollaries. Our results generalized and improve similar existing fixed point results and help to find applications of fixed point results for pair of self mapping in different fields.

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