International Journal of Scientific Research in Mathematical and Statistical Sciences **Survey Paper**
Volume-6, Issue-1, pp.323-327, February (2019) **E-ISSN:** 2348-4519
DOI: https://doi.org/10.26438/ijsrmss/v6i1.323327

Application of Fixed Point Theorems for Digital Contractive Type Mappings in Digital Metric Space

Neena Gupta[1], Amardeep Singh[2], Geeta Modi[3]

[1]Department of Mathematics, Career College, Barkatullah University, Bhopal, M.P., India
[2]Department of Mathematics, Govt. MVM College, Barkatullah University, Bhopal, M.P., India
[3]Department of Mathematics, Govt. MVM College, Barkatullah University, Bhopal, M.P., India

Corresponding author: gneena33@gmail.com

Available online at: www.isroset.org

Received 15/Feb/2019, Accepted: 25/Feb/2019, Online: 28/Feb/2019

Abstract- In this paper we introduce a notion and define **a** contractive type mapping for digital metric spaces. We prove some fixed point theorems in digital metric space by using contractive type mapping. We obtain Banach contraction principle in digital metric space. The purpose of this paper is to associate fixed point theory and digital images. This shows an application of fixed point theory in digital metric space.

Keywords- digital image, digital metric space, Banach contraction principle, finite sequence, increasing sequence, decreasing sequence, θ-contractive.

Subject Classification- 2010 MSC: 47H10, 54E35, 68U10

1. INTRODUCTION

Fixed point theory began from Banach contraction principle of Banach [1] (1922) with complete metric space as a background and went back to Brouwer fixed point theorem of Brouwer [2, 3] (1910) with R_n as background. The area of fixed point theory is very active in many braches of mathematics and other related disciplines such as image processing, computer vision, applied mathematics, etc.

Digital topology is the study of the topological properties of images arrays. The results provide a sound mathematical basis for image processing operations such as image thinning, border following, contour filling and object counting.

Kong [4] then introduced the digital fundamental group of a discrete object. The digital version of the topological concept was given by Boxer [5, 6, 7].

A. Rosenfeld [8] first studied the almost fixed point property of digital images. Ege and Karaca [9, 10] gave relative and reduced Lefschetz fixed point theorem for digital images. They also calculated the degree of antipodal map for the sphere like digital images using fixed point properties. Ege and Karaca [11] defined a digital metric space and proved the famous Banach Contraction Principle for digital images.

In this paper, we introduce $\theta-$contractions and θ-contractive mappings on digital metric spaces. We prove the existence and uniqueness of fixed points in digital metric space.

2. PRELIMINARIES

Let X be a subset of Z^n for a positive integer n where Z^n is the set of lattice points in the n–dimensional Euclidean Space and ℓ represent an adjacency relation for the members of X. A digital image consists of (X, ℓ).

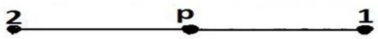
Definition 2.1. (Boxer [6]): Let ℓ, n be positive integers, $1 \leq \ell \leq n$ and p, q be two distinct points $p = (p_1,p_2,,\ldots,p_n)$, $q = (q_1,q_2,,\ldots,q_n) \in Z^n$

p and q are ℓ adjacent if there are at most ℓ indices i such that $|p_i - q_i| < 1$ and for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

The following statements can be obtained from definition 2.1

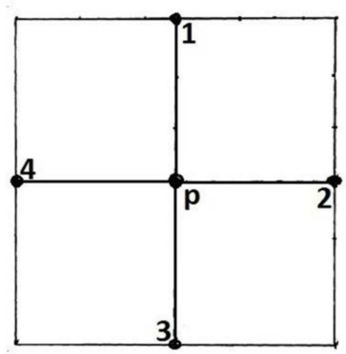
For a given $p\in Z^n$, the number of points $q\in Z^n$ which are ℓ adjacent to p is denoted by $k = k(\ell, n)$. It may be noted that $k(\ell, n)$ is independent of p.

2.1.1 If $p \in Z$ (i.e. $n = 1$) then ℓ can take only one value $\ell=1$. In this case, $k(1,1) = 2$, since $p-1$ & $p+1$ are the only points 1-adjacent to p in Z. Thus, $k = k(1,1) = 2$ and q is 1-adjacent to p if and only if $|p-q| = 1$.

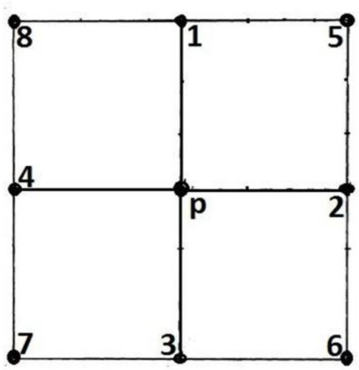


1 - adjacency

2.1.2 If $p \in Z^2$ (i.e. $n = 2$) then ℓ can take values $\ell = 1, 2$.
When $\ell = 1$, the points 1-adjacent to $p = (p_1, p_2)$ are $(p_1 \pm 1, p_2)$, $(p_1, p_2 \pm 1)$
Thus, the number of points 1-adjacent to $p = (p_1, p_2)$ is 4, so that $k = k(1,2) = 4$. (fig. (a))
When $\ell = 2$, the points 2-adjacent to $p = (p_1, p_2)$ are $(p_1 \pm 1, p_2)$, $(p_1, p_2 \pm 1)$, $(p_1 \pm 1, p_2 \pm 1)$
Thus, the number of points 2-adjacent to $p = (p_1, p_2)$ is 8, so that $k = k(2, 2) = 8$. (fig. (b))



(a) 1 - adjacency



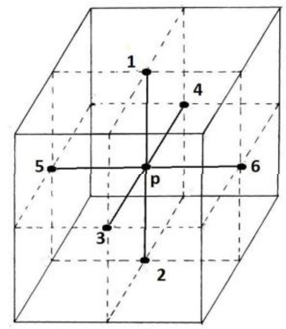
(b) 2 - adjacency

2.1.3 If $p \in Z^n$ (i.e. $n = 3$) then ℓ can take values $\ell = 1, 2, 3$.

When $\ell = 1$, the points 1-adjacent to $p = (p_1, p_2, p_3)$ are

$(p_1 \pm 1, p_2, p_3)$, $(p_1, p_2 \pm 1, p_3)$, $(p_1, p_2, p_3 \pm 1)$

Thus, the number of points 1-adjacent to p is 6, so that $k = k(1, 3) = 6$. (fig. (a))

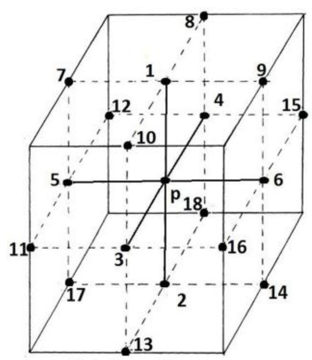


(a) 1 - adjacency

When $\ell = 2$, the points 2-adjacent to $p = (p_1, p_2, p_3)$ are

$(p_1 \pm 1, p_2, p_3)$, $(p_1, p_2 \pm 1, p_3)$, $(p_1, p_2, p_3 \pm 1)$, $(p_1 \pm 1, p_2 \pm 1, p_3)$, $(p_1 \pm 1, p_2, p_3 \pm 1)$, $(p_1, p_2 \pm 1, p_3 \pm 1)$

Thus, number of points 2-adjacent to p is 18, so that $k = k(2, 3) = 18$. (fig. (b))



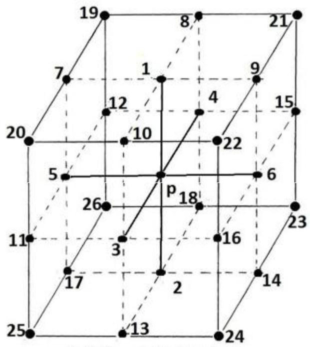
(b) 2 - adjacency

When $\ell = 3$, the points 3-adjacent to $p = (p_1, p_2, p_3)$ are

$(p_1 \pm 1, p_2, p_3)$, $(p_1, p_2 \pm 1, p_3)$, $(p_1, p_2, p_3 \pm 1)$,$(p_1 \pm 1, p_2 \pm 1, p_3)$,$(p_1 \pm 1, p_2, p_3 \pm 1)$, $(p_1, p_2 \pm 1, p_3 \pm 1)$,$(p \pm 1_1, p_2 \pm 1, p_3 \pm 1)$

Thus, the number of points 3-adjacent to p is 26,

so that $k = k(3, 3) = 26$. (fig. (c))



(c) 3 - adjacency

In general to study n-D digital image, if $1 \leq \ell \leq n$ then $k = k(\ell, n)$ is given by the following formula [12].

$$k(\ell, n) = \sum_{i=n-\ell}^{n-1} 2^{n-i} c_i^n ,$$

where $c_i^n = \frac{n!}{(n-i)!\, i!}$

Suppose X is a non-empty subset of Z^n,
$1 \leq \ell \leq n$, $k = k(\ell, n)$
Then (X, ℓ) is called a digital image with ℓ-adjacency (Rosenfeld [13]). We also say that (X, ℓ) is called n-D digital image [8].

Definition 2.2 (Han [14]): Let $X \subset Z$, d be the Euclidean metric on Z^n. (X, d) is a metric space. Suppose (X, ℓ) is a digital image with ℓ−adjacency, then (X, d, ℓ) is called a digital metric space.

Definition 2.3 (Han [14]): A sequence $\{x_n\}$ of points of the digital metric space (X, d, ℓ) is a Cauchy sequence if there is $M \in N$ such that, $d(x_n, x_n) < 1$ for all n, m > M.

Theorem 2.1 (Han [14]): For a digital metric space (X, d, ℓ), if a sequence $\{x_n\} \subset X \subset Z^n$ is a Cauchy sequence, there is $M \in N$ such that for all n, m > M, we have $x_n = x_m$.

Definition 2.4 (Han [14]): A sequence $\{x_n\}$ of points of a digital metric space (X, d, ℓ) converges to a limit $L \in X$, if for all $\epsilon > 0$, there is $M \in N$ such that $d(x_n, L) < \epsilon$ for all n > M

Proposition 2.1 (Han [14]): A sequence $\{x_n\}$ of points of a digital metric space (X, d, ℓ) converges to a limit $L \in X$ if there is $M \in N$ such that $x_n = L$, for all n > M. (i.e. $x_n = x_{n+1} = x_{n+2} = \cdots = L$).

Definition 2.5 (Han [14]): A digital metric space (X, d, ℓ) is complete if any Cauchy sequence $\{x_n\}$ converges to a point L of (X, d, ℓ).

Theorem 2.2 (Han [14]): A digital metric space (X, d, ℓ) is complete.

Definition 2.6 (Han [14]): Let (X, d, ℓ) be a digital metric space and $T : (X, d, \ell) \longrightarrow (X, d, \ell)$ be a self-map. If there exists $\lambda \in [0, 1)$ such that, $d(Tx, Ty) \leq \lambda d(x, y)$, for all $x, y \in X$, then T is called a contraction map.

Proposition 2.2 (Han [14]): Every digital contraction map $T : (X, d, \ell) \longrightarrow (X, d, \ell)$ is ℓ−continuous (Digital continuous).

Lemma 2.1 [15]: Let $X \in Z^n$ and (X, d, ℓ) be digital metric space. Then there does not exist a sequence $\{x_m\}$ of distinct elements in X, such that $d(x_{m+1}, x_m) < d(x_m, x_{m+1})$, for m = 1, 2, 3, …..
i.e. there exist a finite sequence $\{x_m\}$

3. MAIN RESULT

First we introduce a notion.

Notion 3.1: Let $\Theta = \{\theta : [0, \infty) \longrightarrow [0, \infty)\}$ be such that θ is increasing, $\theta(t) < \sqrt{t}$ for $t > 0$, $\theta(t) = 0$ iff $t = 0$

Definition 3.1: Suppose (X, d, ℓ) is a digital metric space, $T : X \longrightarrow X$ and $\theta \in \Theta$. Suppose $d(Tx, Ty) \leq \theta(d(x, y))$, $\forall\, x, y \in X$. Then T is called a digital θ-contraction.

Now we prove a fixed point theorem on θ-contraction.

Theorem 3.1

Suppose (X, d, ℓ) is a digital metric space, $T : X \longrightarrow X$ and $\theta \in \Theta$. Suppose $d(Tx, Ty) \leq \theta(d(x,y))$, $\forall$ x, y $\in$ X. T is called a digital θ-contraction. Then T has unique fixed point.

Proof: Let $x_0 \in X$ and suppose $x_{n+1} = Tx_n$, for n = 0, 1, 2, 3,……..
We may suppose that $x_n \neq x_{n+1}$, for n = 0, 1, 2,…….
Otherwise x_n is a fixed point.
Now,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \theta(d(x_n, x_{n-1})) \\ &= \theta(d(x_{n-1}, x_n)) \\ &< \sqrt{d(x_{n-1}, x_n)} \\ &\leq d(x_{n-1}, x_n) \end{aligned}$$

Therefore $d(x_{n+1}, x_n) < d(x_{n-1}, x_n)$, $(\because x_n \neq x_{n-1})$
Therefore $d(x_{n+1}, x_n)$ is strictly decreasing sequence.
Therefore $x_n = x_{n+1}$, for large n. By Lemma (2.1)
Therefore x_n is a fixed point of T, for large n.

Uniqueness of fixed point of T

Suppose u and v are fixed points of T.
Then,

$$\begin{aligned} d(u, v) &= d(Tu, Tv) \\ &\leq \theta(d(u, v)) \\ &< \sqrt{d(u, v)} \\ &\leq d(u, v) \end{aligned}$$

It is a contradiction, if $u \neq v$.
Therefore u = v.
Hence T has unique fixed point.

Corollary 3.1

(Banach Contraction Principle in Digital Metric Spaces)
Let (X, d, ℓ) be a digital metric space and $T : X \longrightarrow X$ be such that $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$ and for some $\lambda \in [0, 1)$. Then, T has unique fixed point.

Proof: Take $\theta(t) = \lambda t$ in Theorem 3.1. Then we get the result.

Theorem 3.2 Let (X, d, ℓ) be a digital metric space and $T : X \longrightarrow X$ such that $d(Tx, Ty) < \mu(x, y)$, $\forall$ x, y ∈ X, x≠ y Where

$$\mu(x, y) = \max\left\{ \frac{1}{2}\left[d(y,Ty)\frac{1+d(x,\ Tx)}{1+d(x,\ y)} + d(Tx,Ty) + d(x,y)\right],\ d(x, Tx)\frac{1+d(y,\ Ty)}{1+d(Tx,\ Ty)}\right\}$$

Then T has unique fixed point.

Proof:

Let $x_{n+1} = Tx_n$ for n = 0, 1, 2, ……

Suppose $x_n \neq x_{n+1}$ for n = 0, 1, 2, ……

Otherwise x_n is a fixed point.

Now,

$d(x_n, x_{n+1})$

$= d(x_{n-1}, x_n)$

$< \mu(x_{n-1}, x_n)$

$$= \max\left\{ \frac{1}{2}\left[\begin{matrix} d(x_n,Tx_n)\frac{1+d(x_{n-1},\ Tx_{n-1})}{1+d(x_{n-1},\ x_n)} \\ +d(Tx_{n-1},Tx_n) + d(x_{n-1},x_n)\end{matrix}\right],\ d(x_{n-1},Tx_{n-1})\frac{1+d(x_n,\ Tx_n)}{1+d(Tx_{n-1},\ Tx_n)}\right\}$$

$$=\max\left\{ \frac{1}{2}\left[\begin{matrix} d(x_n,\ x_{n+1})\frac{1+d(x_{n-1},\ x_n)}{1+d(x_{n-1},\ x_n)} \\ + d(x_n,\ x_{n+1}) + d(x_{n-1},\ x_n)\end{matrix}\right],\ d(x_{n-1},x_n)\frac{1+d(x_n,\ x_{n+1})}{1+d(x_n,\ x_{n+1})}\right\}$$

$$\leq \max\{\tfrac{1}{2}[d(x_n,\ x_{n+1}) + d(x_{n-1},\ x_{n+1})],\ d(x_{n-1},\ x_n)\}$$

$$\leq \max\{\tfrac{1}{2}[d(x_{n-1},\ x_n)], d(x_{n-1},\ x_n)\}$$

$$= d(x_{n-1},\ x_n)$$

Hence,

$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for n = 0, 1, 2, ……

Hence by Lemma (2.1)

T has a fixed point.

Uniqueness of fixed point of T

Let u and v are two fixed points of T.

Therefore Tu = u and Tv = v

Now,

d(u, v)

= d(Tu, Tv)

$< \mu(u, v)$

$$=\max\left\{ \frac{1}{2}\left[d(v,Tv)\frac{1+d(u,\ Tu)}{1+d(u,\ v)} + d(Tu,Tv) + d(u,v)\right],\ d(u,Tu)\frac{1+d(v,\ Tv)}{1+d(Tu,\ Tv)}\right\}$$

$$=\max\left\{ \frac{1}{2}\left[d(v,v)\frac{1+d(u,\ u)}{1+d(u,\ v)} + d(u,v) + d(u,v)\right],\ d(u, u)\frac{1+d(v,\ v)}{1+d(u,\ v)}\right\}$$

$= \max\{\frac{1}{2}[d(u,v) + d(u,v)], 0\}$

$= \max\{d(u,v), 0\}$

$= d(u,v)$

⇨ $d(u,v) < d(u,v)$, for u ≠ v

which is contradiction.

Therefore u = v

Hence there exists unique fixed point.

This completes the proof.

4. CONCLUSION

Our purpose is to give the digital version of Banach fixed point theorem by introducing θ-contractive type mapping. These results are the applications of fixed point theory in digital metric space. It will be useful for digital topology and fixed point theory. In the future, we will also use the fixed point theory to solve some problems in digital images.

REFERENCES

[1] Banach S,"Sur les operations dans les ensembles abstraits et leurs applications aux equations integrals",Fund. Math., No. 3, pp. 133–181, 1922.

[2] Brouwer LEJ, "ubereineindeutige,stetiger Transformation envon Flaachen in Sich", Math. Ann., 69, pp. 176-180, 1910.

[3] Brouwer LEJ,"uberAbbildungen Von Mannigfaltigkeiton", Math. Ann., 77,pp. 97-115, 1912.

[4] Kong TY, "A Digital Fundamental Group, Computers and Graphics", 13, pp. 159-166, 1989.

[5] Boxer L, "Digitally Continuous Functions",Pattern Recognition Letters, 15, pp. 833- 839, 1994.

[6] Boxer L, "A Classical Constructions for The Digital Fundamental Group", J.Math. Imaging Vis., 10, pp. 51-62,1999.

[7] Boxer L, "Properties of Digital Homotopy", J. Math. Imaging Vis., 22,pp. 19-26, 2005.

[8] Rosenfeld A,"Continuous functions on digital pictures", Pattern Recognition Letters 4, pp. 177-184, 1986.

[9] Ege O, Karaca I, "Applications of The Lefschetz Number to Digital Images",Bull. Belg. Math. Soc. Simon Stevin,21,pp.823-839,2014.

[10] Ege O, Karaca I, "Banach Fixed Point Theorem for Digital Images", J. Nonlinear Sci. Appl. 8, pp. 237-245, 2015.

[11] Ege O, Karaca I, "Lefschetz Fixed Point Theorem for Digital Images",Fixed Point Theory, Appl., 13 pages, 2013.
[12] Han S.E,"Non-Product property of the digital fundamental group", Inform. Sci., 171, pp. 73-91, 2005.
[13] Rosenfeld A, "Digital Topology",Amer. Math. Monthly, 86, pp. 76-87,1979.
[14] Han S. E, "Banach fixed point Theorem from the viewpoint of digital topology", J. Nonlinear sci. Appl., 9, pp. 895-905, 2016.
[15] Sridevi K., M. V. R. Kameshwari And D. M. K. Kiran,"Fixed Point Theorem for Digital Contractive Type Mappings in Digital Metric Space" International Journal of Mathematics Trends and Technology, Vol. 48, No. 3, pp. 159-167, 2017

