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Research Article

Electromagnetic Radiation through an Ideal Gas and its Analytic Solution Using Fourier Transform and Greens Function

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Abstract—This paper explores the significance of solving physical problems in an infinite domain using mathematical techniques. While physical problems are never truly infinite, discussing them in an infinite domain allows us to weaken the influence of boundaries and obtain solutions that provide valuable insights. Among various analytic approaches to solve partial differential equations, explicit solutions hold particular importance as they offer a clearer understanding of the problem.

Keywords— Fourier Transform, Wave operator, Dirac delta function, Green function, Green's formula, Electromagnetic radiation

1. Introduction

In this study, we focus on finding explicit solutions for the three-dimensional homogeneous wave equation in an infinite domain. We employ two different methods: the Fourier transform method and the Green's function method. Both approaches demonstrate their elegance in providing explicit representations of the solutions for the wave equation in three dimensions. Furthermore, we apply these methods to address a physical scenario involving electromagnetic radiation traveling through an ideal gas and interacting with the surrounding gas molecules' electrons, leading to energy loss and continuous radiation production.

The three-dimensional homogeneous wave equation:[1] The three-dimensional homogeneous wave equation in an infinite space will satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad (1)$$

In an infinite domain there are only initial conditions satisfied by the equation (1).

Let

$$u(X,0) = f(X), \frac{\partial u}{\partial t}(X,0) = g(X) \quad (1-\mathrm{a,b})$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $X = (x, y, z)$ and $X_0 = (x_0, y_0, z_0)$.

Fourier transform:[2,6] Three dimensional Fourier transform relationship is given as:

$$F^{-1}(F(\mu)) = f(X) = \iiint F(\mu) e^{-i\mu\cdot X} d^3\mu$$

And

$$F(\mu) = \frac{1}{(2\pi)^3} \iiint f(X) e^{i\mu\cdot X} d^3 X$$

where $F(\mu)$ denotes the Fourier transform of piece-wise continuous function $f(X)$ with $\mu = (\mu_1, \mu_2, \mu_3)$ and $\mu \cdot X = \mu_1 x + \mu_2 y + \mu_3 z$.

Green's function: [1,5] The Green function, denoted by $G(X,t;X_0,t_0)$ associated to the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$, represents the reaction at X at time t of the source at X_0 acting at time t_0 and it satisfying the equation

$$\frac{\partial^2 G}{\partial t^2} = c^2 \nabla^2 G + \delta(X - X_0)\delta(t - t_0) \quad (2)$$

where $\delta(X - X_0)$ is the three-dimensional Dirac delta function, which can be expressed as the product of one-dimensional Dirac delta functions $\delta(X - X_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$[1]

2. Literature Review

As a part of my interest in partial differential equations and its solution, I have reviewed many books written by Richard haberman, Roach G F , Pinsky M A etc. In this review I found that there are many analytical methods like Variable separable method, Method of Eigen function expansion, Fourier Transform method, Laplace transform method, method of characteristics, Greens function method using method of images to solve partial differential equations but among all these methods Greens function method having its own importance to solve physical problem modeled by partial differential equations. So I thorough tried to understand about the Greens function explained by Roach G F , Pinsky M A and Haberman R in their books.

Further literature of Pinsky M A and Muhlenbruch T. & Raji W, enables me to cracked the physical scenario as discussed in the paper

3. Experimental Method/Procedure/Design

Nowadays there is trend to solve problems using numerical methods but the analytic approach to solve the problem has always plays vital role and advisable as it provides the exact solution. Moreover, the solution in its explicit form has its own importance and beauty to understand the problem.

So First I have shown the detailed work to get explicit solution of three dimensional wave equation in an infinite domain using Fourier Transform method and Greens function method to help the reader to understand it in a better way and then applied both the method to encounter the physical scenario modelled by three dimensional wave equation.

4. Results and Discussion

4.1. Fourier transforms method

To solve the initial value problem (1) using Fourier transform, let $U(\mu,t)$ be the Fourier transform of $u(X,t)$. [2,6]

Then by definition

$$u(X,t) = \iiint U(\mu,t)e^{-i\mu\cdot X}d^3\mu$$

Where

$$U(\mu,t) = \frac{1}{(2\pi)^3}\iiint u(X,t)e^{i\mu\cdot X}d^3X$$

We have

$$\nabla^2 u(X,t) = -\iiint U(\mu,t)\mu^2 e^{-i\mu\cdot X}d^3\mu$$

and

$$\frac{\partial^2 u}{\partial t^2}(X,t) = \iiint \frac{d^2U(\mu,t)}{dt^2}e^{-i\mu\cdot X}d^3\mu$$

It follows from $\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$, we have

$$\iiint \left[\frac{d^2U(\mu,t)}{dt^2} + c^2\mu^2 U(\mu,t)\right]e^{-i\mu\cdot X}d^3\mu = 0$$

$$\Rightarrow \frac{d^2U(\mu,t)}{dt^2} + c^2\mu^2 U(\mu,t) = 0 \qquad (3)$$

Solving upon (3);

$$U(\mu,t) = C_1(\mu)\cos c\mu t + C_2(\mu)\sin c\mu t \qquad (4)$$

At $t=0$, Let $U(\mu,0) = F(\mu)$ and $U_t(\mu,0) = G(\mu)$ be the Fourier transform of $u(X,0) = f(X)$ and $u_t(X,0) = g(X)$ respectively.

Using (4) we get

$$C_1(\mu) = F(\mu), C_2(\mu) = \frac{G(\mu)}{c\mu}$$

Substituting $C_1(\mu)$ and $C_2(\mu)$ in (4), the Fourier transform of $u(X,t)$ is

$$U(\mu,t) = F(\mu)\cos c\mu t + \frac{G(\mu)}{c\mu}\sin c\mu t \qquad (5)$$

where $F(\mu)$ and $G(\mu)$ are Fourier transforms of initial position $f(X)$ and initial velocity $g(X)$ respectively.

Thus, we have

$$u(X,t) = \iiint \left(F(\mu)\cos c\mu t + \frac{G(\mu)}{c\mu}\sin c\mu t\right)e^{-i\mu\cdot X}\ d^3\mu \qquad (6)$$

$$u(X,t) = \iiint \left(F(\mu)\cos c\mu t + \frac{G(\mu)}{c\mu}\sin c\mu t\right)e^{-i\mu\cdot X}\ d^3\mu$$

It is to be observed that the volume integral and surface integral on the sphere of radius R are related as [2,5]

$$\iiint_{|X|\le R} f(X)dV = \int_{R_0=0}^{R}\left(\iint_{|X|=R_0} f(X)dS\right)dr$$

$$\Rightarrow \iint_{|X|=R_0} f(X)dS = \frac{d}{dR}\left(\iiint_{|X|\le R} f(X)dx\,dy\,dz\right) \qquad (7)$$

Use $f(X) = e^{i\mu\cdot X}$ in (7) one gets

$$\iint_{|X|=R_0} e^{i\mu\cdot X}dS = \frac{d}{dR}\left(\iiint_{|X|\le R} e^{i\mu\cdot X}dx\,dy\,dz\right) \qquad (8)$$

To evaluate $\iiint_{|X|\le R} e^{i\mu\cdot X}dx\,dy\,dz$, we consider the sphere whose radius R and centre is at $X=0$. Also the angle $\emptyset = 0$ is to be taken in the direction of μ.

So, we have $0 < R_0 < R, 0 < \theta < 2\pi, 0 < \emptyset < \pi$.

Let $R_0 = |X|$, then $\mu\cdot X = |\mu||X|\cos\emptyset = \mu R_0\cos\emptyset$ and $d^3X = R_0{}^2\sin\emptyset\ d\emptyset\,d\theta\,dr$

Hence

$$\iiint_{|X|\le R} e^{i\mu\cdot X}dx\,dy\,dz = \int_{R_0=0}^{R}\int_{\emptyset=0}^{\pi}\int_{\theta=0}^{2\pi} e^{i\mu R_0\cos\emptyset}R_0{}^2\sin\emptyset\ d\emptyset\,d\theta\,dr$$

$$\Rightarrow \iint_{|X|=R} e^{i\mu\cdot X}dS = \frac{d}{dR}\left(\frac{4\pi}{\mu}\left(\frac{\sin\mu R}{\mu^2} - \frac{R\cos\mu R}{\mu}\right)\right)$$

$$\Rightarrow \iint_{|X|=R} e^{i\mu\cdot X}dS = 4\pi R^2\frac{\sin\mu R}{\mu R} \qquad (9)$$

Let $R = ct$ yields

$$\frac{\sin c\mu t}{c\mu} = \frac{1}{4\pi c^2 t}\iint_{|X_0|=ct} e^{i\mu\cdot X_0}dS \qquad (10)$$

$$\Rightarrow \iiint G(\mu)\frac{\sin c\mu t}{c\mu}e^{-i\mu\cdot X}\ d^3\mu = \iiint G(\mu)\left(\frac{1}{4\pi c^2 t}\iint_{|X_0|=ct} e^{i\mu\cdot X_0}\,dS\right)e^{-i\mu\cdot X}\ d^3\mu$$

Interchanging the order of integration it will be

$$\iiint G(\mu)\frac{\sin c\mu t}{c\mu}e^{-i\mu\cdot X}\ d^3\mu = \frac{1}{4\pi c^2 t}\iint_{|X_0|=ct}\left\{\iiint G(\mu)e^{-i\mu\cdot(X-X_0)}\ d^3\mu\right\}dS$$

$$\Rightarrow \iiint G(\mu)\frac{\sin c\mu t}{c\mu}e^{-i\mu\cdot X}\ d^3\mu = \frac{1}{4\pi c^2 t}\iint_{|X_0|=ct} g(X - X_0)dS \quad (11)$$

Differentiating equation (10) with respect to t, one obtains

$$\cos c\mu t = \frac{d}{dt}\left(\frac{1}{4\pi c^2 t}\iint_{|X_0|=ct} e^{i\mu\cdot X_0}\,dS\right)$$

Based on the similar argument provided earlier, it can be deduced that

$$\iiint F(\mu)\cos c\mu t\ e^{-i\mu\cdot X}\ d^3\mu = \frac{d}{dt}\left(\frac{1}{4\pi c^2 t}\iint_{|X_0|=ct} f(X - X_0)dS\right) \quad (12)$$

By employing the integrals described in equations (11) and (12) into the equation (6), the solution of the wave equation (1) can be given in its explicit form as

$$u(X,t) = \frac{1}{4\pi c^2 t}\iint_{|X_0|=ct} g(X - X_0)dS + \frac{d}{dt}\left(\frac{1}{4\pi c^2 t}\iint_{|X_0|=ct} f(X - X_0)dS\right) \quad (13)$$

This explicit solution is in terms of the surface integration of initial velocity and the derivative of the surface integration of the initial position on the sphere of the radius ct.

4.2 Green's function method

To implement the theory of Green's function, using the linear differential wave operator, $L = \frac{\partial^2}{\partial t^2} - c^2\nabla^2$ [1]

The equations (1) and (2) are reduces to

$$L(u) = 0 \quad (14)$$

$$L(G) = \delta(X - X_0)\,\delta(t - t_0) \quad (15)$$

It follows from Green's formula that

$$\int_{t_i}^{t_f}\iiint [uL(v) - vL(u)]\ d^3x\,dt = \iiint\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)_{t_i}^{t_f} d^3x - c^2\int_{t_i}^{t_f}\left(\oiint (u\nabla v - v\nabla u).\hat{n}\ dS\right)dt \quad (16)$$

where $\iiint d^3x$ represents the three dimensional integration on the space and $\oiint dS$ represents surface integration on its boundary. Also it is to be noted that t_i is the initial time and t_f is the final time. [1]

Consider the function $v = G(X,t;\ X_0,t_0)$ with the reciprocity of the Greens function $G(X,t_0;\ X_0,t) = G(X_0,t_0;\ X,t)$. By setting $t_i = 0$ and $t_f = t_{0^+}$, we can obtain the following relationship:

$$\int_0^{t_{0^+}}\iiint [uL(G(X,t_0;X_0,t)) - G(X,t_0;X_0,t)L(u)]\ d^3x\,dt = \iiint\left(u\frac{\partial G}{\partial t} - G\frac{\partial u}{\partial t}\right)_0^{t_{0^+}} d^3x - c^2\int_0^{t_{0^+}}\left(\oiint (u\nabla G - G\nabla u)\cdot\hat{n}\ dS\right)dt$$

Using equations (14) and (15), together with the conditions (1-a) and (1-b), the above expression simplifies to

$$\int_0^{t_{0^+}}\iiint u(X,t)\delta(X - X_0)\delta(t - t_0)d^3x\,dt = \iiint\left(u\frac{\partial G}{\partial t} - G\frac{\partial u}{\partial t}\right)_0^{t_{0^+}} d^3x - c^2\int_0^{t_{0^+}}\left(\oiint (u\nabla G - G\nabla u)\cdot\hat{n}\ dS\right)dt$$

At $t = t_{0^+}$, $G = 0$ and since G is source varying function $\frac{\partial}{\partial t}G = 0$. Thus we get the solution of the equation (1) in terms of the Green's function as

$$u(X,t) = \iiint\left[\frac{\partial u}{\partial t_0}(X_0,0)G(X,t;X_0,0) - u(X_0,0)\frac{\partial}{\partial t_0}G(X,t;X_0,0)\right]d^3X_0 - c^2\int_0^{t^+}\left(\oiint (u(X_0,t_0)\nabla G(X,t\,;X_0,t_0) - G(X,t\,;X_0,t_0)\nabla u(X_0,t_0))\cdot\hat{n}\ dS_0)dt_0\right) \quad (17)$$

To find the Green's function $G(X,t\,;X_0,t_0)$ in an infinite space, applying Fourier transform to equation (2) we have

$$F\left(\frac{\partial^2 G}{\partial t^2}\right) = F(c^2\nabla^2 G) + F\big(\delta(X - X_0)\delta(t - t_0)\big) \quad (18)$$

Using the characteristic of Delta function [4]

$$\int_{-\infty}^{\infty}\delta(X - X_0)f(X)\,d^3X = f(X_0)$$

It can be observed that

$$F\big(\delta(X - X_0)\big) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\delta(X - X_0)e^{i\mu\cdot X}d^3X = \frac{e^{i\mu\cdot X_0}}{(2\pi)^3}$$

It follows from (18) that

$$\frac{\partial^2 \bar{G}}{\partial t^2} + c^2\mu^2\bar{G} = \frac{e^{i\mu X_0}}{(2\pi)^3}\delta(t-t_0) \qquad (19)$$

Where $\bar{G}(\mu, t;\ X_0, t_0)$ is Fourier transform of the Green function and $\mu^2 = \mu \cdot \mu$.
As we know that $G(X, t; X_0, t_0)$ represents the response on X at a time t for the source at X_0 at time t_0.
This means $G(X, t;\ X_0, t_0) = 0$ for $t < t_0$
It follows that $\bar{G}(\mu, t;\ X_0, t_0) = 0$ for $t < t_0$
Hence for $t > t_0$ we have

$$\frac{\partial^2 \bar{G}}{\partial t^2} + c^2\mu^2\bar{G} = 0 \qquad (\because \delta(t-t_0) = 0)$$

Solving upon it we get

$$\bar{G}(\mu, t\ ; X_0\ , t_0) = \frac{e^{i\mu X_0}}{c\mu\,(2\pi)^3} \sin c\mu(t-t_0)$$

It follows from the definition of the Fourier transform that

$$G(X, t; X_0, t_0) = \frac{1}{(2\pi)^3}\iiint \bar{G}(\mu, t\ ; X_0\ , t_0) e^{-i\mu X} d^3\mu$$

$$\Rightarrow G(X, t; X_0, t_0) = \frac{1}{(2\pi)^3}\iiint \frac{e^{-i\mu(X-X_0)}}{c\mu} \sin c\mu(t-t_0)\, d^3\mu$$

Using the spherical coordinates, with center at $\mu = 0$ and angle $\emptyset = 0$ in the direction of $X - X_0$, we have $0 < \mu < \infty, 0 < \theta < 2\pi, 0 < \emptyset < \pi$.
Let $r = |X - X_0|$.
Then $\mu \cdot (X - X_0) = |\mu||X - X_0| cos\ \emptyset = \mu r\ cos\ \emptyset$ and $d^3\mu = \mu^2\ sin\ \emptyset\ \ d\emptyset\, d\theta\, d\mu$.

Hence

$$G(X, t; X_0, t_0) = \frac{1}{(2\pi)^3}\int_{\mu=0}^{\infty}\int_{\emptyset=0}^{\pi}\int_{\theta=0}^{2\pi} e^{-i\mu r\cos\emptyset}\frac{\sin c\mu(t-t_0)}{c\mu}\mu^2 \sin\emptyset\ d\emptyset\, d\theta\, d\mu$$

After applying the integration on the variable $\emptyset$ and θ and using Euler's formula one has

$$G(X, t; X_0, t_0) = \frac{1}{(2\pi)^2 cr}\int_{\mu=0}^{\infty}\{\cos\left(r - c(t-t_0)\right)\mu - \cos\left(r + c(t-t_0)\right)\mu\}\ d\mu$$

To understand the above integral we need to refocus on the definition of Fourier transform.
Note that the Fourier transform of Dirac delta function (usually known as impulse function) $\delta(x - x_0)$ is given by

$$F(\delta(x-x_0)) = \frac{1}{2\pi}\int \delta(x-x_0)\, e^{i\mu\cdot x} dx = \frac{e^{i\mu\cdot X_0}}{2\pi}$$

So its inverse Fourier transform will be given by

$$\delta(x-x_0) = \int \frac{e^{i\mu\cdot X_0}}{2\pi} e^{-i\mu\cdot X} d\mu = \frac{1}{2\pi}\int e^{-i\mu\cdot(x-x_0)}\, d\mu$$

Let $x_0 = 0$ we have

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu x}\, d\mu$$

$$\Rightarrow \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\cos\mu x + i\sin\mu x)\, d\mu = \frac{1}{\pi}\int_{0}^{\infty}\cos\mu x\ d\mu$$

By comparing this integral expression of $\delta(x)$ with

$$G(X, t; X_0, t_0) = \frac{1}{(2\pi)^2 cr}\int_{\mu=0}^{\infty}\{\cos\left(r - c(t-t_0)\right)\mu - \cos\left(r + c(t-t_0)\right)\mu\}\ d\mu$$

$$G(X, t; X_0, t_0) = \frac{1}{4\pi cr}\left[\delta\left(r - c(t-t_0)\right) - \delta\left(r + c(t-t_0)\right)\right]$$

As $r > 0$ and $t - t_0 > 0$, $\delta\left(r + c(t-t_0)\right) = 0$.
Hence

$$G(X, t; X_0, t_0) = \frac{1}{4\pi cr}\delta\left(r - c(t-t_0)\right) \qquad (20)$$

This represents the Green's function for the three dimensional wave equation, is a spherical shell impulse spreading out from $r = 0$($i.e\ X = X_0$) at radial velocity c with amplitude diminishing proportional to $\frac{1}{r}$. [1]
As there are no boundaries, the solution shown in equation (17) is transformed into

$$u(X, t) = \iiint\left[\frac{\partial u}{\partial t_0}(X_0, 0)G(X, t; X_0, 0) - u(X_0, 0)\frac{\partial}{\partial t_0}G(X, t; X_0, 0)\right]d^3X_0$$

Using the Green's function shown in (20) we have

$$u(X, t) = \iiint G(X_0)\frac{1}{4\pi cr}\delta(r - ct)d^3X_0 - \iiint f(X_0)\left[\frac{\partial}{\partial t_0}\frac{1}{4\pi cr}\delta\left(r - c(t-t_0)\right)\right]_{t_0=0} d^3X_0 \quad (21)$$

This is the another form of explicit solution in terms of the volume integral of the given problem (1) which reveals the impact of the initial conditions $f(X)$ and $g(X)$ individually. This illustrates the elegance and effectiveness of the Green's function method. [1]
Now, we employ both theories to the following problem.

5. Implementation

Problem:
Let the gas molecules travels through an ideal gas. Then the electromagnetic radiations of the gas molecules satisfies the three-dimensional wave equation: [2,3]

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \qquad (22)$$

with initial conditions,

$$u(X, 0) = 0, \frac{\partial u}{\partial t}(X, 0) = \begin{cases} T & if\ a^2 > x^2 + y^2 + z^2 \\ 0 & otherwise \end{cases}$$

where c is speed of electromagnetic radiation in free space.
The problem aims to study the behavior of the electromagnetic radiation as it travels through an ideal gas, interacting with the surrounding electrons of gas molecules, resulting in energy loss and continuous radiation production from the surrounding electrons.

Solution: To find the solution to this problem, one can use the Fourier transform method and the theory of Green's function described earlier to derive an expression that reveals

the effect of the initial conditions $f(X) = u(X,0)$ and $g(X) = \frac{\partial u}{\partial t}(X,0)$ on the radiation's behavior.
Here we have the initial position $f(X) = 0$
And the initial velocity

$$g(X) = \begin{cases} T & \text{if } a^2 > x^2 + y^2 + z^2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Fourier Transform method

As $f(X) = 0$, it follows from the solution in its explicit form shown in the equation (13)

$$u(X,t) = \frac{1}{4\pi c^2 t} \iint_{|X_0|=ct} g(X - X_0)\, dS \qquad (23)$$

Where

$$g(X - X_0) = \begin{cases} T & \text{if } a^2 > (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \\ 0 & \text{otherwise} \end{cases}$$

To find the surface integration of the function $g(X - X_0)$ on the sphere of radius ct, we consider the sphere of radius ct with centre at $X = (x, y, z)$ and if we let $|X - X_0| = \xi$, then according to the definition of the function $g(X - X_0)$,we have the following cases to understand the effect of the initial velocity of the source X_0 on the point $X = (x, y, z)$.
$0 < ct < a - \xi$, $a - \xi < ct < a + \xi$, $ct > a + \xi$

Case 1: $0 < ct < a - \xi$
In such case, the sphere of radius ct centred at X is entirely covered in sphere of radius a centred at X_0 shown in the below figure 1.

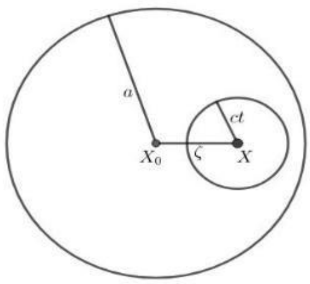

Figure 1. Sphere of radius a containing the sphere of radius $ct < a - \xi$

Hence, $\xi^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < a^2$,
Hence the solution will be

$$u(X,t) = \frac{1}{4\pi c^2 t} \iint_{|X_0|=ct} g(X - X_0) dS = \frac{1}{4\pi c^2 t} \iint_{|X_0|=ct} T\, dS$$

Using polar coordinates, one has

$$u(X,t) = \frac{1}{4\pi c^2 t} \int_{\emptyset=0}^{\pi} \int_{\theta=0}^{2\pi} T(ct)^2 \sin\emptyset \; d\emptyset\, d\theta$$

$$= \frac{2\pi T(ct)^2}{4\pi c^2 t} \int_{\emptyset=0}^{\pi} \sin\emptyset \; d\emptyset$$

$$= \frac{Tt}{2} (-\cos\emptyset)_0^{\pi} = \frac{Tt}{2} 2 = Tt$$

$$\Rightarrow u(X,t) = Tt \qquad (24)$$

Case 2: $a - \xi < ct < a + \xi$

The sphere of radius ct centred at X is intersecting the sphere of radius a centred at X_0 as shown in figure 2.

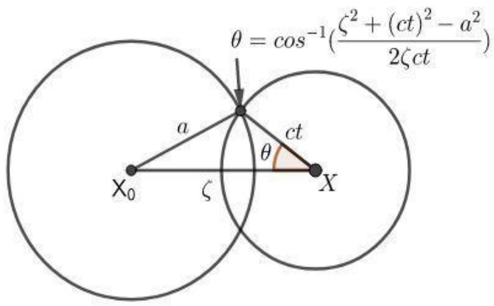

Figure 2 . Intersection of two sphere of radius a and ct

The surface area of intersecting part of the sphere of radius ct will be affected by $g(X - X_0)$. On the required surface area, the angle $\emptyset$ varies from $\emptyset = 0$ to $\emptyset = \cos^{-1}\left(\frac{(ct)^2 + \xi^2 - a^2}{2\, ct\, \xi}\right)$ with symmetry and θ varies from 0 to 2π.
Hence, using (23) the solution will be

$$u(X,t) = \frac{1}{4\pi c^2 t} \iint_{|X_0|=ct} g(X - X_0) dS$$

$$\Rightarrow u(X,t) = \frac{1}{4\pi c^2 t}\; 2\int_{\emptyset=0}^{\cos^{-1}\left(\frac{(ct)^2+\xi^2-a^2}{2\,ct\,\xi}\right)} \int_{\theta=0}^{2\pi} T(ct)^2 \sin\emptyset \; d\emptyset\, d\theta$$

$$\Rightarrow u(X,t) = \frac{2\pi T(ct)^2}{4\pi c^2 t}\; 2\int_{\emptyset=0}^{\cos^{-1}\left(\frac{(ct)^2+\xi^2-a^2}{2\,ct\,\xi}\right)} \sin\emptyset \; d\emptyset$$

$$\Rightarrow u(X,t) = Tt \int_{\emptyset=0}^{\cos^{-1}\left(\frac{(ct)^2+\xi^2-a^2}{2\,ct\,\xi}\right)} \sin\emptyset \; d\emptyset$$

$$\Rightarrow u(X,t) = Tt\; (-\cos\emptyset)_0^{\cos^{-1}\left(\frac{\xi^2+r^2-a^2}{2\,\xi\, r}\right)}$$

$$\Rightarrow u(X,t) = Tt\; \left(-\cos\left[\cos^{-1}\left(\frac{\xi^2 + r^2 - a^2}{2\,\xi\, r}\right)\right] + \cos 0\right)_0^{\cos^{-1}\left(\frac{\xi^2+r^2-a^2}{2\,\xi\, r}\right)}$$

$$\Rightarrow u(X,t) = Tt\left(1 - \frac{\xi^2 + r^2 - a^2}{2\xi\, r}\right)$$

$$\Rightarrow u(X,t) = \frac{T}{2\, c\, \xi}(a^2 - (\xi - ct)^2) \qquad (25)$$

Case 3: $ct > a + \xi$

In this case, we have
$\xi^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 > a^2$ as shown in the following figure.

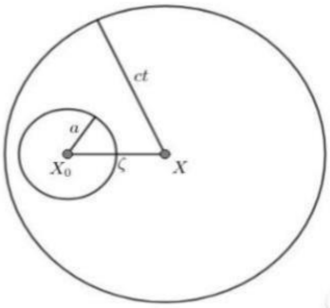

Figure 3 Sphere of radius $ct > a + r$ containing the sphere of radius a

It follows from the definition of g, $g(X - X_0) = 0$.
∴The solution will be

$$u(X,t) = \frac{1}{4\pi c^2 t} \iint_{|X_0|=ct} g(X - X_0) dS = 0 \qquad (26)$$

Hence the electromagnetic radiations $u(X,t)$ of the gas molecules at X at time t is given by

$$u(X,t) = \begin{cases} Tt & 0 < ct < a - \xi \\ \frac{T}{2c\xi}[a^2 - (\xi - ct)^2] & 0 < a - \xi < ct < a + \xi \\ 0 & ct > a + \xi \end{cases}$$

Where $|X - X_0| = \xi$ (27)

(b) Green's function method

By utilizing the concept of Green's function $G(X,t;\, X_0, t_0)$ and the solution shown in the equation (21) in terms of Green''s function, as $f(X) = 0$, the solution will be

$$u(X,t) = \iiint g(X') \frac{1}{4\pi c r} \delta(r - ct)\, d^3X' \qquad (28)$$

where the integration has been taken with respect to all the sources X' which effects on X at time t. Again let $\xi = |X - X_0|$, the distance between the source X_0 and X and let $r = |X - X'|$, where X' influences on X at the distance $r = ct$. Here X_0 is the fixed source and X' is the varying source.
To understand the effect of initial velocity at the source X_0 on X at time t, we consider the sphere with radius r, centre at X. Also, the angle $\emptyset = 0$ is to be taken in the direction of $X - X_0$ and polar angle θ is to be considered on the circle with radius $r \cdot sin\phi$.
According to the distance r, again we have the same cases discussed below in the figure 4 as

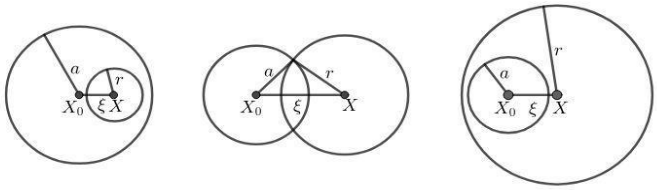

Figure 4. Spheres of radius
$0 < r < a - \xi, a - \xi < r < a + \xi,\ r > a + \xi$

Case 1: If $0 < r < a - \xi$

In this case, the sphere of radius r centred at X is entirely covered in sphere of radius a centred at X_0 shown in Fig. 4. Using spherical coordinates,

$$u(X,t) = \int_0^{a-\xi} \int_{\emptyset=0}^{\pi} \int_{\theta=0}^{2\pi} g(X') \frac{1}{4\pi c r} \delta(r - ct) r^2 \sin\emptyset \; d\emptyset\, d\theta\, dr$$

$$\Rightarrow u(X,t) = \int_0^{a-\xi} \int_{\emptyset=0}^{\pi} \int_{\theta=0}^{2\pi} T \frac{1}{4\pi c r} \delta(r - ct) r^2 \sin\emptyset \; d\emptyset\, d\theta\, dr$$

$$= \int_0^{a-\xi} \frac{Tr}{2c} \delta(r - ct)\,(-\cos\emptyset)_0^{\pi}\; dr$$

$$= \int_0^{ct} \frac{Tr}{c} \delta(r - ct)\, dr = \frac{T(ct)}{c}$$

$$\Rightarrow u(X,t) = Tt \qquad (29)$$

Case 2: If $a - \xi < r < a + \xi$

Then sphere of radius r centred at X is intersecting the sphere of radius a centred at X_0 as shown in Figure 4.
In this case, one gets

$$u(X,t) = 2\int_{a-\xi}^{a+\xi} \int_{\emptyset=0}^{\cos^{-1}\left(\frac{\xi^2 + r^2 - a^2}{2\xi r}\right)} \int_{\theta=0}^{2\pi} g(X') \frac{1}{4\pi c r} \delta(r - ct) r^2 \sin\emptyset \; d\emptyset\, d\theta\, dr$$

$$\Rightarrow u(X,t) = \int_{a-\xi}^{a+\xi} \int_{\emptyset=0}^{\cos^{-1}\left(\frac{\xi^2 + r^2 - a^2}{2\xi r}\right)} T\frac{r}{c} \delta(r - ct) \sin\emptyset\; d\emptyset\; dr$$

$$= \int_{a-\xi}^{a+\xi} \frac{Tr}{c} \delta(r - ct)\,(-\cos\emptyset)_0^{\cos^{-1}\left(\frac{\xi^2 + r^2 - a^2}{2\xi r}\right)} dr$$

$$\Rightarrow u(X,t) = \int_{a-\xi}^{a+\xi} \frac{Tr}{c}\left(1 - \frac{\xi^2 + r^2 - a^2}{2\xi\, r}\right) \delta(r - ct)\, dr$$

$$= \int_{a-\xi}^{a+\xi} \frac{Tr}{c}\left(\frac{a^2 - (\xi - r)^2}{2\xi r}\right) \delta(r - ct)\, dr$$

$$\Rightarrow u(X,t) = \int_{a-\xi}^{a+\xi} \frac{T}{2\xi c}(a^2 - (\xi - r)^2)\delta(r - ct)\, dt$$

Again using the property of Delta function[4] , we get

$$u(X,t) = \frac{T}{2\xi c}(a^2 - (\xi - ct)^2) \qquad (30)$$

Case 3: If $r > a + \xi$
The influence of source X_0 at the position X is zero because.
In this case, X_0 never effects on X.
Hence $g(X_0) = 0$ and

$$u(X,t) = \iiint g(X')\frac{1}{4\pi cr}\delta(r-ct)d^3X' = 0 \qquad (31)$$

Thus, using the Green's function also the electromagnetic radiations $u(X,t)$ at X at time t will be

$$u(X,t) = \begin{cases} Tt & 0 < ct < a-\xi \\ \frac{T}{2c\xi}\left[a^2-(\xi-ct)^2\right] & 0 < a-\xi < ct < a+\xi \\ 0 & ct > a+\xi \end{cases}$$

Where $|X-X_0| = \xi$ (32)

Hence using both the method one can see that how we are elegantly getting the analytic solution of the given initial value problem. The solution represents that the electromagnetic radiation $u(X,t)$ is depends upon the initial radial velocity a of the source X_0 as well as the distance between source X from X_0.
This approach enables us to take into account the influence of the initial velocity and the spatial distribution of the source X_0 on the behavior of the electromagnetic radiation $u(X,t)$ at any point X and time t.
.

6. Conclusion

(a) Using the solution of the problem shown in (32), analytically we can demonstrate the self-sustaining nature of the electromagnetic wave front, as well as how the spherical wave front of the electric field E influences the wave front of the magnetic field B in accordance with Huygens 'Principle. This understanding is crucial in comprehending the continuous propagation and energy conservation in electromagnetic radiation as it travels through free space.

(b) In the Green's function method, the solution is expressed as an integral involving the Green's function, which represents the response of the system to a point source or impulse. The Green's function takes into account the influence of the initial position and velocity of the source on the entire domain, and its integration provides a clearer physical interpretation of how these initial conditions affect the electromagnetic radiation at different points and times. On the other hand, while the Fourier transform method can also provide the solution to the wave equation, it may not offer a direct physical interpretation of the effects of the initial position and velocity.

(c) The effect of the initial position and velocity of the source in the solution can be better understood and visualized using the Green's function method compared to the Fourier transform method.

Future Scope

(1) The explicit forms of the solutions obtained using Fourier transform and Green's function methods not only reveal the individual effects of initial position and initial velocity but also make the solutions amenable to programming and numerical analysis. This adds to the beauty and practicality of these methods in understanding and simulating the behavior of electromagnetic radiation and other wave phenomena in different contexts.

(2) The advantage of explicit representation of the solution is that, in many cases one can solve the problem using bounded continuous functions f and g for which Fourier transform is undefined.

Data Avaibility
None

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There is no any conflict of interest in my research work as I am the only author for the research work and there is no any funding source for it.

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