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Fixed Point Theorem in Partial Metric Spaces

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Abstract—In this paper, we proved the common fixed point theorems for sequence of mappings in Partial Metric Spaces.

Keywords—*Partial Metric Space, Complete Patial Metric Space, Coincidence Point, Weakly Compatible, Fixed Point.*

I. INTRODUCTION

The study of fixed point theorems of maps satisfying contractive type conditions in partial metric spaces has been a very active field of research activity recently. Partial metric spaces were introduced by Mathews[15] in 1992, and proved common fixed point theorems for compatible maps in partial metric spaces. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation.

Definition 1.1: A partial metric on a nonempty set X is a function $p: X \times X \to R_+$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(p2) $p(x,x) \le p(x,y)$,

(p3) $p(x,y) = p(y,x)$,

(p4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Remark 1.2: It is clear that, if $p(x,y) = 0$, then from (p1) and (p2), $x = y$. But if $x = y, p(x,y)$ may not be 0. A basic example of a partial metric space is the pair (R_+, p), where $p(x,y) = max\{x,y\}$ for all $x, y \in R^+$. Each partial metric p on Xgenerates a T_0topology τ_p on X which has a base the family of open $p-$ balls $\{B^p(x,\varepsilon), x \in X, \varepsilon > 0\}$ where $B^p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to R^+$ given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a metric on X.

Definition1.3: let (X,p) be a partial metric space and $\{x_n\}$be a sequence in X. Then

(1) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$

(2) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$.

Definition1.4: A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that

$$p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$$

Remark1.5: It is easy to see that every closed subset of a complete partial metric space is complete.

Lemma 1.6: Let (X,p) be a partial metric space. Then

(1) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s),

(2) (X,p) is complete if and only if the metric space (X,p^s) is complete. Further more $\lim_{n\to+\infty} p^s(x_n,x) = 0$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

Mathews[15] obtained the following Banach fixed point theorem on complete partial metric spaces.

Theorem 1.7: Let f be a mapping of a complete partial metric space (X,p) into itself such that there is a real number c with $0 \le c < 1$, satisfying for all $x,y \in X$, $p(fx,fy) \le cp(x,y)$. Then f has a unique fixed point.

II. MAIN RESULT

Definition 2.1: Let X be a non-empty set and $T_1, T_2: X \to X$ are given self maps on X. If $w = T_1x = T_2x$ for some $x \in X$, then x is called a coincidence point of T_1 and T_2 and w is called a point of coincidence of T_1 and T_2.

Definition 2.2: Let X be a non-empty set and $T_1, T_2: X \to X$ are given self maps on X. The pair $\{T_1, T_2\}$ is said to be weakly compatible if $T_1T_2t = T_2T_1t$, whenever $T_1t = T_2t$ for some t in X.

Our main result is the following:

Theorem 2.3: Suppose that $\{A_i\}, \{A_j\}(i \neq j). S, T$ are self maps of a complete partial metric space (X, p) such that $A_iX \subseteq TX, A_jX \subseteq SX \ (i \neq j)$and for all $x, y \in X$, where $\phi \in \phi$

$$p(A_ix, A_jy) \leq \phi\big(max\{p(Sx, Ty), p(A_ix, Sx), p(A_jy, Ty)\}\big) \tag{2.1}$$

If one the ranges A_iX, A_jX, TX and SX is a closed subset of (X, p), then

1. (1) $A_i$and S have a coincidence point, $(i \neq j)$
2. (2) $A_j$and T have a coincidence point. Moreover, if the pairs $\{A_i, S\}$and $\{A_j, T\}(i \neq j)$ are weakly compatible, then $A_i, A_j(i \neq j)$, T and S have a unique common fixed point.

Proof: let x_0 be an arbitrary point in X. Since $A_iX \subseteq TX$, there exists $x_1 \in X$ such that $Tx_1 = A_ix_0$. Since $A_jX \subseteq SX$, there exists $x_2 \in X$ such that $Sx_2 = A_jx_1(i \neq j)$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Tx_{2n+1} = A_ix_{2n}, y_{2n+1} = Sx_{2n+2} = A_jx_{2n+1} \tag{2.2}$$

For every $n \in N(i \neq j)$.

We claim that $\{y_n\}$ is a Cauchy sequence in the partial metric space (X, p).

We have : for $(i \neq j)$,

$$\begin{aligned} p(y_{2p}, y_{2p+1}) &= p(A_ix_{2p}, A_jx_{2p+1}) \\ &\leq \phi\big(max\{p(Sx_{2p}, Tx_{2p+1}), p(A_ix_{2p}, Sx_{2p}), p(A_jx_{2p+1}, Tx_{2p+1})\}\big) \\ &\leq \phi\big(max\{p(y_{2p-1}, y_{2p}), p(y_{2p}, y_{2p-1}), p(y_{2p+1}, y_{2p})\}\big) \\ &\leq \phi\big(\max\{p(y_{2p-1}, y_{2p}), p(y_{2p+1}, y_{2p})\}\big) \end{aligned}$$

Now, we get

$$p(y_{2p}, y_{2p+1}) \leq \phi\big(\max\{p(y_{2p-1}, y_{2p}), p(y_{2p+1}, y_{2p})\}\big) \tag{2.3}$$

Similarly, we obtain

$$p(y_{2p+1}, y_{2p+2}) \leq \phi\big(\max\{p(y_{2p}, y_{2p+1}), p(y_{2p+1}, y_{2p+2})\}\big) \tag{2.4}$$

Therefore, from (2.3) and (2.4),

$p(y_n, y_{n+1}) \leq \phi(\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\})$ for sufficiently large n.

Suppose that there exists $p \in N$ such that $p(y_{2p-1}, y_{2p}) = 0$. Then we have $y_{2p-1} = y_{2p}$ and from (2.3), we obtain $p(y_{2p}, y_{2p+1}) \leq \phi\left(p(y_{2p}, y_{2p+1})\right)$.

Since $\phi(t) < t$ for each $t > 0$, the above inequality implies that $p(y_{2p}, y_{2p+1}) = 0$ and then $y_{2p} = y_{2p+1}$. From (2.4), we get $p(y_{2p+1}, y_{2p+2}) \leq \phi\left(p(y_{2p+1}, y_{2p+2})\right)$, which implies that $y_{2p+1} = y_{2p+2}$.

Hence, we have $y_{2p-1} = y_{2p} = y_{2p+1} = y_{2p+2} = \cdots$.

Then $\{y_n\}$ is a Cauchy sequence in (X, p). The same conclusion holds if we suppose that there exists $p \in N$ such that $p(y_{2p}, y_{2p+1}) = 0$.

Now, we assume that

$$p(y_n, y_{n+1}) > 0, \text{ for sufficiently large } n. \tag{2.6}$$

Then from (2.5), as $\phi(t) < t$ for each $t > 0$, we have

$p(y_n, y_{n+1}) \leq \max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\}$

Hence we get $p(y_n, y_{n+1}) < p(y_{n-1}, y_n)$.

Therefore, $\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\} = p(y_{n-1}, y_n)$ for sufficiently large n.

Thus from (2.5),

$$p(y_n, y_{n+1}) \leq \phi\big(p(y_{n-1}, y_n)\big) \text{ for sufficiently large } n. \tag{2.7}$$

Repeating this inequality n time we obtain

$$p(y_n, y_{n+1}) \leq \phi^n\big(p(y_0, y_1)\big). \tag{2.8}$$

By the properties (p2) and (p3) we have

$\max\{p(y_n, y_n), p(y_{n+1}, y_{n+1})\} \leq p(y_n, y_{n+1})$.

Thus from (2.8),

$$\max\{p(y_n,y_n),p(y_{n+1},y_{n+1})\} \le \phi^n\big(p(y_0,y_1)\big). \quad (2.9)$$

Therefore,

$$p^s(y_n,y_{n+1}) = 2p(y_n,y_{n+1}) - p(y_n,y_n) - p(y_{n+1},y_{n+1}) \le 2p(y_n,y_{n+1}) + p(y_n,y_n) + p(y_{n+1},y_{n+1}) \le 4\phi^n\big(p(y_0,y_1)\big)$$

Now by triangle inequality for the metric p^s and (2.9) for any $k,n \in N^*$ we have

$$\begin{aligned} p^s(y_n,y_{n+k}) &\le p^s(y_n,y_{n+1}) + p^s(y_{n+1},y_{n+2}) + \cdots + p^s(y_{n+k-1},y_{n+k}) \\ &\le 4\phi^n\big(p(y_0,y_1)\big) + 4\phi^{n+1}\big(p(y_0,y_1)\big) + \cdots + 4\phi^{n+k-1}\big(p(y_0,y_1)\big) \\ &\le 4\left(\sum_{i=n}^{n+k-1} \phi^i\big(p(y_0,y_1)\big)\right) \\ &\le 4\left(\sum_{i=n}^{\infty} \phi^i\big(p(y_0,y_1)\big)\right) \end{aligned}$$

Hence and from the property (b) of ϕ we conclude that for an arbitrary $\varepsilon > 0$ there is a positive integer n_0 such that $p^s(y_n,y_{n+k}) < \varepsilon$ for every $n \ge n_0$ and all $k \in N$.

Thus we proved that $\{y_n\}$ is a Cauchy sequence in the metric space (X,p^s). Since $(X,p)$is complete, then from lemma (1.6) (X,p^s) is a complete metric space. Therefore, the sequence $\{y_n\}$ converges to some $y \in X$, that is, $\lim_{n\to+\infty} p^s(y_n,y) = 0$.

From the properties (b) in above lemma, we have

$$p(y,y) = \lim_{n\to+\infty} p(y_n,y) = \lim_{m\ge n\to+\infty} p(y_n,y_m) \quad (2.10)$$

Moreover, since $\{y_n\}$ is a Cauchy sequence in the metric space (X,p^s), then $\lim_{n,m\to+\infty} p^s(y_n,y_m) = 0$ and so from (2.9) and the property (b) of lemma (1.6) we have

$$\lim_{n\to+\infty} p(y_n,y_n) = 0. \quad (2.11)$$

Thus from the definition of p^s and (2.11), we have

$$\lim_{m\ge n\to+\infty} p(y_n,y_m) = 0.$$

Therefore, from (2.10), we have

$$p(y,y) = \lim_{n\to+\infty} p(y_n,y) = \lim_{m\ge n} p(y_n,y_m) = 0. \quad (2.12)$$

This implies that

$$\lim_{n\to+\infty} p(y_{2n},y) = \lim_{n\to+\infty} p(y_{2n-1},y) = 0 \quad (2.13)$$

Thus from (2.13) we have

$$\lim_{n\to+\infty} p(A_i x_{2n},y) = \lim_{n\to+\infty} p(Tx_{2n+1},y) = 0 \quad (2.14)$$

And $\lim_{n\to+\infty} p\big(A_j x_{2n-1},y\big) = \lim_{n\to+\infty} p(Sx_{2n},y) = 0$ (2.15)

Now we can suppose, without loss of generality, that SX is a closed subset of the partial metric space (X,p). From (2.15), there exists $u \in X$ such that $y = Su$. We claim that $p(A_i u,y) = 0$. Suppose, to the contrary, that $p(A_i u,y) > 0$.

$$\begin{aligned} p(y,A_i u) &\le p\big(y,A_j x_{2n+1}\big) + p\big(A_i u,A_j x_{2n+1}\big) - p\big(A_j x_{2n+1},A_j x_{2n+1}\big) \text{ for } (i \ne j) \\ &\le p\big(y,A_j x_{2n+1}\big) + p\big(A_i u,A_j x_{2n+1}\big) \\ &\le p\big(y,A_j x_{2n+1}\big) + \phi(\max\{p(y,y_{2n}),p(A_i u,y),p(y_{2n+1},y_{2n})\}) \end{aligned}$$

Since ϕ is continuous, from (2.12), and letting $n \to \infty$ we obtain

$$\begin{aligned} p(y,A_i u) &\le \lim_{n\to+\infty} [p(y_n,y_{2n+1}) + \phi(\max\{p(y,y_{2n}),p(A_i u,y),p(y_{2n+1},y_{2n})\})] \\ &= \lim_{n\to+\infty} p(y_n,y_{2n+1}) + \phi\left(\lim_{n\to+\infty} \max\{p(y,y_{2n}),p(A_i u,y),p(y_{2n+1},y_{2n})\}\right) \\ &= \phi\big(p(A_i u,y)\big) \end{aligned}$$

Hence, as we supposed that $p(Au,y) > 0$and as $\phi(t) < t$ for $t > 0$,

We have $p(y,A_i u) < p(y,A_i u)$ which is a contraction.

Therefore,

$$p(A_i u,y) = 0 \Rightarrow y = A_i u. \quad (2.16)$$

Since $y = Su$, then $A_i u = Su$, that is a coincidence point of A_i and S. Hence the proof of (i). Since $A_i x \subseteq TX$ and (2.16), we have $y \in TX$. Therefore there exists $v \in X$ such that $y = Tv$. We claim that $p\big(A_j v,y\big) = 0$. Suppose, to the contrary, that $p\big(A_j v,y\big) > 0$. From (2.1) and here $y = Su = A_i u = Tv$

we have

$$\begin{aligned} p(y,A_i v) &= p(A_i u,A_i v) \\ &\le \phi\big(\max\{p(Su,Tv),p(A_i u,Su),p\big(A_j v,Tv\big)\}\big) \end{aligned}$$

$$\leq \phi\big(\max\{p(y,y),p(y,y),p(A_jv,y)\}\big)$$
$$\leq \phi\left(p\big(A_jv,y\big)\right)$$
$$< p\big(A_jv,y\big)$$
This is contradiction. Then, we deduce that $p\big(A_jv,y\big)=0$ and
$$y=A_jv=Tv. \tag{2.17}$$
Therefore v is a coincidence point of A_j and T, then (ii) holds. Since the pair $\{A_i,S\}$is weakly compatible, from (2.16), we have $A_iy=A_iSu=SA_iu=Sy$. We claim that $p(A_iy,y)=0$. Suppose, to the contrary, that $p(A_iy,y)>0$. We have
$$p(A_iy,y) \leq p(A_iy,y_{2n+1})+p(y_{2n+1},y)$$
$$= p\big(A_iy,A_jx_{2n+1}\big)+p(y_{2n+1},y)$$
$$\leq \phi\big(\max\{p(Sy,Tx_{2n+1}),p(A_iy,Sy),p\big(A_jx_{2n+1},Tx_{2n+1}\big)\}\big)+p(y_{2n+1},y)$$
$$\leq \phi(\max\{p(A_iy,y_{2n}),p(A_iy,A_iy),p(y_{2n+1},y_{2n})\})+p(y_{2n+1},y)$$
$$\leq \phi(\max\{p(A_iy,y),p(A_iy,A_iy),0\})$$
$$\leq \phi\big(p(A_iy,y)\big)$$
$$\leq p(A_iy,y) \tag{2.18}$$
Which is a contradiction. Then we deduce that
$$p(A_iy,y)=0 \text{ and } A_iy=Sy=y \tag{2.19}$$
Since the pair $\{A_j,T\}$ is weakly compatible, from (2.17), we have $A_jy=A_jTv=TA_jv=Ty$. We claim that $p(A_jy,y)=0$. Suppose, to the contrary, that $p(A_jy,y)>0$, then by (2.1) and (2.19), we have
$$p\big(y,A_jy\big)=p\big(A_iy,A_jy\big)$$
$$\leq \phi\big(\max\{p(Sy,Ty),p(A_iy,Sy),p\big(A_jy,Ty\big)\}\big)$$
$$\leq \phi\big(\max\{p\big(y,A_jy\big),p(y,y),p\big(A_jy,A_jy\big)\}\big)$$
$$\leq p\big(A_jy,y\big)$$
This is a contradiction. We deduce that
$$p(A_jy,y)=0 \text{ and } A_jy=Ty=y. \tag{2.20}$$
Now, combining (2.19) and (2.20), we obtain
$$y=A_iy=A_jy=Sy=Ty, i\neq j.$$
That is, y is a common fixed point of A_i, A_j, S and T.

Uniqueness:

Let us suppose that $z\in X$ is a common fixed point of A_i, A_j, S and T with $p(z,y)>0$.
Using(2.1), we get
$$p(y,z)=p\big(A_iy,A_jz\big)$$
$$\leq \phi\big(\max\{p\big(A_iy,A_jz\big),p\big(A_iy,A_jy\big),p\big(A_jz,A_jz\big)\}\big)$$
$$= \phi(\max\{p(y,z),p(y,y),p(z,z)\})$$
$$= \phi\big(p(y,z)\big) < p(y,z)$$
Which is contradiction. Then we deduce that $z=y$. Therefore , the uniqueness of the common fixed point is proved.

Corollary 2.4: Suppose that A,B,S and T are self maps of a complete partial metric space (X,p) such that $AX\subseteq TX, BX\subseteq SX$ and
$$p(Ax,By)\leq \phi(\max\{p(Sx,Ty),p(Ax,Sx),p(By,Ty)\})$$
For all $x,y\in X$ where $\phi\in\Phi$ if one of the ranges AX,BX,TX and SX is a closed subset of (X,p), then (i) $A$and S have coincidence point. Moreover, if the pairs$\{A,S\}$ and $\{B,T\}$ are weakly compatible, then A,B,S and T have a unique common fixed point.

Corollary 2.5: Suppose that S and T are self maps of a complete partial metric space (X,p) such that $TX\subseteq SX$ and for all $x,y\in X$ where $\phi\in\Phi$ and
$$p(Tx,Ty)\leq \phi\left(\max\left[p(Sx,Sy),\frac{\big(p(Tx,Sx)+p(Ty,Sy)\big)}{2}+\frac{\big(p(Ty,Sx)+p(Tx,Sy)\big)}{2}\right]\right)$$
if one of the ranges TX and SX is a closed subset of (X,p), then (i) $S$and T have coincidence point. (ii) Moreover, if the pairs$\{S,T\}$ is weakly compatible, then S and T have a unique common fixed point.

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