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Bayesian Estimation via numerical approximations under Progressive Type II Censoring

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Abstract **-**Classical and Bayesian estimation of the unknown parametric functions for power generalized Weibull distribution under progressive Type II censoring scheme are undertaken in the present paper. Newton Raphson iterative procedure is used for computation of maximum likelihood estimates which are not obtained in closed form. Asymptotic and bootstrap confidence intervals are also obtained. Squared error and general entropy loss functions are considered for Bayes estimation under the assumption of two independent gamma priors. The approximate Bayes estimates are obtained using Tierney-Kadane approximation. Alternatively, Metropolis Hastings algorithm is run under Gibbs sampler environment to generate Bayes estimates. Computed Bayes estimates are compared with the classical maximum likelihood estimates based a simulated data and a real data set.

Keywords**:** Progressive Type II censoring scheme, Boot-*p* and Boot-*t* intervals, Tierney and Kadane method, Markov Chain Monte Carlo.

I. INTRODUCTION

Censored data is a common feature in life-testing and survival studies. Type I (*time*) censoring involves termination of a life-test experiment at a prescribed time *T*, Type II (*failure*) censoring is the one where the life testing experiment will be terminated when a prespecified number of failures r, $r \le n$ is observed and hybrid censoring is a mixture of these two schemes such that the experiment is terminated at min (T, r), whichever occurs earlier. Review of work on the various lifetime distributions carried out under these two censorings schemes is summarized in [1] . A serious drawback of these schemes is that other experimental units cannot be removed from the experiment before the final termination point of the experiment, which could be cost and time intensive. [2] Introduced a more general censoring scheme known as progressive Type II censoring in which intermittent removal of the experimental units is allowed during the experiment. The progressive Type II censoring scheme is briefly described as follows: Out of n life-test units, randomly selected r_1 out of *n-1* surviving units are withdrawn from the life-test at the first observed failure time, r_2 out of $n-r_1-2$ surviving units are withdrawn from the experiment at the time of occurrence of the second failure. Continuing thus, at the time of the final m^{th} ($1 \le m \le n$) failure, the remaining surviving units r_m with $r_m = n-m-r_1-r_2-...-r_{m-1}$ are withdrawn from the experiment. The observed sample is referred to as progressive Type II censored sample of size m. This scheme maintains optimum balance between experimental time and effective sample size used in the trials. Type II censoring scheme is a particular case of this scheme with $r_1 = r_2 = ... = r_{m-1} = 0,\ r_m = n-m$ and the complete sampling corresponds to the case when $r_1 = r_2 ..... = r_{m-1} = r_m = 0$. [3] provide the likelihood function under progressive Type II right censored sample $x_1, x_2, ..., x_m$ as under

$$f(\underline{x}) = C\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{r_i}$$

where, $C = n(n-r_1-1)(n-r_1-r_2-2)...$
$...(n-r_1-....-r_{m-1}-m+1)$ is constant.

The power generalized Weibull distribution (PGWD) is an extension of the Weibull distribution which was first proposed by [4] as a lifetime distribution to accommodate non-monotone hazard rates in addition to constant and monotone hazard pattern. Goodness of fit [5] and an application to cancer data [6] for PGWD have already been discussed in literature. More recently [7] have undertaken the Bayesian analysis of PGWD under Type II censoring.

Any estimation procedure involves loss due to estimation as invariably some gap remains between the true value and the estimator. A good estimator is characterized by the least mean squared errors (MSEs). Squared Error Loss Function (SELF) is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. General Entropy Loss Function (GELF) given by [8] which is asymmetric and useful for the situations where it is worse to underestimate (or overestimate) the potentiality of an event than to overestimate (underestimate) the unknown parameters. Tierney and Kadane (T-K) method [9] and Markov Chain Monte Carlo (MCMC) technique are suitable for giving fairly approximate solution to complex posterior distribution functions. MCMC procedure is adaptable and compliant for the purpose of data generation and subsequent parameter estimation (see, [10] and [11] for instance).

The rest of this paper is organized as follows. The maximum likelihood estimators of the parameters are presented in Section 2. Section 3 is devoted to determination of the interval estimation for the unknown parametric function. Bayes estimation and construction of credible intervals using the T-K and MCMC techniques are undertaken in Section 4. Numerical examples are presented in Section 5. A real data set based analysis is presented in Section 6 to illustrate the methods of inference developed in the paper. Finally, concluding remarks are made in Section 7.

II. THE POWER GENERALIZED WEIBULL DISTRIBUTION

A random variable X follows PGWD with shape parameters $\alpha > 0, \beta > 0$ and scale parameter $\lambda > 0$, if its probability density function (*pdf*) is given by

$$f(x;\alpha,\beta,\lambda) = \frac{\beta}{\alpha\lambda^{\beta}} x^{\beta-1}\left(1+\left(\frac{x}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1} \times \exp\left(1-\left(1+\left(\frac{x}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right); x>0, \alpha,\beta,\lambda>0 \qquad (2.1)$$

λ is the scale parameter while shape parameters are α and β . Fig. 1 shows the plot of PGWD(α, β, λ) for various values of α when $\beta = 3$ is fixed. It is seen that as α is increased the density curve flattens out and spreads over wider interval exhibiting larger variance.

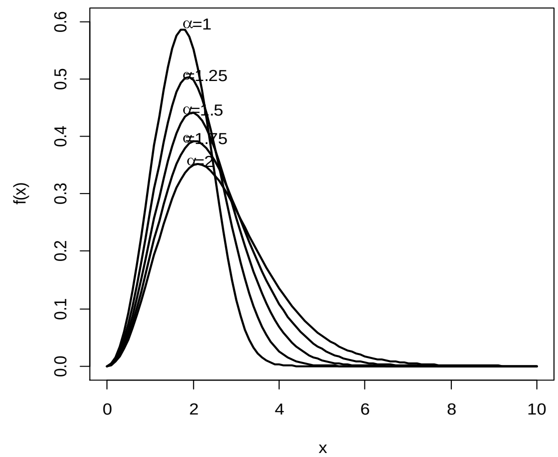

Fig. 1: *pdf* of the PGWD (α, β) for $\beta = 3$.

2.1 Some Distributional Properties

***Mean* (*M*)**

$$E(t) = \int_0^{\infty} R(t)\,dt = \int_0^{\infty} \exp\left(1-\left(1+\left(\frac{t}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)dt \qquad (2.2)$$

By using the substitution $y = \left(\frac{t}{\lambda}\right)^{\beta}$, equation (2.2) reduces to (2.3) as under,

$$\mu = \frac{\lambda}{\beta}\int_0^{\infty} y^{(1-\beta)/\beta} \exp\left(1-(1+y)^{1/\alpha}\right)dy \qquad (2.3)$$

Which converges for $\alpha > 1$, thus implying a finite expected lifetime for PGWD.

***Median Time to System Failure* (*M*e)**

Since mean time to system failure does not assume closed form for PGWD, we therefore consider the median time to system failure (MTSF) given by

$$Me = MTSF = \lambda\left((1+\log 2)^{\alpha} - 1\right)^{1/\beta} \qquad (2.4)$$

***Mode* (*M*o)**

Mode of PGWD is the solution of the following non-linear equation

$$\alpha(\beta-1) - \left(\frac{x}{\lambda}\right)^{\beta}\left[\alpha + \beta\left(1+\left(1+\left(\frac{x}{\lambda}\right)^{\beta}\right)^{1/\alpha}\right)\right] = 0 \qquad (2.5)$$

Table 1: Mean, Median and Mode of PGWD for different values of α and β

α	1	1	1	2	2	2	3	3	3
β	2	2.5	3	2	2.5	3	2	2.5	3
M	1.772	1.775	1.786	3.272	2.859	2.638	5.561	4.279	3.648
*M*e	1.6	1.7	1.7	2.7	2.5	2.4	3.9	3.4	3.1

	65	27	7	33	67	63	26	31	36
Mo	0.8 63	1.1 08	1.2 72	1.1 27	1.3 89	1.5 48	1.2 87	1.5 59	1.7 14

The numerical values of mean, median and mode are given in Table 1 and it is observed that *Mode* < *Median* < *Mean*. Thus from Table 1 and Figure 1, PGWD is seen to be a positively skewed distribution.

2.2 Reliability Characteristics

Reliability function and failure rate function of PGWD are respectively given by (2.6) and (2.7).

$$R(t)=\exp\left(1-\left(1+\left(\frac{t}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)\;;\; t\geq 0,\alpha,\beta>0 \qquad (2.6)$$

$$h(t)=\frac{f(t)}{R(t)}=\frac{\beta}{\alpha\lambda^{\beta}}t^{\beta-1}\left(1+\left(\frac{t}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1}$$
$$;t\geq 0,\alpha,\beta>0 \qquad (2.7)$$

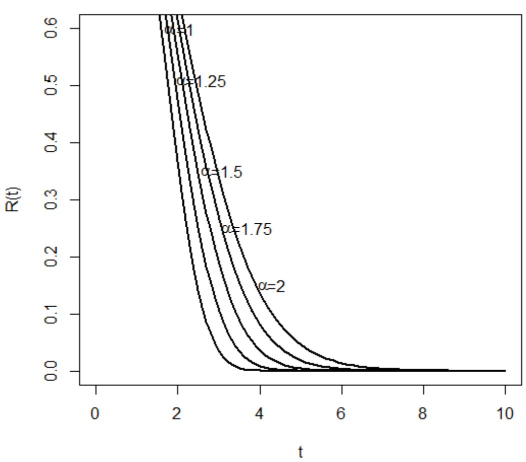

Fig. 2: *R*(t) of the PGWD (*α*, *β*) for *β*=3.

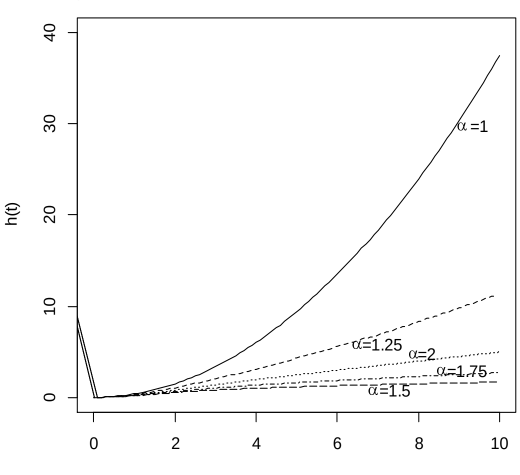

Fig. 3: *h*(t) of the PGWD (*α*, *β*) for *β*=3

III. MAXIMUM LIKELIHOOD ESTIMATION

Suppose that $X_{1:m:n}, X_{2:m:n},\ldots, X_{m:m:n}$ is a progressive Type II censored sample of size *m* from a sample of size *n* taken from (2.1) by removing r_i random units, at each respective failure *i* for *i* = *1*, *2*,…, *m*, for the test. The likelihood function based on the progressive Type II censored sample is given by

$$f(x_{1:m:n},x_{2:m:n},\ldots,x_{m:m:n})=C\prod_{i=1}^{m} f_{X_{i:m:n}}(x_{i:m:n})$$
$$\times[1-F_{X_{i:m:n}}(x_{i:m:n})]^{r_i}$$

Where,

$$C=n(n-r_1-1)(n-r_1-r_2-2)\ldots(n-r_1-\ldots.-r_{m-1}-m+1)$$

$$l(\alpha,\beta|x)=C\frac{\beta^{m}}{\alpha^{m}\lambda^{m\beta}}\prod_{i=1}^{m}x_i^{\beta-1}\left(\prod_{i=1}^{m}\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1}\right)$$

$$\times\exp\left(\sum_{i=1}^{m}(r_i+1)\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)\right) \qquad (3.1)$$

Now, the log likelihood function is given by,

$$\log l=\log C+m\log\beta-m\log\alpha-m\beta\log\lambda$$
$$+(\beta-1)\sum_{i=1}^{m}\log x_i+\left(\frac{1}{\alpha}-1\right)\sum_{i=1}^{m}\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)$$
$$+\sum_{i=1}^{m}(r_i+1)\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right) \qquad (3.2)$$

MLE of *α* is obtained as solution of the first partial derivatives of (3.2) with respect to *α* which is given as,

$$m\alpha+\sum_{i=1}^{m}\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)-\sum_{i=1}^{m}(r_i+1)$$
$$\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{1/\alpha}\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)=0 \qquad (3.3)$$

MLE of *β* is provided by the solution of the first partial derivatives of equation (3.2) with respect to *β* given by,

$$\frac{m}{\beta}-m\log(\lambda)+\sum_{i=1}^{m}\log(x_i)+\left(\frac{1}{\alpha}-1\right)$$

$$\sum_{i=1}^{m}\left(\frac{x_i}{\lambda}\right)^{\beta}\log\left(\frac{x_i}{\lambda}\right)\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{-1}$$

$$-\frac{1}{\alpha}\sum_{i=1}^{m}(r_i+1)\left(\frac{x_i}{\lambda}\right)^{\beta}\log\left(\frac{x_i}{\lambda}\right)\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1}=0 \quad (3.4)$$

(3.3) and (3.4) cannot be solved analytically. Therefore, we use Newton Raphson (N-R) iteration method to obtain $\hat{\alpha}$ and $\hat{\beta}$.

Remark 1: MLE*s* of the reliability and hazard rate, at a given time *t* are given respectively by

$$R(t)=\exp\left(1-\left(1+\left(\frac{t}{\lambda}\right)^{\hat{\beta}}\right)^{\frac{1}{\hat{\alpha}}}\right);\ t>0 \quad (3.5)$$

and

$$h(t)=\frac{\hat{\beta}}{\hat{\alpha}\,\lambda^{\hat{\beta}}}t^{\hat{\beta}-1}\left(1+\left(\frac{t}{\lambda}\right)^{\hat{\beta}}\right)^{\frac{1}{\hat{\alpha}}-1};\ t>0 \quad (3.6)$$

IV. INTERVAL ESTIMATIONS

The exact distribution of Maximum Likelihood Estimates (MLE) for the unknown parameters α and β cannot be obtained explicitly. Therefore, we evaluate Asymptotic Confidence Interval (ACI) and Bootstrap Confidence Intervals (BCI) for α and β in the following subsections.

4.1 Asymptotic Confidence intervals

The asymptotic variances and covariance of the MLE*s* for the unknown parameters α and β are given by the elements of Fisher's information matrix. The large sample approach is to assume that the MLE*s* ($\hat{\alpha},\hat{\beta}$) are approximately bivariate normal with mean (α, β) and covariance matrix $I(\phi)$, where $I(\phi)$ is the inverse of the observed information matrix defined as

$$I_0\left(\hat{\alpha},\hat{\beta}\right)=\begin{bmatrix}-\frac{\partial^2\log l}{\partial\alpha^2} & -\frac{\partial^2\log l}{\partial\alpha\,\partial\beta}\\ -\frac{\partial^2\log l}{\partial\beta\,\partial\alpha} & -\frac{\partial^2\log l}{\partial\beta^2}\end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\beta})} \quad (4.1)$$

$\left(\hat{\alpha},\hat{\beta}\right)\sim N\left((\alpha,\beta),I_0\left(\hat{\alpha},\hat{\beta}\right)\right)$ and the respective $100\left(1-\frac{\xi}{2}\right)\%$ ACI are given by

$$\hat{\alpha}\pm Z_{\xi/2}\sqrt{\mathrm{var}(\hat{\alpha})}\ ,\ \hat{\beta}\pm Z_{\xi/2}\sqrt{\mathrm{var}(\hat{\beta})}$$

where, $\mathrm{var}(\hat{\alpha})$ and $\mathrm{var}(\hat{\beta})$ are the elements on the main diagonal of the covariance matrix $I_0\left(\hat{\alpha},\hat{\beta}\right)$ and $Z_{\xi/2}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\xi}{2}$.

In order to find the approximate estimates of the variance of $\hat{R}(t)$ and $\hat{h}(t)$ delta method (see [12], [13]) is chosen.

Let $G_1=\left(\frac{\partial R(t)}{\partial\alpha},\frac{\partial R(t)}{\partial\beta}\right)$ and $G_2=\left(\frac{\partial h(t)}{\partial\alpha},\frac{\partial h(t)}{\partial\beta}\right)$, where $\frac{\partial R(t)}{\partial\alpha},\frac{\partial R(t)}{\partial\beta},\frac{\partial h(t)}{\partial\alpha}$ and $\frac{\partial h(t)}{\partial\beta}$ are the first derivatives of the $R(t)$ and $h(t)$ with respect to the parameters α and β, respectively. The approximate asymptotic variances of $\hat{R}(t)$ and $\hat{h}(t)$ are given respectively by

$\hat{V}ar(\hat{R}(t))=\left[G_1^T I_0 G_1\right]_{(\hat{\alpha},\hat{\beta})}$ and $\hat{V}ar(\hat{h}(t))=\left[G_2^T I_0 G_2\right]_{(\hat{\alpha},\hat{\beta})}$

where I_0 is given by (3.1) and G_l^T is the transpose of G_l, l = 1, 2. These results yield the approximate confidence intervals for $R(t)$ and $h(t)$ respectively as

$$\hat{R}(t)\pm Z_{\xi/2}\sqrt{\mathrm{var}(\hat{R}(t))}\ \text{ and }\ \hat{h}(t)\pm Z_{\xi/2}\sqrt{\mathrm{var}(\hat{h}(t))}$$

4.2 Bootstrap confidence intervals

Bootstrap is a computationally intensive method based on the concept of resampling from an observed data set which are applicable without theoretical assumption of normality ([14], [15]). Computational steps for estimation of confidence intervals of the unknown quantities are presented as under,

Boot-p Confidence Interval Algorithm

1. Generate a bootstrap sample $X_1^*,X_2^*,\ldots\ldots,X_m^*$ using $X_1,X_2,\ldots\ldots,X_m$ and compute the estimate $\hat{\theta}^*$ of the parameter θ (in our case, θ could be α or β) using the bootstrap sample.
2. Repeat step 1, N times.
3. Suppose that $\hat{F}_1(x)=P(\hat{\theta}^*\le x)$ is the cumulative distribution function of $\hat{\theta}^*$. Then, define $\hat{\theta}_{Boot-p}(x)=\hat{F}_1^{-1}(x)$ for a given x. The approximate 100(1- ξ)% confidence interval for θ is given by $\left(\hat{\theta}_{Boot-p}\left(\frac{\xi}{2}\right),\hat{\theta}_{Boot-p}\left(1-\frac{\xi}{2}\right)\right)$

Boot-t Confidence Interval Algorithm

1. Generate the sample $X_1, X_2, \ldots\ldots, X_m$ from equation (2.1) and compute the estimate of the unknown parametric function $\hat{\theta}$.
2. Draw a bootstrap sample $X_1^*, X_2^*, \ldots\ldots, X_m^*$ using $\hat{\theta}$. Then compute the estimate $\hat{\theta}^*$ and $\hat{V}(\hat{\theta}^*)$ and compute the T^* statistics, $T^* = \dfrac{\hat{\theta}^* - \hat{\theta}}{\sqrt{\hat{V}(\hat{\theta}^*)}}$
3. Repeat step 2, N times.
4. Let $\hat{F}_2(x) = P(T^* \le x)$ be cumulative distribution function of T^* Define $\hat{\theta}_{Boot-t}(x) = \hat{\theta} + \sqrt{\hat{V}(\hat{\theta}^*)}\hat{F}_2^{-1}(x)$ for a given x. The approximate 100(1- ξ)% confidence interval for θ is given by,

$$\left(\hat{\theta}_{Boot-t}\left(\frac{\xi}{2}\right), \hat{\theta}_{Boot-t}\left(1-\frac{\xi}{2}\right)\right)$$

V. BAYESIAN ESTIMATION

Gamma distribution can accommodate variety of shapes depending upon parameter values. This flexibility makes them suitable candidate for priors. We consider two independent gamma priors for α and β as gamma (γ, σ) and gamma (μ, η) respectively, as under

$$g(\alpha) = \frac{\gamma^{\sigma}}{\Gamma(\sigma)}\alpha^{\sigma-1}\exp(-\gamma\alpha) \quad ; \alpha > 0, \gamma, \sigma > 0,$$

$$g(\beta) = \frac{\mu^{\eta}}{\Gamma(\eta)}\beta^{\eta-1}\exp(-\mu\beta) \quad ; \beta > 0, \mu, \eta > 0$$

The joint prior density is thus given by

$$g(\alpha,\beta) = \frac{\gamma^{\sigma}\mu^{\eta}}{\Gamma(\sigma)\Gamma(\eta)}\alpha^{\sigma-1}\beta^{\eta-1}\exp(-\gamma\alpha-\mu\beta);$$
$$\alpha,\beta > 0,\ \sigma,\eta,\gamma,\mu > 0 \qquad (5.1)$$

Based on the likelihood function (4.1) of the observed sample and the joint prior (5.1), the joint posterior density of the unknown parameters α and β given the data is obtained as (5.2)

$$\Pi(\alpha,\beta \mid \underline{x}) = \frac{\beta^{m+\eta-1}e^{-(\gamma\alpha+\mu\beta)}}{K\lambda^{m\beta}\alpha^{m-\sigma+1}}\prod_{i=1}^{m}x_i^{\beta-1}\left(\prod_{i=1}^{m}\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1}\right)$$

$$\times\exp\left(\sum_{i=1}^{m}(r_i+1)\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)\right) \qquad (5.2)$$

Since $\Pi(\alpha,\beta \mid \underline{x})$ is analytically intractable, therefore T-K method is used to obtain estimates.

5.1 Tierney and Kadane Approximation

T-K method approximates posterior expectation of a parametric function $w(\alpha,\beta)$ that is expressible as a ratio of two integrals. Let

$$\tilde{w} = E(w(\alpha,\beta)|\underline{x}) = \int\int w(\alpha,\beta)\Pi(\alpha,\beta \mid \underline{x})\,d\alpha\,d\beta$$

$$= \frac{\int\int e^{n\delta_w^*(\alpha,\beta)}d\alpha\,d\beta}{\int\int e^{n\delta(\alpha,\beta)}d\alpha\,d\beta} \qquad (5.3)$$

where,

$$\delta(\alpha,\beta) = \frac{[l(\alpha,\beta|\underline{x}) + \rho(\alpha,\beta)]}{n},$$

$$\delta_w^*(\alpha,\beta) = \delta(\alpha,\beta) + \frac{\log w(\alpha,\beta)}{n} \qquad (5.4)$$

Here $l(\alpha,\beta|x)$ denotes the log-likelihood and $\rho(\alpha,\beta) = \log\, g(\alpha,\beta)$ such that $g(\alpha,\beta)$ represents the joint prior distribution. An application of T-K approximation suggests that equation (5.3) is given as,

$$\tilde{w} = \sqrt{\frac{|\Sigma_w^*|}{|\Sigma|}}\exp\left[n\left\{\delta^*\left(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*}\right) - \delta\left(\hat{\alpha}_{\delta},\hat{\beta}_{\delta}\right)\right\}\right] \qquad (5.5)$$

Here, $|\Sigma|$ and $|\Sigma_w^*|$ denote the determinant of negative inverse Hessians of $\delta(\alpha,\beta)$ and $\delta_w^*(\alpha,\beta)$. Writing

$$\delta(\alpha,\beta) = \frac{1}{n}\Big[m\log\beta - m\log\alpha - m\beta\log\lambda$$
$$+\sigma\log\lambda + \eta\log\mu - \log(\Gamma\sigma) - \log(\Gamma\eta)$$
$$+(\sigma-1)\log\alpha + (\eta-1)\log\beta - \lambda\alpha - \mu\beta$$
$$+(\beta-1)\sum_{i=1}^{m}\log x_i + \left(\frac{1}{\alpha}-1\right)\sum_{i=1}^{m}\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)$$
$$+\sum_{i=1}^{m}(r_i+1)\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)\Bigg]$$

Taking derivatives,

$$\frac{\partial \delta}{\partial \alpha} = -\lambda + \frac{(\sigma-1)}{\alpha} - \frac{m}{\alpha} + \frac{1}{\alpha^2}\sum_{i=1}^{m}\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)$$

$$-\frac{1}{\alpha^2}\sum_{i=1}^{m}(r_i+1)\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{1/\alpha}\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)$$

and

$$\frac{\partial \delta}{\partial \beta} = -\mu + \frac{(\eta-1)}{\beta} + \frac{m}{\beta} - m\log(\lambda) + \sum_{i=1}^{m}\log(x_i)$$

$$\times\log\left(\frac{x_i}{\lambda}\right)\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{-1} + \left(\frac{1}{\alpha}-1\right)\sum_{i=1}^{m}\left(\frac{x_i}{\lambda}\right)^{\beta}$$

$$-\frac{1}{\alpha}\sum_{i=1}^{m}(r_i+1)\left(\frac{x_i}{\lambda}\right)^{\beta}\times\log\left(\frac{x_i}{\lambda}\right)\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1}$$

Likewise, the corresponding second-order derivatives are obtained as

$$\frac{\partial^2 \delta}{\partial \alpha^2} = -\frac{(\sigma-1)}{\alpha^2}\frac{m}{\alpha^2} + \frac{2}{\alpha^3}\sum_{i=1}^{m}\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)$$

$$-\frac{1}{\alpha^4}\sum_{i=1}^{m}(r_i+1)\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\left(\frac{1}{\alpha}\right)}$$

$$\times\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)\left[\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)+2\alpha\right]$$

$$\frac{\partial^2 \delta}{\partial \alpha\,\partial \beta} = \frac{\partial^2 \delta}{\partial \alpha\,\partial \beta}$$

$$= -\frac{1}{\alpha^2}\sum_{i=1}^{m}\frac{\left(\frac{x_i}{\lambda}\right)^{\beta}\log\left(\frac{x_i}{\lambda}\right)}{\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)}\left[1-\frac{1}{\alpha}(r_i+1)\right.$$

$$\left.\times\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\left(\frac{1}{\alpha}\right)}\left(\log\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)+\alpha\right)\right]$$

$$\frac{\partial^2 \delta}{\partial \beta^2} = -\frac{(\eta-1)}{\beta} - \frac{m}{\beta^2} + \left(\frac{1}{\alpha}-1\right)$$

$$\times\sum_{i=1}^{m}\left(\frac{x_i}{\lambda}\right)^{\beta}\log\left(\frac{x_i}{\lambda}\right)^2\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{-2}$$

$$-\frac{1}{\alpha^2}\sum_{i=1}^{m}(r_i+1)\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\left(\frac{1}{\alpha}-2\right)}\left(\frac{x_i}{\lambda}\right)^{\beta}$$

$$\times\log\left(\frac{x_i}{\lambda}\right)^2\left(\alpha+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)$$

Thus, we obtain $|\Sigma|$ as

$$|\Sigma| = \left[\frac{\partial^2 \delta}{\partial \alpha^2}\frac{\partial^2 \delta}{\partial \beta^2} - \frac{\partial^2 \delta}{\partial \alpha\,\partial \beta}\frac{\partial^2 \delta}{\partial \beta\,\partial \alpha}\right]^{-1}$$

In order to compute $\left|\Sigma_w^*\right|$, we first compute the following expressions:

$$\frac{\partial \delta^*}{\partial \alpha} = \frac{\partial \delta}{\partial \alpha} + \frac{w_\alpha}{n\,w(\alpha,\beta)}$$

$$\frac{\partial \delta^*}{\partial \beta} = \frac{\partial \delta}{\partial \beta} + \frac{w_\beta}{n\,w(\alpha,\beta)}$$

$$\frac{\partial^2 \delta^*}{\partial \alpha^2} = \frac{\partial^2 \delta}{\partial \alpha^2} + \frac{1}{n}\left[\frac{w(\alpha,\beta)w_{\alpha\alpha}-(w_\alpha)^2}{\{w(\alpha,\beta)\}^2}\right]$$

$$\frac{\partial^2 \delta^*}{\partial \beta^2} = \frac{\partial^2 \delta}{\partial \beta^2} + \frac{1}{n}\left[\frac{w(\alpha,\beta)w_{\beta\beta}-(w_\beta)^2}{\{w(\alpha,\beta)\}^2}\right]$$

$$\frac{\partial^2 \delta^*}{\partial \alpha\,\partial \beta} = \frac{\partial^2 \delta}{\partial \beta\,\partial \alpha} + \frac{1}{n}\left[\frac{w(\alpha,\beta)w_{\alpha\beta}-w_\alpha w_\beta}{\{w(\alpha,\beta)\}^2}\right]$$

It is thus seen that

$$\left|\Sigma_w^*\right| = \left[\frac{\partial^2 \delta^*}{\partial \alpha^2}\frac{\partial^2 \delta^*}{\partial \beta^2} - \frac{\partial^2 \delta^*}{\partial \alpha\partial \beta}\frac{\partial^2 \delta^*}{\partial \beta\,\partial \alpha}\right]^{-1}$$

Finally, we consider the parametric estimation under SELF and GELF. In order to compute Bayes estimates of α, β, $R(t)$ and $h(t)$ under SELF, we take $w(\alpha,\beta)=\alpha$, $w(\alpha,\beta)=\beta$, $w(\alpha,\beta)=R(t)$ and $w(\alpha,\beta)=h(t)$. Accordingly the function $\delta_w^*(\alpha,\beta)$ (see equation (5.4)) becomes

$$\delta_\alpha^*(\alpha,\beta) = \delta(\alpha,\beta) + \frac{1}{n}\log\alpha$$

$$\delta_\beta^*(\alpha,\beta) = \delta(\alpha,\beta) + \frac{1}{n}\log\beta$$

and

$$\delta_{rt}^{*}(\alpha,\beta)=\delta(\alpha,\beta)+\frac{1}{n}\left(1-\left(1+\left(\frac{t}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)$$

$$\delta_{ht}^{*}(\alpha,\beta)=\delta(\alpha,\beta)+\frac{1}{n}\log\left(\frac{\beta}{\alpha\lambda^{\beta}}t^{\beta-1}\left(1+\left(\frac{t}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1}\right)$$

under GELF, $q > 0$ represents overestimation ($\tilde{\alpha}_{BG_2T}$, $\tilde{\beta}_{BG_2T}$, $\tilde{R}_{BG_2T}, \tilde{h}_{BG_2T}$) and $q < 0$ represents underestimation ($\tilde{\alpha}_{BG_1T}$, $\tilde{\beta}_{BG_1T}$, $\tilde{R}_{BG_1T}, \tilde{h}_{BG_1T}$). To study the behaviour of our proposed estimators we take q = 2 and -2 to represent the two situations. We compute Bayes estimators of $w(\alpha,\beta)=\alpha^{-q}, w(\alpha,\beta)=\beta^{-q}, w(\alpha,\beta)=R^{-q}(t)$ and $w(\alpha,\beta)=h^{-q}(t)$, where $\hat{w}(\alpha,\beta)=\left(E_{w(\alpha,\beta)}\left(\{w(\alpha,\beta)\}^{-q}\right)\right)^{-1/q}$ and accordingly the function $\delta_{w}^{*}(\alpha,\beta)$ becomes

$$\delta_{\alpha}^{*}(\alpha,\beta)=\delta(\alpha,\beta)-\frac{q}{n}\log(\alpha),$$

$$\delta_{\beta}^{*}(\alpha,\beta)=\delta(\alpha,\beta)-\frac{q}{n}\log(\beta),$$

$$\delta_{rt}^{*}(\alpha,\beta)=\delta(\alpha,\beta)-\frac{q}{n}\left(1-\left(1+\left(\frac{t}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right) \text{ and}$$

$$\delta_{ht}^{*}(\alpha,\beta)=\delta(\alpha,\beta)-\frac{q}{n}\log\left(\frac{\beta}{\alpha\lambda^{\beta}}t^{\beta-1}\left(1+\left(\frac{t}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1}\right)$$

The desired Bayes estimators of the parameters α, β, $R(t)$ and $h(t)$ under SELF are found to be

$$\tilde{\alpha}_{BST}=\left\{\sqrt{\frac{|\Sigma_{\alpha}^{*}|}{|\Sigma|}}\exp\left[n\left\{\delta_{\alpha}^{*}\left(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}}\right)-\delta\left(\hat{\alpha}_{\delta},\hat{\beta}_{\delta}\right)\right\}\right]\right\} \quad (5.6)$$

$$\tilde{\beta}_{BST}=\left\{\sqrt{\frac{|\Sigma_{\beta}^{*}|}{|\Sigma|}}\exp\left[n\left\{\delta_{\beta}^{*}\left(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}}\right)-\delta\left(\hat{\alpha}_{\delta},\hat{\beta}_{\delta}\right)\right\}\right]\right\} \quad (5.7)$$

$$\tilde{R}_{BST}(t)=\left\{\sqrt{\frac{|\Sigma_{rt}^{*}|}{|\Sigma|}}\exp\left[n\left\{\delta_{rt}^{*}\left(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}}\right)-\delta\left(\hat{\alpha}_{\delta},\hat{\beta}_{\delta}\right)\right\}\right]\right\} \quad (5.8)$$

and

$$\tilde{h}_{BST}(t)=\left\{\sqrt{\frac{|\Sigma_{ht}^{*}|}{|\Sigma|}}\exp\left[n\left\{\delta_{ht}^{*}\left(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}}\right)-\delta\left(\hat{\alpha}_{\delta},\hat{\beta}_{\delta}\right)\right\}\right]\right\} \quad (5.9)$$

Similarly, the desired Bayes estimators of the unknown parameters α, β, $R(t)$ and $h(t)$ under GELF are found to be

$$\tilde{\alpha}_{BGT}=\left\{\sqrt{\frac{|\Sigma_{\alpha}^{*}|}{|\Sigma|}}\exp\left[n\left\{\delta_{\alpha}^{*}\left(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}}\right)-\delta\left(\hat{\alpha}_{\delta},\hat{\beta}_{\delta}\right)\right\}\right]\right\}^{\frac{-q}{2}} \quad (5.10)$$

$$\tilde{\beta}_{BGT}=\left\{\sqrt{\frac{|\Sigma_{\beta}^{*}|}{|\Sigma|}}\exp\left[n\left\{\delta_{\beta}^{*}\left(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}}\right)-\delta\left(\hat{\alpha}_{\delta},\hat{\beta}_{\delta}\right)\right\}\right]\right\}^{\frac{-q}{2}} \quad (5.11)$$

$$\tilde{R}_{BGT}(t)=\left\{\sqrt{\frac{|\Sigma_{rt}^{*}|}{|\Sigma|}}\exp\left[n\left\{\delta_{rt}^{*}\left(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}}\right)-\delta\left(\hat{\alpha}_{\delta},\hat{\beta}_{\delta}\right)\right\}\right]\right\}^{\frac{-q}{2}} \quad (5.12)$$

and

$$\tilde{h}_{BGT}(t)=\left\{\sqrt{\frac{|\Sigma_{ht}^{*}|}{|\Sigma|}}\exp\left[n\left\{\delta_{ht}^{*}\left(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}}\right)-\delta\left(\hat{\alpha}_{\delta},\hat{\beta}_{\delta}\right)\right\}\right]\right\}^{\frac{-q}{2}} \quad (5.13)$$

5.2 Markov Chain Monte Carlo Technique

The conditional posterior distributions of the parameters α and β are respectively given by

$$\Pi_1(\alpha\,|\,\underline{x},\beta)=\frac{e^{-\gamma\alpha}}{\alpha^{m-\sigma+1}}\int_0^{\infty}\frac{\beta^{m+\eta-1}e^{-\mu\beta}}{K\lambda^{m\beta}}\times\prod_{i=1}^{m}x_i^{\beta-1}\left(\prod_{i=1}^{m}\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1}\right)\times\exp\left(\sum_{i=1}^{m}(r_i+1)\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)\right)d\beta \quad (5.14)$$

$$\Pi_2(\beta\,|\,\underline{x},\alpha)=\frac{\beta^{m+\eta-1}e^{-\mu\beta}}{\lambda^{m\beta}}\int_0^{\infty}\frac{e^{-\gamma\alpha}}{K\alpha^{m-\sigma+1}}\times\prod_{i=1}^{m}x_i^{\beta-1}\left(\prod_{i=1}^{m}\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}-1}\right)\times\exp\left(\sum_{i=1}^{m}(r_i+1)\left(1-\left(1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)\right)d\alpha \quad (5.15)$$

We apply Metropolis-Hastings (M-H) algorithm [16] under Gibbs sampler environment, to generate sample from the full conditional of α and β given by (5.14) and (5.15), respectively. To simulate Bayes estimator the following iterative algorithm is proposed.

Step 1: Start with an initial guess value $\omega_0 = (\alpha_0, \beta_0) \in (0,1)$.

Step 2: Set i = 1.

Step 3: Generate a candidate point α^* and β^* form the respective proposal distributions $\alpha^* \sim N(\hat{\alpha}, I^{-1}(\hat{\omega}))$ and $\beta^* \sim N(\hat{\beta}, I^{-1}(\hat{\omega}))$ and a point u from $U(0, 1)$. Then

$$\alpha^{(i+1)} = \begin{cases} \alpha^* & \text{with probability } \kappa_1(\alpha^*, \alpha^{(i)}) \text{ if } \kappa_1 \le u \\ \alpha^{(i)} & \text{with probability } 1-\kappa_1(\alpha^*, \alpha^{(i)}) \text{ if } \kappa_1 > u \end{cases}$$

and

$$\beta^{(i+1)} = \begin{cases} \beta^* & \text{with probability } \kappa_2(\beta^*, \beta^{(i)}) \text{ if } \kappa_2 \le u \\ \beta^{(i)} & \text{with probability } 1-\kappa_2(\beta^*, \beta^{(i)}) \text{ if } \kappa_2 > u \end{cases}$$

Where the respective M-H acceptance probabilities are

$$\kappa_1(\alpha^*, \alpha^{(i)}) = \min\left\{ \frac{\Pi_1(\alpha^* | \underline{x}, \beta^{(i)})}{\Pi_1(\alpha^{(i)} | \underline{x}, \beta^{(i)})}, 1 \right\} \text{ and}$$

$$\kappa_2(\beta^*, \beta^{(i)}) = \min\left\{ \frac{\Pi_2(\beta^* | \underline{x}, \alpha^{(i)})}{\Pi_2(\beta^{(i)} | \underline{x}, \alpha^{(i)})}, 1 \right\}$$

Step 4: Set i = i+1

Step 5: Repeating the steps 2-4, N times, where N is a very large number, to obtain the sample observations

$$\omega^{(1)}, \omega^{(2)}, ..., \omega^{(N)} = (\alpha^{(1)}, \beta^{(1)}), (\alpha^{(2)}, \beta^{(2)}), ..., (\alpha^{(N)}, \beta^{(N)})$$

Rapid convergence of the generated sequence is facilitated by choosing appropriate starting values. Influence of the initial value is removed by dropping the first M simulated variates. Then the corresponding selected samples are α_i, β_i, $R_i(t)$ and $h_i(t)$, $i = M+1, \ldots, N$, for sufficiently large N, which represent approximate posterior sample based on SELF and GELF, the Bayes estimates of the unknown parameters α, β, R(t) and h(t) function under SELF are given by,

$$\tilde{\alpha}_{BSMC} = \frac{1}{N-M} \sum_{i=M+1}^{N} \alpha_i ,$$

$$\tilde{\beta}_{BSMC} = \frac{1}{N-M} \sum_{i=M+1}^{N} \beta_i$$

$$\tilde{R}_{BSMC} = \frac{1}{N-M} \sum_{i=M+1}^{N} R_i(t)$$

and

$$\tilde{h}_{BSMC} = \frac{1}{N-M} \sum_{i=M+1}^{N} h_i(t)$$

Also, the approximate Bayes estimates for α, β, $R(t)$ and $h(t)$, under GELF are given by

$$\tilde{\alpha}_{BGMC} = \left(\frac{1}{N-M} \sum_{i=M+1}^{N} \alpha_i^{-q} \right)^{\frac{-q}{2}} ,$$

$$\tilde{\beta}_{BGMC} = \left(\frac{1}{N-M} \sum_{i=M+1}^{N} \beta_i^{-q} \right)^{\frac{-q}{2}} ,$$

$$\tilde{R}_{BGMC} = \left(\frac{1}{N-M} \sum_{i=M+1}^{N} R_i^{-q}(t) \right)^{\frac{-q}{2}}$$

and

$$\tilde{h}_{BGMC} = \left(\frac{1}{N-M} \sum_{i=M+1}^{N} h_i^{-q} \right)^{\frac{-q}{2}}$$

$q > 0$ represents overestimation ($\tilde{\alpha}_{BG_2MC}, \tilde{\beta}_{BG_2MC}, \tilde{R}_{BG_2MC}, \tilde{h}_{BG_2MC}$) and $q < 0$ represents underestimation ($\tilde{\alpha}_{BG_1MC}, \tilde{\beta}_{BG_1MC}, \tilde{R}_{BG_1MC}$, $\tilde{h}_{BG_1MC}$). To study the empirical behaviour of our proposed estimators we take q = 2 and -2 to represent the two situations.

5.3 Bayesian Intervals

Credible intervals and HPD intervals are obtained by following [17] based on samples generated from the full conditionals of the parametric functions.

Bayesian Credible Intervals (BCI)

(i) Order the sample observations generated through M-H algorithm,

$$\alpha_{(1)} \le \alpha_{(2)} \le ... \le \alpha_{(N-M)} \text{ and } \beta_{(1)} \le \beta_{(2)} \le ... \le \beta_{(N-M)}$$

(ii) Subsequently $100(1-\xi)\%$ BCI for α and β are determined as,

$$\left(\alpha_{\left(\frac{\xi}{2}(N-M)\right)}, \alpha_{\left(\frac{1-\xi}{2}(N-M)\right)} \right) \text{ and } \left(\beta_{\left(\frac{\xi}{2}(N-M)\right)}, \beta_{\left(\frac{1-\xi}{2}(N-M)\right)} \right)$$

Highest Posterior Density (HPD) Intervals

Empirical HPD interval estimation for the unknown parametric function is undertaken based on [18] algorithm.

(i) Based on the obtained ordered values, we compute the $100(1-\xi)\%$ credible intervals with their respective lengths such that $d_l^{\alpha} = \alpha_{(l+[(1-\xi)(N-M)])} - \alpha_{(l)}$ and $d_l^{\beta} = \beta_{(l+[(1-\xi)(N-M)])} - \beta_{(l)}$ where, $l = 1, 2, ..., \xi(N-M)$.

(ii) Search for the credible intervals having smallest length, $l^{\alpha} = \min(d_l^{\alpha})$ and $l^{\beta} = \min(d_l^{\beta})$. The smallest length credible interval is the required HPD interval for the unknown parameters α and β.

VI. SIMULATION STUDY

Numerical illustrations of the theoretical contributions are aimed at comparison of proposed Bayes estimates of the unknown parameters α and β. A random sample of size n from PGWD with parameters α = 1.1, β = 2.4 and λ = 5 (known) is generated. Choose m = 10, 20 for n = 20; m = 10, 15, 20, 30 for n = 30 and m = 20, 30, 40, 40 for n = 50. Progressive Type II censored samples are accordingly extracted under these censoring schemes (see, Table 1). The associated MLE*s* are computed using N-R iteration method. For Bayesian study, we consider the arbitrary values of hyper parameters as $\gamma = \sigma = 2$ and $\eta = \mu = 4$. MSE*s* and Bayes estimates of unknown parameters are derived with respect to two different loss functions, namely SELF and GELF using T-K method and MCMC technique under progressive Type II censoring scheme. 5000 replications for progressive Type II censored samples of size m from a sample of size n are drawn. In each case, Bayes estimates with respect to the loss functions SELF and GELF are computed for distinct combinations of *(n,m)*. Finally, the computed estimates obtained under different samples are compared on the basis of their MSE*s*. MLE*s*, Bayes estimates and their respective MSE*s* for the unknown parameters α, β, reliability function $R(t)_{t\,=\,7}$ and hazard function $h(t)_{t\,=\,5}$ are presented for various *(n,m)* censoring schemes in Tables 2-5. MLE*s* compete quite well with the corresponding Bayes estimates. However, GELF1 estimate under MCMC shows superior performance compared with the rest of the six estimates. Broadly, MCMC based estimates have least MSE*s* and MLE*s* have highest. With the increase in effective sample sizes m, the MSE*s* of all the proposed estimates tend to decrease.

Table 1: Progressive Censoring Schemes (CS) used in the simulation study

n	*m*	Censoring Scheme	CS
20	10	10, 0, 0, 0, 0, 0, 0, 0, 0, 0	CS[1]
		2, 2, 2, 2, 2, 0, 0, 0, 0, 0	CS[2]
		0, 0, 0, 0, 0, 2, 2, 2, 2, 2	CS[3]
		0, 0, 0, 0, 0, 0, 0, 0, 0, 10	CS[4]
	20	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	CS[5]
30	20	10, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	CS[6]
		2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	CS[7]
		0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 2	CS[8]
		0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 , 10	CS[9]
	25	5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	CS[10]
		0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3	CS[11]
		1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2	CS[12]
		0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5	CS[13]
	30	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	CS[14]
50	30	20, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	CS[15]
		4, 4, 4, 4, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	CS[16]

		0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 4, 4, 4, 4	CS[17]
		0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 20	CS[18]
	40	10,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0	CS[19]
		2,2,2,2,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0	CS[20]
		0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,2,2,2,2	CS[21]
		0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,10	CS[22]
	50	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	CS[23]

Table 2: MLE*s* and Bayes Estimates of Parameter α and their MSE*s* (in brackets) for various (n, m)

CS	$\hat{\alpha}_{ML}$	$\tilde{\alpha}_{BST}$	$\tilde{\alpha}_{BG_1T}$	$\tilde{\alpha}_{BG_2T}$	$\tilde{\alpha}_{BSMC}$	$\tilde{\alpha}_{BG_1MC}$	$\tilde{\alpha}_{BG_2MC}$
CS[1]	1.22143	1.25547	1.21862	1.27213	1.12713	1.32614	1.49576
	(0.05916)	(0.05246)	(0.05024)	(0.05327)	(0.04823)	(0.03945)	(0.05174)
CS[2]	1.28371	1.30248	1.24369	1.33258	1.13276	1.32485	1.61769
	(0.05128)	(0.05082)	(0.04625)	(0.05143)	(0.04425)	(0.03533)	(0.04721)
CS[3]	1.35746	1.36791	1.28451	1.37257	1.16247	1.50361	1.68653
	(0.04626)	(0.04716)	(0.04201)	(0.04825)	(0.04136)	(0.03082)	(0.04243)
CS[4]	1.37961	1.40257	1.32415	1.43250	1.19254	1.09735	1.32587
	(0.03725)	(0.04128)	(0.03624)	(0.04252)	(0.03625)	(0.02764)	(0.03976)
CS[5]	1.27136	1.29824	1.24273	1.36964	1.20147	1.17548	1.23461
	(0.05243)	(0.05106)	(0.04128)	(0.05296)	(0.03647)	(0.03102)	(0.04127)
CS[6]	1.21367	1.23746	1.20851	1.25291	1.19364	1.17246	1.23741
	(0.04216)	(0.04052)	(0.03746)	(0.04286)	(0.03421)	(0.03175)	(0.03652)
CS[7]	1.23586	1.24285	1.21846	1.26423	1.21073	1.19642	1.26789
	(0.04052)	(0.03728)	(0.03425)	(0.04013)	(0.03375)	(0.03085)	(0.03512)
CS[8]	1.25761	1.26761	1.23871	1.28487	1.22413	1.21536	1.27134
	(0.03826)	(0.03419)	(0.03378)	(0.03821)	(0.03216)	(0.02814)	(0.03301)
CS[9]	1.26375	1.29642	1.27645	1.31072	1.23854	1.22743	1.29138
	(0.03246)	(0.03125)	(0.03052)	(0.03286)	(0.03017)	(0.02712)	(0.03176)
CS[10]	1.19634	1.22463	1.21423	1.24698	1.19864	1.17246	1.21243
	(0.03463)	(0.03404)	(0.03241)	(0.03514)	(0.02974)	(0.02736)	(0.03105)
CS[11]	1.21863	1.24137	1.22761	1.25841	1.20143	1.19365	1.22763
	(0.03146)	(0.03276)	(0.03179)	(0.03475)	(0.02758)	(0.02674)	(0.03015)
CS[12]	1.24632	1.26841	1.24694	1.26176	1.22861	1.21761	1.23148
	(0.02963)	(0.31056)	(0.03074)	(0.03274)	(0.02586)	(0.02587)	(0.02874)
CS[13]	1.24876	1.27536	1.26854	1.28964	1.23854	1.23894	1.24856
	(0.02834)	(0.02986)	(0.02736)	(0.03181)	(0.02476)	(0.02215)	(0.02601)
CS[14]	1.26524	1.28068	1.26524	1.28745	1.23761	1.21842	1.25697
	(0.02965)	(0.03127)	(0.02863)	(0.03385)	(0.02496)	(0.02376)	(0.02681)
CS[15]	1.11635	1.12471	1.11094	1.13257	1.11065	1.10581	1.12476
	(0.02158)	(0.02101)	(0.02085)	(0.02274)	(0.01958)	(0.01654)	(0.02147)
CS[16]	1.13854	1.12786	1.11875	1.13846	1.11847	1.11257	1.12814

CS	$\hat{\alpha}_{ML}$	$\tilde{\alpha}_{BST}$	$\tilde{\alpha}_{BG_1T}$	$\tilde{\alpha}_{BG_2T}$	$\tilde{\alpha}_{BSMC}$	$\tilde{\alpha}_{BG_1MC}$	$\tilde{\alpha}_{BG_2MC}$
	(0.02114)	(0.02019)	(0.01847)	(0.02104)	(0.01874)	(0.01524)	(0.02016)
CS[17]	1.14237	1.21854	1.12537	1.14257	1.12436	1.11876	1.13175
	(0.01947)	(0.01836)	(0.01725)	(0.01985)	(0.01758)	(0.01447)	(0.01958)
CS[18]	1.14856	1.22746	1.12630	1.14847	1.12846	1.12573	1.13764
	(0.01763)	(0.01796)	(0.01689)	(0.01826)	(0.01625)	(0.01412)	(0.01824)
CS[19]	1.10690	1.12476	1.11583	1.13864	1.10584	1.09654	1.12694
	(0.01826)	(0.01813)	(0.01425)	(0.02046)	(0.01246)	(0.01074)	(0.01357)
CS[20]	1.11846	1.13846	1.12317	1.14174	1.11584	1.10569	1.12824
	(0.01756)	(0.01763)	(0.01524)	(0.01952)	(0.01204)	(0.00936)	(0.01304)
CS[21]	1.12134	1.14261	1.12576	1.14258	1.11179	1.11873	1.12957
	(0.01613)	(0.01546)	(0.01425)	(0.01840)	(0.01116)	(0.00913)	(0.01256)
CS[22]	1.12384	1.46810	1.12879	1.14759	1.12753	1.12863	1.13785
	(0.01659)	(0.01501)	(0.01464)	(0.01725)	(0.01028)	(0.00876)	(0.01221)
CS[23]	1.12463	1.14865	1.12854	1.15169	1.11463	1.01054	1.13470
	(0.01763)	(0.01675)	(0.01463)	(0.01652)	(0.01263)	(0.01185)	(0.01496)

Table 3: MLE*s* and Bayes Estimates of Parameter β and their MSE*s* (in brackets) for various (n, m)

CS	$\hat{\beta}_{ML}$	$\tilde{\beta}_{BST}$	$\tilde{\beta}_{BG_1T}$	$\tilde{\beta}_{BG_2T}$	$\tilde{\beta}_{BSMC}$	$\tilde{\beta}_{BG_1MC}$	$\tilde{\beta}_{BG_2MC}$
CS[1]	2.11934	2.68053	2.38235	2.79582	2.27841	1.79203	2.47246
	(0.07877)	(0.07232)	(0.06963)	(0.07638)	(0.07024)	(0.06861)	(0.07684)
CS[2]	1.66085	1.78284	1.68368	1.82671	1.63468	1.57259	1.86729
	(0.08635)	(0.08071)	(0.07692)	(0.08465)	(0.07526)	(0.07225)	(0.08244)
CS[3]	1.52807	1.64672	1.51934	1.76913	1.46927	1.42337	1.54732
	(0.09547)	(0.09157)	(0.08441)	(0.09535)	(0.08612)	(0.081295)	(0.08863)
CS[4]	1.34335	1.43507	1.37648	1.62584	1.32667	1.27691	1.59058
	(0.11661)	(0.09724)	(0.09249)	(0.10269)	(0.09237)	(0.08653)	(0.09634)
CS[5]	2.10563	2.13674	1.68153	2.62471	1.76584	1.4836	2.37584
	(0.06636)	(0.06325)	(0.05273)	(0.06885)	(0.05674)	(0.04726)	(0.06386)
CS[6]	1.58976	1.63258	1.54327	1.67641	1.58245	1.47263	1.63872
	(0.05648)	(0.05376)	(0.04786)	(0.05485)	(0.05195)	(0.04637)	(0.05426)
CS[7]	1.38238	1.45763	1.374863	1.55746	1.38254	1.33864	1.48925
	(0.06155)	(0.05726)	(0.05211)	(0.05964)	(0.05571)	(0.05186)	(0.05752)
CS[8]	1.30774	1.36852	1.31746	1.43058	1.25631	1.23514	1.33264
	(0.06402)	(0.06125)	(0.05763)	(0.06274)	(0.05863)	(0.05374)	(0.05876)
CS[9]	1.19187	1.24662	1.17965	1.27483	1.23854	1.16523	1.27464
	(0.06959)	(0.06376)	(0.05963)	(0.06672)	(0.06157)	(0.05548)	(0.06557)
CS[10]	1.47383	1.51328	1.48362	1.56254	1.46852	1.42637	1.53271
	(0.06779)	(0.06354)	(0.05638)	(0.06638)	(0.05523)	(0.05278)	(0.05849)
CS11]	1.42562	1.46674	1.43265	1.52137	1.41639	1.37654	1.48365
	(0.06942)	(0.06614)	(0.05854)	(0.06824)	(0.05857)	(0.05625)	(0.06217)
CS[12]	1.39675	1.40527	1.34506	1.44583	1.38504	1.32476	1.40587
	(0.07334)	(0.06942)	(0.06352)	(0.07168)	(0.06425)	(0.06118)	(0.06785)
CS[13]	1.35258	1.37135	1.29562	1.42637	1.34675	1.26849	1.37692
	(0.07732)	(0.07421)	(0.06786)	(0.07652)	(0.06962)	(0.06569)	(0.07153)
S[14]	1.34332	1.38157	1.48367	1.59258	1.38351	1.48236	1.59158
	(0.05723)	(0.04252)	(0.04076)	(0.05381)	(0.04252)	(0.04276)	(0.05281)

CS[15]	1.34336	1.39654	1.32856	1.47364	1.37078	1.28631	1.44257
	(0.04660)	(0.04352)	(0.04063)	(0.04581)	(0.04126)	(0.03765)	(0.04695)
CS[16]	1.12456	1.23254	1.19374	1.31594	1.20638	1.16825	1.26769
	(0.05176)	(0.04624)	(0.04317)	(0.04776)	(0.04652)	(0.04214)	(0.04725)
CS[17]	1.05199	1.14342	1.12451	1.15694	1.13589	1.10846	1.16385
	(0.05713)	(0.05248)	(0.04682)	(0.05520)	(0.04812)	(0.04563)	(0.05162)
CS[18]	1.24438	1.27635	1.23492	1.32869	1.24654	1.21367	1.33587
	(0.05905)	(0.05561)	(0.05168)	(0.05832)	(0.05321)	(0.04752)	(0.05695)
CS[19]	1.19187	1.24716	1.18364	1.31857	1.21624	1.14829	1.23761
	(0.04959)	(0.04321)	(0.04158)	(0.04732)	(0.04527)	(0.04185)	(0.05119)
CS[20]	1.12456	1.17254	1.13612	1.23673	1.13251	1.12574	1.20458
	(0.05574)	(0.04755)	(0.04472)	(0.05325)	(0.04876)	(0.04691)	(0.05475)
CS[21]	1.14554	1.20861	1.16583	1.27812	1.18653	1.15792	1.21691
	(0.05876)	(0.05456)	(0.04773)	(0.05753)	(0.05124)	(0.04967)	(0.05723)
CS[22]	1.08622	1.12935	1.10692	1.24236	1.12276	1.10364	1.15743
	(0.06314)	(0.05873)	(0.05472)	(0.06192)	(0.05381)	(0.05036)	(0.05868)
CS[23]	1.08627	1.12671	1.10546	1.17365	1.1224	1.09584	1.25761
	(0.05589)	(0.04725)	(0.04365)	(0.05172)	(0.04436)	(0.04063)	(0.04764)

Table 4: MLE*s* and Bayes Estimates of Reliability Function $R(t)_{t=7}$ and their MSE*s* (in brackets) for various (n, m)

CS	$\hat{R}_{ML}$	$\tilde{R}_{BST}$	$\tilde{R}_{BG_1T}$	$\tilde{R}_{BG_2T}$	$\tilde{R}_{BSMC}$	$\tilde{R}_{BG_1MC}$	$\tilde{R}_{BG_2MC}$
CS[1]	0.42134	0.42768	0.41036	0.43258	0.38562	0.36574	0.39854
	(0.03257)	(0.03148)	(0.02985)	(0.03252)	(0.02746)	(0.02514)	(0.02963)
CS[2]	0.42657	0.43156	0.42160	0.43764	0.39584	0.37153	0.41237
	(0.03124)	(0.03014)	(0.02814)	(0.03146)	(0.02642)	(0.02476)	(0.02685)
CS[3]	0.43257	0.44581	0.43257	0.44175	0.40127	0.38416	0.42165
	(0.02985)	(0.02814)	(0.02716)	(0.02958)	(0.02425)	(0.02214)	(0.02349)
CS[4]	0.44713	0.45213	0.43856	0.44823	0.42138	0.39158	0.43571
	(0.02736)	(0.02416)	(0.02358)	(0.02714)	(0.02310)	(0.02107)	(0.02214)
CS[5]	0.43257	0.44861	0.43571	0.46275	0.41573	0.40564	0.43287
	(0.02763)	(0.02647)	(0.02413)	(0.02786)	(0.02137)	(0.01986)	(0.02198)
CS[6]	0.41238	0.42865	0.40136	0.43524	0.39584	0.37196	0.40197
	(0.02854)	(0.02745)	(0.02574)	(0.02846)	(0.02316)	(0.02214)	(0.02416)
CS[7]	0.41758	0.43168	0.41263	0.44518	0.40698	0.38165	0.41856
	(0.02759)	(0.02673)	(0.02496)	(0.02794)	(0.02247)	(0.02143)	(0.02371)
CS[8]	0.42637	0.43856	0.42168	0.44879	0.41693	0.38675	0.42749
	(0.02685)	(0.02496)	(0.02374)	(0.02649)	(0.02108)	(0.02101)	(0.02269)
CS[9]	0.43254	0.45138	0.42859	0.45563	0.42691	0.39572	0.43854
	(0.02476)	(0.02384)	(0.02168)	(0.02574)	(0.01905)	(0.01957)	(0.02109)
CS[10]	0.41846	0.43527	0.41095	0.44257	0.40967	0.38467	0.41632
	(0.02925)	(0.02714)	(0.02574)	(0.02852)	(0.02385)	(0.02143)	(0.02486)
CS[11]	0.42179	0.44861	0.41685	0.45248	0.41257	0.39854	0.42163
	(0.02814)	(0.02692)	(0.02463)	(0.02714)	(0.02109)	(0.01985)	(0.02376)
CS[12]	0.42864	0.45138	0.42861	0.44263	0.42864	0.40528	0.42286
	(0.02746)	(0.02486)	(0.02384)	(0.02574)	(0.02285)	(0.01865)	(0.02106)
CS[13]	0.43254	0.45894	0.43158	0.42965	0.43852	0.42857	0.42746

	(0.02549)	(0.02372)	(0.02269)	(0.02413)	(0.02164)	(0.01764)	(0.02085)
CS[14]	0.42842 (0.02769)	0.43291 (0.02651)	0.41697 (0.02413)	0.44216 (0.02714)	0.40163 (0.02374)	0.38463 (0.02174)	0.42136 (0.02485)
CS[15]	0.38473 (0.02184)	0.41583 (0.01985)	0.40254 (0.01724)	0.43864 (0.02196)	0.39854 (0.01658)	0.37416 (0.01413)	0.41857 (0.01825)
CS[16]	0.39571 (0.02015)	0.42563 (0.01826)	0.41132 (0.01695)	0.44213 (0.02045)	0.40864 (0.01523)	0.38435 (0.01317)	0.43125 (0.01726)
CS[17]	0.40367 (0.01846)	0.43852 (0.01769)	0.41867 (0.01618)	0.44969 (0.01958)	0.41856 (0.01426)	0.41096 (0.01395)	0.43852 (0.01625)
CS[18]	0.42138 (0.01763)	0.44864 (0.01689)	0.42857 (0.01526)	0.45125 (0.01846)	0.42857 (0.01475)	0.43842 (0.01284)	0.44246 (0.01592)
CS[19]	0.39746 (0.02219)	0.41238 (0.02075)	0.36254 (0.01856)	0.43257 (0.02274)	0.35864 (0.01625)	0.33746 (0.01426)	0.38254 (0.01863)
CS[20]	0.41238 (0.02136)	0.43854 (0.01863)	0.38461 (0.01746)	0.44163 (0.02141)	0.36842 (0.01541)	0.34851 (0.01316)	0.39584 (0.01726)
CS[21]	0.42864 (0.02015)	0.45231 (0.01765)	0.39254 (0.01674)	0.44673 (0.02014)	0.38263 (0.01598)	0.35864 (0.01256)	0.41263 (0.01523)
CS[22]	0.43068 (0.01987)	0.45769 (0.01703)	0.41357 (0.01536)	0.46258 (0.01856)	0.41368 (0.01466)	0.37851 (0.01196)	0.42854 (0.01486)
CS[23]	0.42863 (0.02126)	0.43257 (0.02114)	0.41267 (0.02019)	0.44257 (0.02286)	0.41213 (0.01981)	0.38462 (0.01874)	0.42137 (0.02087)

Table 5: MLE*s* and Bayes Estimates of Hazard rate Function $h(t)_{t=5}$ and their MSE*s* (in brackets) for various (*n*, *m*)

CS	$\hat{h}_{ML}$	$\tilde{h}_{BST}$	$\tilde{h}_{BG_1T}$	$\tilde{h}_{BG_2T}$	$\tilde{h}_{BSMC}$	$\tilde{h}_{BG_1MC}$	$\tilde{h}_{BG_2MC}$
CS[1]	0.26432 (0.00989)	0.31357 (0.00862)	0.27363 (0.00716)	0.38648 (0.01052)	0.28514 (0.00732)	0.24675 (0.00628)	0.34527 (0.00814)
CS[2]	0.17108 (0.01266)	0.23673 (0.01025)	0.21563 (0.00842)	0.26574 (0.01264)	0.25675 (0.00868)	0.19356 (0.00746)	0.29486 (0.00953)
CS[3]	0.13966 (0.01457)	0.18546 (0.01227)	0.17472 (0.00972)	0.24752 (0.01331)	0.18576 (0.00934)	0.16782 (0.00871)	0.26386 (0.01032)
CS[4]	0.12545 (0.01764)	0.21778 (0.01322)	0.21324 (0.01016)	0.25586 (0.01563)	0.15337 (0.01064)	0.1356 (0.00946)	0.22458 (0.01243)
CS[5]	0.28779 (0.00763)	0.32531 (0.00645)	0.27654 (0.00606)	0.35763 (0.00735)	0.28457 (0.00627)	0.23536 (0.00573)	0.31764 (0.00694)
CS[6]	0.23863 (0.01165)	0.28736 (0.00943)	0.24826 (0.00823)	0.33657 (0.01074)	0.26657 (0.00854)	0.22636 (0.00676)	0.27383 (0.00975)
CS[7]	0.17959 (0.01325)	0.19357 (0.01157)	0.17567 (0.00956)	0.23568 (0.01287)	0.18564 (0.00924)	0.14347 (0.00835)	0.21471 (0.01257)
CS[8]	0.15619 (0.01643)	0.16334 (0.01462)	0.14348 (0.01175)	0.19354 (0.01635)	0.13642 (0.01134)	0.12452 (0.00986)	0.16258 (0.01376)
CS[9]	0.11595 (0.01937)	0.12867 (0.01648)	0.11467 (0.01376)	0.15538 (0.01821)	0.11368 (0.01486)	0.09328 (0.01147)	0.14748 (0.01568)

CS[10]	0.20681 (0.01245)	0.25753 (0.01076)	0.22718 (0.00895)	0.32358 (0.01183)	0.23743 (0.00972)	0.19245 (0.00893)	0.26574 (0.01047)
CS[11]	0.19253 (0.01436)	0.23643 (0.01252)	0.18366 (0.01027)	0.27563 (0.01357)	0.21367 (0.01168)	0.15937 (0.00926)	0.23258 (0.01249)
CS[12]	0.17964 (0.01725)	0.18537 (0.01534)	0.15749 (0.01345)	0.24376 (0.01667)	0.16543 (0.01341)	0.13644 (0.01173)	0.21547 (0.01463)
CS[13]	0.16859 (0.01904)	0.17684 (0.01678)	0.12856 (0.01470)	0.19534 (0.01843)	0.14163 (0.01513)	0.12746 (0.01367)	0.16921 (0.01768)
CS[14]	0.16753 (0.01467)	0.18645 (0.01257)	0.12836 (0.01064)	0.24548 (0.01343)	0.17463 (0.01146)	0.10645 (0.00976)	0.21768 (0.01268)
CS[15]	0.23868 (0.01247)	0.26943 (0.01156)	0.22672 (0.00926)	0.28735 (0.01286)	0.22415 (0.01076)	0.18657 (0.00874)	0.24758 (0.01175)
CS[16]	0.20659 (0.01452)	0.24675 (0.01273)	0.19575 (0.01046)	0.25968 (0.01369)	0.18734 (0.01164)	0.16494 (0.00977)	0.22849 (0.01293)
CS[17]	0.17327 (0.01647)	0.21834 (0.01418)	0.15941 (0.01264)	0.23764 (0.01484)	0.16943 (0.01332)	0.13825 (0.01167)	0.18294 (0.01387)
CS[18]	0.14249 (0.01769)	0.19047 (0.01579)	0.12764 (0.01483)	0.20342 (0.01631)	0.13941 (0.01463)	0.11532 (0.01391)	0.17596 (0.01485)
CS[19]	0.20774 (0.01185)	0.25861 (0.00954)	0.22652 (0.00793)	0.28632 (0.01084)	0.21892 (0.00863)	0.17684 (0.00776)	0.23582 (0.00985)
CS[20]	0.18974 (0.01365)	0.22435 (0.01025)	0.20365 (0.00827)	0.24745 (0.01236)	0.19675 (0.00964)	0.14483 (0.00885)	0.21368 (0.01147)
CS[21]	0.17867 (0.01478)	0.18674 (0.01291)	0.17541 (0.01149)	0.21547 (0.01381)	0.15264 (0.01205)	0.12879 (0.01176)	0.18726 (0.01489)
CS[22]	0.15305 (0.01573)	0.15879 (0.01387)	0.13943 (0.01276)	0.18964 (0.01478)	0.13761 (0.01316)	0.11975 (0.01234)	0.16992 (0.01407)
CS[23]	0.18929 (0.01468)	0.23346 (0.01207)	0.19745 (0.01096)	0.27523 (0.01357)	0.20479 (0.01153)	0.14917 (0.00964)	0.24751 (0.01253)

VII. REAL DATA STUDY

The following data from [19] report repair times (*in hours*) for 46 failures of an airborne communications receiver.

Table 6: Real data on repair times (*in hours*) for 46 failures

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7
0.7	0.8	0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5
1.5	1.5	1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0
3.3	3.3	4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0
7.5	8.8	9.0	10.3	22.0	24.5				

Fig. 4 shows an empirical fit to PGWD. Two alternative popular lifetime models, namely Weibull and gamma are also fitted to the selected data sets.

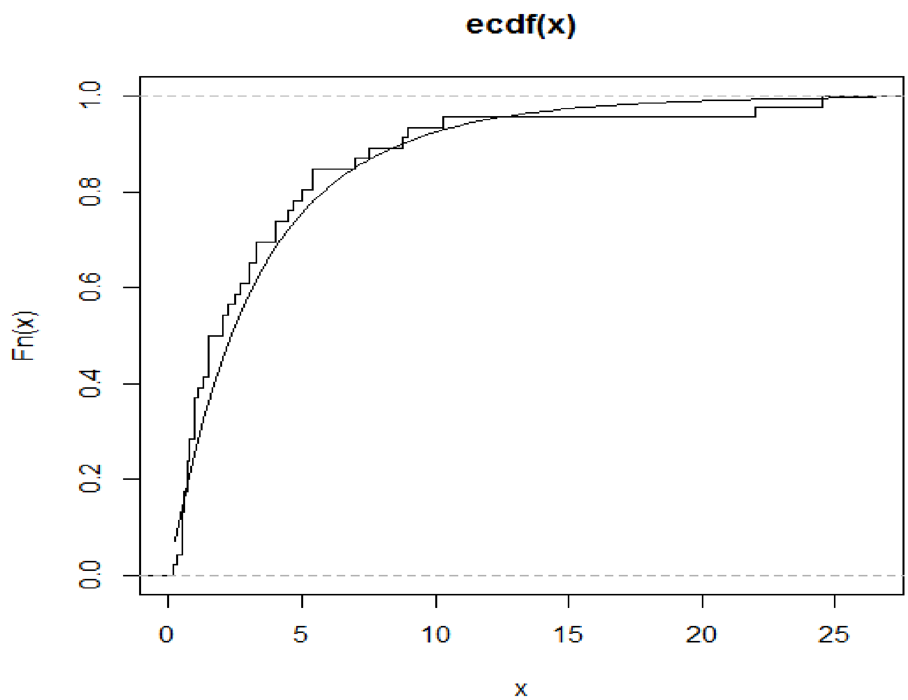

Fig. 4: Fitted Distribution Functions for Real data

Extracted Type progressive II censored samples extracted from real data of Table 6 are presented in Table 7. MLE*s* and Bayes estimates of unknown parameters *α*, *β*, reliability and hazard rate functions with two loss functions (SELF and GELF) under T-K and MCMC methods are given in the Tables 8-11. The ACI, Boot-*p*, Boot-*t* CI*s*, BCI and HPD for the parametric estimates are presented in the Tables 12 and 13. GELF under MCMC is found to be the best estimate in terms of minimum MSE*s*, for all the parametric functions. It is observed that Boot-*p* intervals are mostly shorter as compared to Boot-*t* intervals for all the four parametric functions. The ACI *s* compete quite well with Boot-*t* intervals. However, these two interval estimates are incomparable across all censored samples in a similar way. As for some instances ACI*s* have shorter average length and in some cases, the opposite is true. HPD intervals are superior to all other interval estimates as they exhibit shortest average length. The average length of all confidence intervals tends to decrease with the increase in effective sample sizes *m*

Table 7: Progressive Censoring Schemes (C-S) for Real Data

n	*m*	Censoring Scheme	C-S
46	30	2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 0, 0, 0, 0, 0	C-S[1]
	36	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0	C-S[2]
	40	2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	C-S[3]
	46	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,	C-S[4]

Table 8: Estimated Values of Parameter α for different Progressive Type II Censoring Scheme for Real data

CS	MLE*s*	T-K method			MCMC technique		
		SELF	GELF		SELF	GELF	
			GELF1	GELF2		GELF1	GELF2
C-S[1]	2.8547	2.8124	2.4512	2.8645	2.8227	2.4654	2.8245
C-S[2]	2.0353	2.0125	1.8954	2.1214	2.0225	1.8854	2.1451
C-S[3]	1.8584	1.8458	1.8102	1.9754	1.8624	1.8265	2.1210
C-S[4]	1.4993	1.4954	1.4521	1.5214	1.4921	1.4754	1.5621

*GELF1- underestimation and GELF2- overestimation

Table 9: Estimated Values of Parameter β for different Progressive Type II Censoring Scheme for Real data

CS	MLE*s*	T-K method			MCMC technique		
		SELF	GELF		SELF	GELF	
			GELF1	GELF2		GELF1	GELF2
C-S[1]	1.6761	1.6645	1.5954	1.6954	1.6595	1.5854	1.6901
C-S[2]	1.3742	1.3721	1.3452	1.4212	1.3820	1.3485	1.4385
C-S[3]	1.2329	1.2254	1.1954	1.2513	1.2301	1.2041	1.2425
C-S[4]	1.0918	1.0911	1.0754	1.1125	1.1025	1.0821	1.1114

*GELF1- underestimation and GELF2- overestimation

Table 10: Estimated Values of Reliability Function $R(t)$ for different Progressive Type II Censoring Scheme for Real data

CS	MLE*s*	T-K method			MCMC technique		
		SELF	GELF		SELF	GELF	
			GELF1	GELF2		GELF1	GELF2
C-S[1]	0.4345	0.4340	0.4012	0.4521	0.4315	0.4125	0.4496
C-S[2]	0.3332	0.3312	0.3102	0.3485	0.3294	0.3162	0.3456
C-S[3]	0.3213	0.3200	0.3025	0.3314	0.3301	0.3056	0.3312
C-S[4]	0.2462	0.2412	0.2354	0.2612	0.2545	0.2317	0.2598

*GELF1- underestimation and GELF2- overestimation

Table 11: Estimated Values of Hazard rate Function $h(t)$ for different Progressive Type II Censoring Scheme for Real data

CS	MLE*s*	T-K method			MCMC technique		
		SELF	GELF		SELF	GELF	
			GELF1	GELF2		GELF1	GELF2
C-S[1]	0.1772	0.1759	0.1412	0.1954	0.1759	0.1454	0.2088
C-S[2]	0.2208	0.2189	0.2014	0.2419	0.2189	0.2045	0.2432
C-S[3]	0.2141	0.2230	0.2031	0.2436	0.2230	0.1987	0.2387
C-S[4]	0.2557	0.2542	0.2345	0.2781	0.2542	0.2401	0.2710

*GELF1- underestimation and GELF2- overestimation

Table 12: Confidence Intervals for Parameter α for Real data

CS	ACI	Boot-*p*	Boot-*t*	BCI	HPD
C-S[1]	(1.8485, 3.8609)	(1.8945, 3.5422)	(1.8012, 4.0126)	(2.165, 3.3421)	(2.254, 3.3025)
C-S[2]	(1.3951, 2.6755)	(1.4215, 2.452)	(1.3821, 2.9085)	(1.5421, 2.3122)	(1.8242, 2.2125)
C-S[3]	(1.3053, 2.4114)	(1.3215, 2.2651)	(1.2941, 2.5412)	(1.2205, 1.4325)	(1.2154, 1.5521)
C-S[4]	(1.0949, 1.9036)	(1.1295, 1.7854)	(1.1121, 1.7912)	(1.2402, 1.6942)	(1.3002, 1.6442)

Table 13: Confidence Intervals for Parameter β for Real data

CS	ACI	Boot-*p*	Boot-*t*	BCI	HPD
C-S[1]	(1.1198, 2.2325)	(1.2001, 2.0252)	(1.1021, 2.2426)	(1.3121, 1.9854)	(1.4212, 1.8540)
C-S[2]	(0.9429, 1.8055)	(0.9625, 1.7855)	(0.9541, 1.8124)	(0.9985, 1.6592)	(1.0125, 1.6503)
C-S[3]	(0.8580, 1.6078)	(0.8812, 1.5954)	(0.9021, 1.6950)	(0.9521, 1.5251)	(0.9851, 1.4850)
C-S[4]	(0.7784, 1.4051)	(0.7541, 1.354)	(0.7654, 1.4211)	(0.8521, 1.2540)	(0.9125, 1.2425)

VIII. CONCLUDING REMARKS

In this paper, classical and Bayesian estimators of the unknown parameters, reliability and hazard functions of PGWD assuming progressive Type II censoring scheme are studied. It is observed from Tables 2-5 that Bayesian procedure provides more precise parametric estimates as measured by MSE*s* in comparison to the classical estimators. In the Bayes case, the estimators are obtained under SELF and GELF assuming two independent gamma priors by using T-K and MCMC approaches. Theoretical results are illustrated for a simulated data set. Real data based study reinforces the simulation study findings. Proposed estimates are then compared numerically and it is observed that the Bayes estimates under MCMC approach provide highest precision results which are followed by T-K based approximations. It is seen that as the effective sample sizes increase, MSE*s* of estimates based on progressive Type II censored data decrease for all the considered censoring scenarios. Thus, improved estimates are obtained when sample size is appreciable. The discussed methodology provides an alternative PGWD model which exhibits flexible hazard rates for data analysis under life testing situations and can be recommended for use in medical, engineering and areas beyond where life-testing experiments are regularly conducted under time and cost constraints.

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