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Development of Homotopy Perturbation Method for Solving Nonlinear Algebraic Equations

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Abstract — In this paper, we proposed expanding the application of the He's homotopy perturbation method to solve nonlinear algebraic equations using the first seven terms of Taylor's series. The main objective of our research is to find an approximate solution with high accuracy for solving non-linear algebraic equations. In the proposed hybrid scheme, we combined the homotopy perturbation method with the Taylor's series for an analytical function, by truncating the first seven terms of Taylor's series and then apply the homotopy perturbation method. We demonstrated the efficacy of the proposed method by finding an approximate solution with high accuracy. This solution is the intersection of the graph with the horizontal axis. We also found a new iterative formula that gives, in every iteration, an approximate solution that converges from the exact solution faster and with small error. Numerical examples are given to show the efficiency and accuracy of the proposed iterative scheme.

Keywords — *Homotopy Perturbation Method, Iterative Scheme, Nonlinear Algebraic Equations, Taylor's Series.*

I. INTRODUCTION

Sometimes it is difficult to find the exact solution of many algebraic equations, so we need to develop and improve some numerical methods.

Homotopy perturbation method has been proposed firstly by the Chinese Mathematician Ji-Huan He, in the year 1997 and systematical description in the year 2000 [1]. In fact, it is a coupling between the traditional perturbation method and homotopy in topology [2]. This method was applied to solve nonlinear partial differential equations [3], and to solve nonlinear boundary value problems [4].

T. Liu. *et al.*, applied Cauchy methods to solve nonlinear algebraic equations [5]. S. Abbasband proposed modified Adomian method to solve nonlinear algebraic equations [6].

P.K. Bera. *et al.,* used the applications of the Aboodh Transform and the homotopy perturbation method to the nonlinear Oscillators [7]. Also P.K. Bera. *et al.*, studied the nonlinear vibration of Euler-Bernoulli Beams Using coupling between the Aboodh transform and the homotopy perturbation method [8].

The main of our work is to truncate the first seven terms of Taylor's series and then apply the homotopy perturbation method to get approximate solutions with high accuracy and converge from the exact solution faster.

Rest of the paper is organized as follows: Section I contains the introduction of the paper and some related works, in Section II a description of the proposed iterative method. Section III presents the application of the new iterative scheme to solve some nonlinear algebraic equations, section IV concludes the research work.

II. PROPOSED ITERATIVE SCHEME

Consider the general nonlinear algebraic equation:

$$f(x)=0\ ,\ x\in R \tag{1}$$

Where f is a continuously differentiable function on the real range $[a,b]$ and we assume that $\alpha\in[a,b]$ is a simple zero of Eqn. (1) and we denote β to an initial assumption sufficiently close to exact solution α.

By expanding $f(x)$ in Eqn. (1) into a Taylor's series around β, we have:

$$\begin{aligned} f(x)&=f(\beta)+(x-\beta)f'(\beta)+\frac{1}{2!}(x-\beta)^2f^{(2)}(\beta)\\ &+\frac{1}{3!}(x-\beta)^3f^{(3)}(\beta)+\frac{1}{4!}(x-\beta)^4f^{(4)}(\beta)\\ &+\frac{1}{5!}(x-\beta)^5f^{(5)}(\beta)+\frac{1}{6!}(x-\beta)^6f^{(6)}(\beta)=0 \end{aligned} \tag{2}$$

Then we have:

$$\begin{aligned} x&=\beta-\frac{f(\beta)}{f'(\beta)}-\frac{1}{2!}(x-\beta)^2\frac{f^{(2)}(\beta)}{f'(\beta)}\\ &-\frac{1}{3!}(x-\beta)^3\frac{f^{(3)}(\beta)}{f'(\beta)}-\frac{1}{4!}(x-\beta)^4\frac{f^{(4)}(\beta)}{f'(\beta)}\\ &-\frac{1}{5!}(x-\beta)^5\frac{f^{(5)}(\beta)}{f'(\beta)}-\frac{1}{6!}(x-\beta)^6\frac{f^{(6)}(\beta)}{f'(\beta)} \end{aligned} \tag{3}$$

Where

$$f'(\beta) \neq 0$$

We construct a homotopy $H: R\times[0,1]\rightarrow R$ for Eqn. (3), which satisfies:

$$H(v,p)=v-\beta+\frac{f(\beta)}{f'(\beta)} -p\begin{bmatrix} -\frac{1}{2!}(v-\beta)^2\frac{f^{(2)}(\beta)}{f'(\beta)} & -\frac{1}{3!}(v-\beta)^3\frac{f^{(3)}(\beta)}{f'(\beta)} \\ -\frac{1}{4!}(v-\beta)^4\frac{f^{(4)}(\beta)}{f'(\beta)} & -\frac{1}{5!}(v-\beta)^5\frac{f^{(5)}(\beta)}{f'(\beta)} \\ -\frac{1}{6!}(v-\beta)^6\frac{f^{(6)}(\beta)}{f'(\beta)} & \end{bmatrix}=0 \quad (4)$$

Where $v \in$, $p\in[0,1]$ is embedding parameter.

It is obvious that:

$$H(v,0)=v-\beta+\frac{f(\beta)}{f'(\beta)}=0$$

$$H(v,1)=v-\beta+\frac{f(\beta)}{f'(\beta)} -\begin{bmatrix} -\frac{1}{2!}(v-\beta)^2\frac{f^{(2)}(\beta)}{f'(\beta)} & -\frac{1}{3!}(v-\beta)^3\frac{f^{(3)}(\beta)}{f'(\beta)} \\ -\frac{1}{4!}(v-\beta)^4\frac{f^{(4)}(\beta)}{f'(\beta)} & -\frac{1}{5!}(v-\beta)^5\frac{f^{(5)}(\beta)}{f'(\beta)} \\ -\frac{1}{6!}(v-\beta)^6\frac{f^{(6)}(\beta)}{f'(\beta)} & \end{bmatrix}=0$$

The changing of p from 0 to 1, is just that $H(v,p)$ from $H(v,0)$ to $H(v,1)$. In topology, this is called deformation, $H(v,0)$ and $H(v,1)$ are called homotopic.

Applying the perturbation technique, due to the fact that $0\leq p\leq 1$ can be considered as a small parameter, we can assume that the solution for Eqn. (4) can be expressed as a series in p:

$$v=v_0+pv_1+p^2v_2+\cdots=\sum_{i=0}^{\infty}p^iv_i \quad (5)$$

The approximate solution of Eqn. (1):

$$\alpha=\lim_{p\rightarrow 1}v=v_0+v_1+v_2+\cdots \quad (6)$$

By substituting Eqn. (5) into Eqn. (4), yields:

$$\sum_{i=0}^{\infty}p^iv_i-\beta+\frac{f(\beta)}{f'(\beta)} -p\begin{bmatrix} -\frac{1}{2!}\left(\sum_{i=0}^{\infty}p^iv_i-\beta\right)^2\frac{f^{(2)}(\beta)}{f'(\beta)} & -\frac{1}{3!}\left(\sum_{i=0}^{\infty}p^iv_i-\beta\right)^3\frac{f^{(3)}(\beta)}{f'(\beta)} \\ -\frac{1}{4!}\left(\sum_{i=0}^{\infty}p^iv_i-\beta\right)^4\frac{f^{(4)}(\beta)}{f'(\beta)} & -\frac{1}{5!}\left(\sum_{i=0}^{\infty}p^iv_i-\beta\right)^5\frac{f^{(5)}(\beta)}{f'(\beta)} \\ -\frac{1}{6!}\left(\sum_{i=0}^{\infty}p^iv_i-\beta\right)^6\frac{f^{(6)}(\beta)}{f'(\beta)} & \end{bmatrix}=0 \quad (7)$$

By equating the terms with identical powers of p, we have:

$$p^0: v_0=\beta-\frac{f(\beta)}{f'(\beta)} \quad (8)$$

$$p^1: v_1=-\frac{1}{2!}(v_0-\beta)^2\frac{f^{(2)}(\beta)}{f'(\beta)}-\frac{1}{3!}(v_0-\beta)^3\frac{f^{(3)}(\beta)}{f'(\beta)} -\frac{1}{4!}(v_0-\beta)^4\frac{f^{(4)}(\beta)}{f'(\beta)}-\frac{1}{5!}(v_0-\beta)^5\frac{f^{(5)}(\beta)}{f'(\beta)} -\frac{1}{6!}(v_0-\beta)^6\frac{f^{(6)}(\beta)}{f'(\beta)}$$

Then we get

$$v_1=-\frac{1}{2!}\frac{f^2(\beta)f^{(2)}(\beta)}{f'^3(\beta)}+\frac{1}{3!}\frac{f^3(\beta)f^{(3)}(\beta)}{f'^4(\beta)} -\frac{1}{4!}\frac{f^4(\beta)f^{(4)}(\beta)}{f'^5(\beta)}+\frac{1}{5!}\frac{f^5(\beta)f^{(5)}(\beta)}{f'^6(\beta)} -\frac{1}{6!}\frac{f^6(\beta)f^{(6)}(\beta)}{f'^7(\beta)}=\varphi_1(\beta)$$

$$p^2: v_2=-(v_0-\beta)v_1\frac{f^{(2)}(\beta)}{f'(\beta)}-\frac{1}{2!}(v_0-\beta)^2v_1\frac{f^{(3)}(\beta)}{f'(\beta)} -\frac{1}{3!}(v_0-\beta)^3v_1\frac{f^{(4)}(\beta)}{f'(\beta)}-\frac{1}{4!}(v_0-\beta)^4v_1\frac{f^{(5)}(\beta)}{f'(\beta)} -\frac{1}{5!}(v_0-\beta)^5v_1\frac{f^{(6)}(\beta)}{f'(\beta)}$$

Then we get

$$p^2: v_2=v_1\begin{bmatrix} \frac{f(\beta)f^{(2)}(\beta)}{f'^2(\beta)}-\frac{1}{2!}\frac{f^2(\beta)f^{(3)}(\beta)}{f'^3(\beta)} \\ +\frac{1}{3!}\frac{f^3(\beta)f^{(4)}(\beta)}{f'^4(\beta)}-\frac{1}{4!}\frac{f^4(\beta)f^{(5)}(\beta)}{f'^5(\beta)} \\ +\frac{1}{5!}\frac{f^5(\beta)f^{(6)}(\beta)}{f'^6(\beta)} \end{bmatrix} =\varphi_1(\beta)\varphi_2(\beta)$$

Where

$$\varphi_2(\beta)=\frac{f(\beta)f^{(2)}(\beta)}{f'^2(\beta)}-\frac{1}{2!}\frac{f^2(\beta)f^{(3)}(\beta)}{f'^3(\beta)} +\frac{1}{3!}\frac{f^3(\beta)f^{(4)}(\beta)}{f'^4(\beta)}-\frac{1}{4!}\frac{f^4(\beta)f^{(5)}(\beta)}{f'^5(\beta)} +\frac{1}{5!}\frac{f^5(\beta)f^{(6)}(\beta)}{f'^6(\beta)}$$

The approximate solution of Eqn. (1) is:

$$\alpha=\lim_{p\rightarrow 1}v=v_0+v_1+v_2+\cdots =\beta-\frac{f(\beta)}{f'(\beta)}+\varphi_1(\beta)+\varphi_1(\beta).\varphi_2(\beta)$$

Then we have the new algorithm: For a given x_0, calculate the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}+\varphi_1(x_n)+\varphi_1(x_n).\varphi_2(x_n) \quad (9)$$

III. TEST PROBLEMS

To verify the accuracy of the proposed method of (9), we have presented some examples.

Where (n) is the number of iterations.

Examples 1. Consider the following equation:

$$f(x)=x^2-(1-x)^5=0$$

with $x_0=0.2$ an assumption initial.

Table 1. Numerical results and absolute errors for Example 1.

n	**Obtained Solution** x_n	$\|f(x)\|$
1	0.34256006171719616928	5.47580 e -003
2	0.34595481490775996077	1.51122 e -009
3	0.345954815848242017961122564769	4.68909 e -021
4	0.345954815848242017958204406446972 38951092639302964	2.57413 e -031
5	0.34595481584824201795820440644713258 53690805694928332377	4.89912 e -051
6	0.34595481584824201795820440644713258 5369080569492830188848409	5.14051 e -056
7	0.34595481584824201795820440644713258 53690805694928301888803998179091754	2.38426 e -061

Examples 2. Consider the equation:

$$f(x) = e^x - 3x^2 = 0$$

with $x_0 = 1$ an assumption initial.

Table 2. Numerical results and absolute errors for Example 2.

n	**Obtained Solution** x_n	$\|f(x)\|$
1	0.91005030099490710404	1.27151 e -004
2	0.9100075724887090640886644	1.02106 e -017
3	0.91000757248870906065733833574478986	1.18985 e -025
4	0.91000757248870906065733829575936794 22212	1.07050 e -035
5	0.91000757248870906065733829575936794 581866792017167	2.91145 e -040
6	0.91000757248870906065733829575936794 5818765761019447506199622	5.15606 e -050
7	0.91000757248870906065733829575936794 58187657610194660431787774301789310	3.25701 e -060

Examples 3. Consider the equation:

$$f(x) = x - 2 - e^{-x} = 0$$

with $x_0 = 2$ an assumption initial.

Table 3. Numerical results and absolute errors for Example 3.

n	**Obtained Solution** x_n	$\|f(x)\|$
1	2.1200280544022551446	2.06741 e -007
2	2.120028238987641229480813	4.34042 e -021
3	2.12002823898764122948468827357	3.34102 e -025
4	2.12002823898764122948468797527216 3954380	3.52483 e -031
5	2.1200282389876412294846879752718 492449391750089600	3.09607 e -041
6	2.1200282389876412294846879752718492 4493914736613687083992236	2.32002 e -051
7	2.1200282389876412294846879752718492 44939147366136868768527626152043965	1.58222 e -061

Examples 4. Consider the equation:

$$f(x) = \cos x - x = 0$$

with $x_0 = 1$ an assumption initial.

Table 4. Numerical results and absolute errors for Example 4.

n	**Obtained Solution** x_n	$\|f(x)\|$
1	0.739197375840288	1.87855 e -004
2	0.739085133215160714296906740245	1.21574 e -016
3	0.73908513321516064165531208767395 20062216	1.31550 e -031
4	0.73908513321516064165531208767387 340401343817104905	4.42037 e -041
5	0.73908513321516064165531208767387340 4013411758900756842086083	1.04246 e -051
6	0.73908513321516064165531208767387340 40134117589007574649656804349914460	3.36031 e -061
7	0.73908513321516064165531208767387340 401341175890075746496568063577 328464631734940	1.43365 e -071

Examples 5. Consider the equation:

$$f(x) = \ln x + e^x - 2x^2 + 1 = 0$$

with $x_0 = 0.1$ an assumption initial.

Table 5. Numerical results and absolute errors for Example 5.

n	**Obtained Solution** x_n	$\|f(x)\|$
1	0.1223514483	6.46184 e -004
2	0.12242478353815371538	9.56867 e -011
3	0.122424783549016327365471488770	2.18934 e -020
4	0.12242478354901632736795688610606 40831627	1.92736 e -030
5	0.12242478354901632736795688610584 528403590044337220	3.83510 e -042
6	0.12242478354901632736795688610584 5284035900007899841213834284	5.15922 e -050
7	0.12242478354901632736795688610584 52840359000078998470707175961729004960	7.94708 e -060

IV. CONCLUSION

In this work, we proposed expanding the application of HPM that based on the Taylor's series to solve nonlinear algebraic equations. The new iterative scheme gives approximate solution very close to the exact solution and with low numerical errors. Thus, the proposed iterative method can be used successfully in finding approximate solutions.

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