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HIGHER ORDER L_P-DERIVATIVE, L_P –CONTINUITY AND L_P –BOUNDEDNESS OF COMPLEX VALUED FUNCTION

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Abstract—Higher order Lp-derivative of Complex valued functions is defined. It is shown that this derivative is more general than ordinary derivative. Also Lp-Continuity and Lp-boundedness is studied for these type of functions and it is proved that L_P-continuity and L_P-boundedness are more general than the ordinary continuity and ordinary boundedness.

Keywords— Lp-derivative, Lp-Continuous, Lp-boundedness..

I. INTRODUCTION

Let μ be Lebesgue measure on real number space R and C be the set of all complex numbers. If $f : R \to C$ is measurable, define for $1 \le p < \infty$,

$$\|f\|_p = \left\{ \int_R | f |^p \, d\mu \right\}^{\frac{1}{p}}$$

Let Lp denote the space of all such f for which $\|f\|_p < \infty$. Let $f \in Lp$ in some neighborhood of a point $x \in R$. If there is a polynomial $P(t) = P_x(t)$ of degree at most k whose coefficients are in C and $t \in R$ such that

$$\left\{ \frac{1}{h} \int_0^h | f(x+t) - P(t) |^p \, dt \right\}^{\frac{1}{p}} = o(h^k) \text{ as } h \to 0 \quad (1)$$

then f is said to have a k-th Lp-derivative at x of order k and if $\frac{\alpha_k}{k!}$ is the coefficient of t^k in $P(t)$ then α_k is called the k-th Lp-derivative of f at x and is denoted by $f_{(k),p}(x)$. Similar to the definition of Lp-derivative for real valued functions. In [1] Calderon and Zygmund introduced Lp-derivative to study elliptic partial differential equation. In [2] Evans studied the properties of Lp-derivative and the relation with Peano derivative for real valued functions. Recently S.N.Mukhopadhyay presented Lp-derivative and its interrelations with other generalized derivative in his book "Higher Order

Derivative" [3]. Here in this short article I have studied the generality of Lp-derivative for complex valued functions. Also Lp-continuity and Lp-boundedness is presented in same line.

It can be shown that if $f_{(k),p}(x)$ exists then it is unique and that all the previous derivative $f_{(i),p}(x), 0 \le i \le k-1$, also exists. Also if the ordinary derivative $f^{(k)}(x)$ exists at x then $f_{(k),p}(x)$ exists at x and $f^{(k)}(x) = f_{(k),p}(x)$.(It can proved using the same technique as it is used in [3] for real valued function.

In this paper there are four sections in which Section-I deal with the introduction of the total work, the definition of L-p derivative of complex valued functions and some preliminary ideas. In Section-II the generality of L_p-derivative of complex valued function is shown. In Section-III we define the L_p-continuity and L_p-boundedness of complex valued function and also the generality is shown. In Section-IV we conclude about its future aspects.

II. GENERALITY OF L_P-DERIVATIVE IN C

Here we are giving an example to show that L_P-derivative for complex valued function is more general than the ordinary derivative.

Example 2.1 Let

$$f(x) = \begin{cases} x^3 e^{ix^{-2}} & if, x \neq 0 \\ 0 & if, x = 0 \end{cases}$$

Then $f'(0) = 0$ and for $x \neq 0$ $f'(x) = 3x^2 e^{ix^{-2}} - 2ie^{ix^{-2}}$ and so $f'(x)$ exists everywhere. But $f''(0)$ does not exists, for if $x_n = \frac{1}{\sqrt{2n\pi}}$ then $e^{ix^{-2}} = e^{i2n\pi} = 1$ and so $f'(x_n) - f'(0) = 3\frac{1}{2n\pi}.1 - 2i.1$. Hence,

$$\frac{f'(x_n) - f'(0)}{x_n - 0} = \sqrt{2n\pi}(\frac{3}{2n\pi} - 2i) = \frac{3}{\sqrt{2n\pi}} - 2i\sqrt{2n\pi}$$

does not tends to finite limit as $n \to \infty$, so $f''(0)$ does not exists. But by taking $P(t) = 0.1 - 0.t - 0.t^2$ we get,

$$\left\{\frac{1}{h}\int_0^h | t^3 e^{it^{-2}} - 0.1 - 0.t - 0.t^2 |^p \, dt\right\}^{\frac{1}{p}} = \left\{\frac{1}{h}\int_0^h | t^3 e^{it^{-2}} |^p \, dt\right\}^{\frac{1}{p}} \le 2\left\{\frac{1}{h}\int_0^h | t^3 |^p \, dt\right\}^{\frac{1}{p}}$$

$$= 2\left\{\frac{1}{h}\frac{h^{3p+1}}{3p+1}\right\}^{\frac{1}{p}} \text{(taking } h > 0\text{)}$$

$$= \frac{2}{(3p+1)^{\frac{1}{p}}} h^3 = o(h^3) \text{ as } h \to 0$$

So $f_{(2),p}(0)$ exists and $f_{(2),p}(0) = 0$.

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III. LP -CONTINUITY AND LP-BOUNDEDNESS:

Suppose $f : R \to C$ is measurable. Then f is said to be Lp-continuous at x if

$$\left\{\frac{1}{h}\int_0^h | f(x+t)-f(x)|^p \, dt\right\}^{\frac{1}{p}} = o(1) \text{ as } h \to 0. \qquad (2)$$

Clearly if f is continuous at x then f is Lp-continuous at x and if f is Lp-continuous at x then $f(x) = f_{(0),p}(x)$. So if Lp-derivative $f_{(k),p}(x)$ exists then $f_{(0),p}(x)$ exists and then f is Lp-continuous if $f_{(0),p}(x) = f(x)$. In particular f is L_1-continuous at x if and only if x is a Lebesgue point of f. For real valued function f it is known that if $f \in L_p,\ p \geq 0$ then f is Lp-continuous almost everywhere [p-6 [4]].

A function f is said to be Lp-bounded of order k at x if $o(h^k)$ in (1) is replaced by $O(h^k)$ as $h \to 0$.

That is f is said to be Lp-bounded of order k at x if ,

$$\left\{\frac{1}{h}\int_0^h | f(x+t)-P(t)|^p \, dt\right\}^{\frac{1}{p}} = O(h^k) \text{ as } h \to 0 \qquad (3)$$

where $P(t) = P_x(t)$ is a polynomial of degree at most k whose coefficients are in C and $t \in R$.

Clearly if f is Lp-continuous at x, if $k = 0$ then $P(t) = P(0) = f_{(0),p}$ and so f is Lp-bounded at x of order 0. It is easy to verify that if f is Lp-bounded of order k at x then $f_{(k-1),p}$ exists and hence f is Lp-bounded of order $k-1$ at x.

(The proof is similar as that of real valued function which is proved in [p-59[3]].

Theorem 3.1. If f is bounded in a neighborhood of x then f is Lp-bounded of order zero.

Proof. Suppose f is bounded at x, then there is $M > 0$ and $\delta > 0$ such that $f(x+t) \leq M$ for $|t| < \delta$. So,

$$\left\{\frac{1}{h}\int_0^h | f(x+t)-f(x)|^p \, dt\right\}^{\frac{1}{p}} \leq 2M = O(1) = O(h^0) \text{ as } h \to 0$$

and so f is Lp-bounded at x of order 0.

The following example shows that the converse of the above theorem is not true.

Example 3.2. Let $S = \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})$ and let f be defined as,

$$f(x) = \begin{cases} x^3 e^{ix^{-2}} & if, x \in S \\ ni & if, x = \frac{1}{n}, n = 1,2,.. \\ 0 & elsewhere \end{cases}$$

Then f is not bounded in any neighborhood of zero. But as in Example 2.1 we can show that

$$\left\{\frac{1}{h}\int_0^h | f(x+t) - f(x)|^p \, dt\right\}^{\frac{1}{p}} = O(h^3) \text{ as } h \to 0.$$

So f is Lp-bounded of order 3 and hence it is Lp-bounded of order zero.

IV. CONCLUSION and Future Scope

In this paper it is shown Lp-derivative, Lp-continuity and Lp-boundedness of complex valued function is a generalization of the ordinary derivative, continuity and boundedness of complex valued function.

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