**International Journal of Scientific Research in Mathematical and Statistical Sciences**
Vol.10, Issue.2, pp.33-37, April 2023
E-ISSN: 2348-4519
Available online at: www.isroset.org

Research Paper

# Application of Jensen's Inequality for Proving Nesbitt's Inequality and Further Generalizing It

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**Received**: 05/Mar/2023; **Accepted**: 03/Apr/2023; **Published**: 30/Apr/2023

***Abstract***— With the help of this article, the author is introducing new generalizations of a very popular inequality that we call Nesbitt's inequality. For obtaining new generalizations, there is a use of another inequality called Jensen's Inequality for convex functions, which is directly linked with derivatives of a function. There is a need for essential calculus and a better understanding of the proofs.

***Keywords***— Jensen's Inequality, Nesbitt's Inequality Generalization, Convex function, Concave Function, Functional Inequalities, Three Variables etc.

## 1. Introduction

The Nesbitt's Inequality says that if $x, y$ and $z$are positive and real quantities then

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}$$

This inequality was first published as a problem in 1903 [1]. We have many proofs for this inequality one can go through Wikipedia for it. Here, we are providing a proof using Jensen's Inequality. So, first we are looking for this inequality. The Jensen's Inequality was first published in 1906 authored by Danish mathematician Johan Jensen [2][3][4]. This inequality states that


Johan Jensen

If $f(x)$ is convex for $x_i$ and $0 < i \leq n$ then,

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i)\geq f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

Now, we can come back on proof of Nesbitt's inequality. So,

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{x}{(x+y+z)-x}+\frac{y}{(x+y+z)-y}+\frac{z}{(x+y+z)-z}$$

Let $(x+y+z)=s$ then,

$$\frac{x}{(x+y+z)-x}+\frac{y}{(x+y+z)-y}+\frac{z}{(x+y+z)-z}=\frac{x}{s-x}+\frac{y}{s-y}+\frac{z}{s-z}$$

Now, consider a function

$$f(t)=\frac{t}{s-t}\Rightarrow f'(t)=\frac{s}{(s-t)^2}\Rightarrow f''(t)=\frac{2s}{(s-t)^3}>0\forall t\in\mathbb{N}$$

Then we can say its convex in nature for all t belongs to natural number. So, here with using Jensen's Inequality for three variables we can say that

$$\frac{f(x)+f(y)+f(z)}{3}\geq f\left(\frac{x+y+z}{3}\right)$$

 

$$\Rightarrow \frac{x}{s-x}+\frac{y}{s-y}+\frac{z}{s-z} \geq 3\frac{\left(\frac{x+y+z}{3}\right)}{s-\left(\frac{x+y+z}{3}\right)}$$

$$\Rightarrow \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geq 3\frac{\left(\frac{x+y+z}{3}\right)}{(x+y+z)-\left(\frac{x+y+z}{3}\right)}$$

$$\Rightarrow \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geq \frac{3}{2}$$

Hence proof.

This paper contains some sections and sub sections. The paper begins with abstract and keywords there is abstract author tried to express the nature of his work and what's the meaning of this work respectively then I. Introduction here author tried to introduce Nesbitt's inequality with what is it and also provide the possible proof using Jensen's inequality for that he also expressed Jensen's inequality for convex function. II. Related work here also mention some of the work that relates with the generalizations of Nesbitt's inequality and its applications. III. Generalizations and their proofs. In this section using Jensen's inequality author first prove some theorems for three variables and then also extend both of them for n numbers of variables. IV Conclusion and further scope. There is equal space for further more generalizations to this inequality using some other methodology or steps and there is a need to found some applications to the generalized inequality. Author is leaving this on readers to encourage for developing their own generalizations and applications respectively. In the last there are references devoted for the word directly used here from there.

## 2. Related Work

Time to time mathematicians and researchers provide their own generalizations to this inequality for now we can take as an exam the paper of Fuhua Wei and Shanhe Wu entitled "Generalizations and analogues of the Nesbitt's inequality" similar kind of paper was also published by Daiyuan Zhang entitled "New Inequalities and applications".

## 3. Main Result

Using below theorems we are introducing generalizations to Nesbitt's Inequality.

**Theorem 1.** If $a, b, c > 0$ and $0 \leq p$ then,

$$\left(\frac{a}{b+c}\right)(\ln a)^p+\left(\frac{b}{c+a}\right)(\ln b)^p+\left(\frac{c}{a+b}\right)(\ln c)^p \geq \frac{3}{2}\left(\ln\left(\frac{a+b+c}{3}\right)\right)^p$$

**Proof**

Suppose a function $f(x)=\left(\frac{x}{s-x}\right)(\ln x)^p \Rightarrow f''(x) > 0 \forall x > 0$ then we can say that $f(x)$ is convex under the given range then for some $a, b, c > 0$, Jensen's Inequality says

$$\frac{f(a)+f(b)+f(c)}{3} \geq f\left(\frac{a+b+c}{3}\right)$$

$$\left(\frac{a}{s-a}\right)(\ln a)^p+\left(\frac{b}{s-b}\right)(\ln b)^p+\left(\frac{c}{s-c}\right)(\ln c)^p \geq 3\left(\frac{\frac{a+b+c}{3}}{s-\frac{a+b+c}{3}}\right)\left(\ln\frac{a+b+c}{3}\right)^p$$

$$\left(\frac{a}{b+c}\right)(\ln a)^p+\left(\frac{b}{c+a}\right)(\ln b)^p+\left(\frac{c}{a+b}\right)(\ln c)^p \geq \frac{3}{2}\left(\ln\left(\frac{a+b+c}{3}\right)\right)^p$$

If we put $p=0$ then we can easily obtain Nesbitt inequality.

$$\left(\frac{a}{b+c}\right)(\ln a)^0+\left(\frac{b}{c+a}\right)(\ln b)^0+\left(\frac{c}{a+b}\right)(\ln c)^0 \geq \frac{3}{2}\left(\ln\left(\frac{a+b+c}{3}\right)\right)^0$$

$$\Rightarrow \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$$

**Theorem 2.** If $a, b, c > 0$ and $0 \leq p \leq 1$ then,

$$\frac{a}{(b+c)^p}+\frac{b}{(c+a)^p}+\frac{c}{(a+b)^p} \geq \frac{3}{2^p}\left(\frac{a+b+c}{3}\right)^{1-p}$$

 

**Proof**

Suppose a function $f(x) = \frac{x}{(s-x)^p} \Rightarrow f''(x) > 0 \forall x > 0$ for all $p\epsilon[0,1]$ then we can say that $f(x)$ is convex under the given range for some $a, b, c > 0$, Jensen's Inequality says

$$\frac{f(a)+f(b)+f(c)}{3} \geq f\left(\frac{a+b+c}{3}\right)$$

$$\frac{a}{(s-a)^p}+\frac{b}{(s-b)^p}+\frac{c}{(s-c)^p} \geq 3\frac{\frac{a+b+c}{3}}{\left(s-\frac{a+b+c}{3}\right)^p}$$

$$\frac{a}{(b+c)^p}+\frac{b}{(c+a)^p}+\frac{c}{(a+b)^p} \geq 3\frac{\frac{a+b+c}{3}}{\left(2\frac{a+b+c}{3}\right)^p}$$

$$\frac{a}{(b+c)^p}+\frac{b}{(c+a)^p}+\frac{c}{(a+b)^p} \geq \frac{3}{2^p}\left(\frac{a+b+c}{3}\right)^{1-p}$$

If we put $p = 1$ then we can easily obtain Nesbitt inequality.

$$\frac{a}{(b+c)^1}+\frac{b}{(c+a)^1}+\frac{c}{(a+b)^1} \geq \frac{3}{2^p}\left(\frac{a+b+c}{3}\right)^{1-1}$$

$$\Rightarrow \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$$

**<u>Generalizations for n number of variables</u>**

**Theorem 3.** If for $0 < i \leq n$ $x_i > 0$ and $0 \leq p$ then,

$$\sum_{i=1}^{n}\frac{x_i(\ln x_i)^p}{S-x_i} \geq \frac{n}{n-1}\left(\ln\frac{1}{n}\sum_{i=1}^{n}x_i\right)^p$$

Where, $\sum_{i=1}^{n} x_i = S$

**Proof**

Suppose a function $f(x) = \left(\frac{x}{s-x}\right)(\ln x)^p \Rightarrow f''(x) > 0 \forall x > 0$ then we can say that $f(x)$ is convex under $(2, S)$ then for some $a, b, c > 0$, Jensen's Inequality says

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\frac{x_i(\ln x_i)^p}{S-x_i} \geq \frac{\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\left(\ln\frac{1}{n}\sum_{i=1}^{n}x_i\right)^p}{S-\frac{1}{n}\sum_{i=1}^{n}x_i}$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\frac{x_i(\ln x_i)^p}{S-x_i} \geq \frac{\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\left(\ln\frac{1}{n}\sum_{i=1}^{n}x_i\right)^p}{\sum_{i=1}^{n}x_i-\frac{1}{n}\sum_{i=1}^{n}x_i}$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\frac{x_i(\ln x_i)^p}{S-x_i} \geq \frac{1}{n-1}\left(\ln\frac{1}{n}\sum_{i=1}^{n}x_i\right)^p$$

$$\sum_{i=1}^{n}\frac{x_i(\ln x_i)^p}{S-x_i} \geq \frac{n}{n-1}\left(\ln\frac{1}{n}\sum_{i=1}^{n}x_i\right)^p$$

If $n = 3$ then,

$$\frac{x_1(\ln x_1)^p}{S-x_1}+\frac{x_2(\ln x_2)^p}{S-x_2}+\frac{x_3(\ln x_3)^p}{S-x_3} \geq \frac{3}{2}\left(\ln\frac{x_1+x_2+x_3}{3}\right)^p$$

 

$$\Rightarrow \frac{x_1(\ln x_1)^p}{x_2+x_3}+\frac{x_2(\ln x_2)^p}{x_3+x_1}+\frac{x_3(\ln x_3)^p}{x_1+x_3} \geq \frac{3}{2}\left(\ln\frac{x_1+x_2+x_3}{3}\right)^p$$

**Theorem 4.** If for $0 < i \leq n$ $x_i > 0$ and $0 \leq p \leq 1$ then,

$$\sum_{i=1}^{n}\frac{x_i}{(S-x_i)^p} \geq \frac{n}{(n-1)^p}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^{1-p}$$

Where, $\sum_{i=1}^{n} x_i = S$

**Proof**

Suppose a function $f(x) = \frac{x}{(s-x)^p} \Rightarrow f''(x) > 0 \forall x > 0$ for all $p\epsilon[0,1]$ then we can say that $f(x)$ is convex under $(0, S)$ then for some $a, b, c > 0$, Jensen's Inequality says

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{(S-x_i)^p} \geq \frac{\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)}{\left(S-\frac{1}{n}\sum_{i=1}^{n}x_i\right)^p}$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{(S-x_i)^p} \geq \frac{1}{(n-1)^p}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^{1-p}$$

$$\sum_{i=1}^{n}\frac{x_i}{(S-x_i)^p} \geq \frac{n}{(n-1)^p}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^{1-p}$$

If $n = 3$ then,

$$\frac{x_1}{(S-x_1)^p}+\frac{x_2}{(S-x_2)^p}+\frac{x_3}{(S-x_3)^p} \geq \frac{3}{2^p}\left(\frac{x_1+x_2+x_3}{3}\right)^{1-p}$$

$$\Rightarrow \frac{x_1}{(x_2+x_3)^p}+\frac{x_2}{(x_3+x_1)^p}+\frac{x_3}{(x_1+x_3)^p} \geq \frac{3}{2^p}\left(\frac{x_1+x_2+x_3}{3}\right)^{1-p}$$

.

## 4. Conclusion and Future Scope

The purpose of this article was to established two new generalizations of Nesbitt's inequality. To establish generalizations to some inequality is off course is not a new idea but yet it's one of the powerful ideas there are many generalizations to it but anywhere I wouldn't able to found a generalization contains logarithmic function other than algebraic with theorem 1st and 3rd this is established now. Inequality 2nd and 4th are also good in some manner but there is a limitation of paper p that should be smaller than 1 so, there is a possibility to extend them for some other indices respectively. Other than it there is scope for finding applications to the generalizations as mentioned in the paper.

**Data Availability**
None

**Conflict of Interest**
The authors declare that they do not have any conflict of interest.

**Funding Source**
None

**Acknowledgements**
Author is thankful to his parents Dr. Lokesh Prakash and Mrs. Seema Prakash for their faith in him that helps to follow his passion in the field of Mathematics, Science and Poetry.

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