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A New Generalization of One Parameter Quasi Exponential Distribution: Properties and Applications

Anwar Hassan[1*], Nazima Akhter[2] and Bilal Ahmad Para[3]

[1,2]Department of Statistics, University of Kashmir, Srinagar, J&K, India
[3]Department of Statistics, Government Degree College Boys Anantnag, J&K, India

Corresponding Author: anwar.hassan2007@gmail.com

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Abstract- In this article, we have introduced a new generalization of one parameter quasi exponential distribution using weighting technique. Important mathematical and statistical properties of the distribution have been derived and discussed. The expression for stochastic ordering, reliability measures and order statistics has been obtained. Maximum likelihood method of estimation along with Monte Carlo simulation has been discussed. Finally, real data set is analyzed to investigate the suitability of the proposed distribution in modeling lifetime data.

Keywords: *Quasi Exponential Distribution, Moments, Stochastic Ordering, Order Statistics, MLE, Lifetime Data*

I. INTRODUCTION

The concept of weighted probability models attracted a lot of researchers to contemplate on and to carry out research on this topic. This concept of weighted models can be traced from the work of Fisher (1934), in connection with his studies, on how methods of ascertainment can influence the form of distribution of recorded observations. Patil and Rao (1978) studied weighted distributions and size biased sampling with applications to wildlife populations and human families. Van Deusen (1986) arrived at size biased distribution theory independently and applied it in fitting assumed distributions to data arising from horizontal point sampling. In fisheries, Taillie, Patil and Hennemuth (1995) modelled populations of fish stocks using weights. In ecology, Dennis and Patil (1984) used stochastic differential equations to arrive at weighted properties of size-biased gamma distribution. Warren (1975) was the first to apply the size biased distributions in connection with sampling wood cells. Gove (2003) reviewed some of the more recent results on size-biased distributions pertaining to parameter estimation in forestry. For survival data analysis, Jing (2010) introduced the weighted inverse Weibull distribution and beta-inverse Weibull distribution as a new lifetime models. Ghitany, Alqallaf, Al-Mutairi and Husain (2011) introduced a two-parameter weighted Lindley distribution with applications to analyse survival data. Ajami and Jahanshahi (2017) introduced weighted rayleigh Distribution as a new generalization of rayleigh distribution and discussed its parameter estimation in broad. Para and Jan (2018) introduced the Weighted Pareto type II Distribution as a new model for handling medical science data and studied its statistical properties and applications. Recently Hassan et al. (2018) introduced two weighted probability models with applications in handling various lifetime data from different applied fields.

Now we proceed with a new generalization of one parameter quasi exponential distribution as a new lifetime distribution using the weighting technique, as there is a need to find more plausible model for fitting the lifetime data from various applied fields.

II. DERIVATION OF WEIGHTED QUASI EXPONENTIAL DISTRIBUTION (WQED)

Let us suppose that X be a continuous random variable of interest such that $X \sim f(x;\theta)$. However, if the sample observations are selected with probability proportional to weighted function $w(x) = x^c$, where $c > 0$ is the weight parameter, then the distribution, with pdf given by:

$$f(x;\theta,c) = \frac{w(x)f(x;\theta)}{E[w(x)]} \tag{1}$$

is called the Weighted Distribution of random variable X.

The probability density function of Quasi Exponential Distribution having rate parameter θ is given by:

$$f(x;\theta) = \frac{2\theta^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} e^{-\theta x^2} \qquad ; x > 0, \theta > 0 \tag{2}$$

Weight function: The weight function is considered $w(x) = x^c$, where $c > 0$ is the weight parameter.
Therefore,

$$E[w(x)] = \frac{\Gamma\left(\frac{c+1}{2}\right)}{\theta^{\frac{c}{2}}\Gamma\left(\frac{1}{2}\right)} \tag{3}$$

Now from the definition (1) we will have the pdf of Weighted Quasi Exponential Distribution as given by (4)

$$f_w(x;c,\theta) = \frac{x^c f(x,\theta)}{E(x^c)},$$

$$f_w(x;c,\theta) = \frac{2x^c\theta^{\left(\frac{c+1}{2}\right)}e^{-\theta x^2}}{\Gamma\left(\frac{c+1}{2}\right)} \quad ; \qquad x > 0, c > 0, \theta > 0 \tag{4}$$

$$where, \qquad E(x^c) = \frac{\Gamma\left(\frac{c+1}{2}\right)}{\theta^{\frac{c}{2}}\Gamma\left(\frac{1}{2}\right)}$$

The graphical overview of pdf of WQED is given in fig.2. Weighted parameter c has direct impact on shape of the distribution. As the c increases, skewness goes on decreasing.

The corresponding cdf of weighted Quasi Exponential Distribution (WQED) is obtained as

$$F_w(x;c,\alpha,\theta) = \int_0^x f_w(x;c,\theta)dx$$

$$= \int_0^x \frac{x^c\theta^{\left(\frac{c+1}{2}\right)}e^{-\theta x^2}}{\Gamma\left(\frac{c+1}{2}\right)}\, dx$$

$$Put \quad \theta x^2 = t, \quad dx = \frac{dt}{2\theta x}$$

$As\, x \to 0, t \to 0 \;\; and \;\; x \to x, t \to \theta x^2$,

After simplification,

$$F_w(x;c,\theta) = \frac{1}{\Gamma\left(\frac{c+1}{2}\right)} \gamma\left(\frac{c+1}{2}, \theta x^2\right) \quad ; \quad x > 0, c > 0, \theta > 0, \tag{5}$$

where θ and c are positive parameters and $\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$ is a lower incomplete gamma function.

The graphical overview of cdf of WQED is given in fig.2.

Fig.1: pdf plot of Weighted Quasi Exponential Distribution

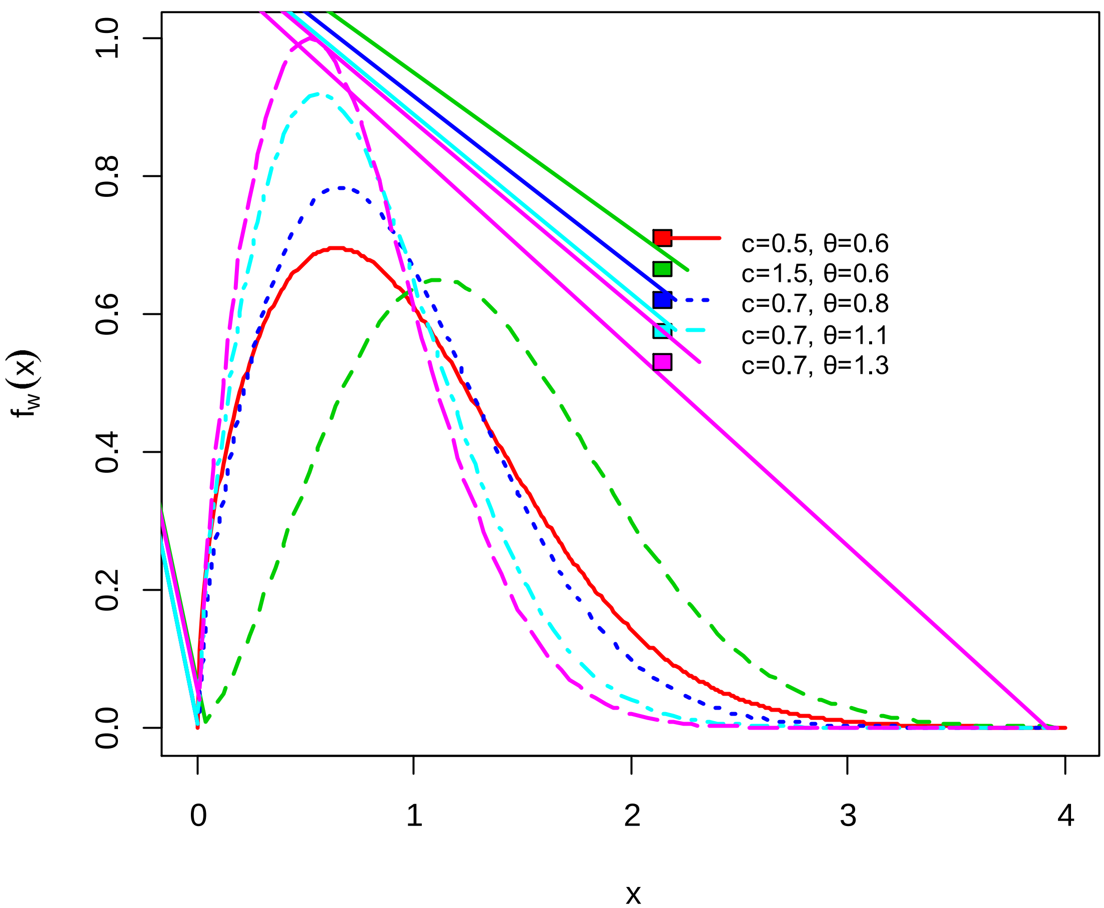
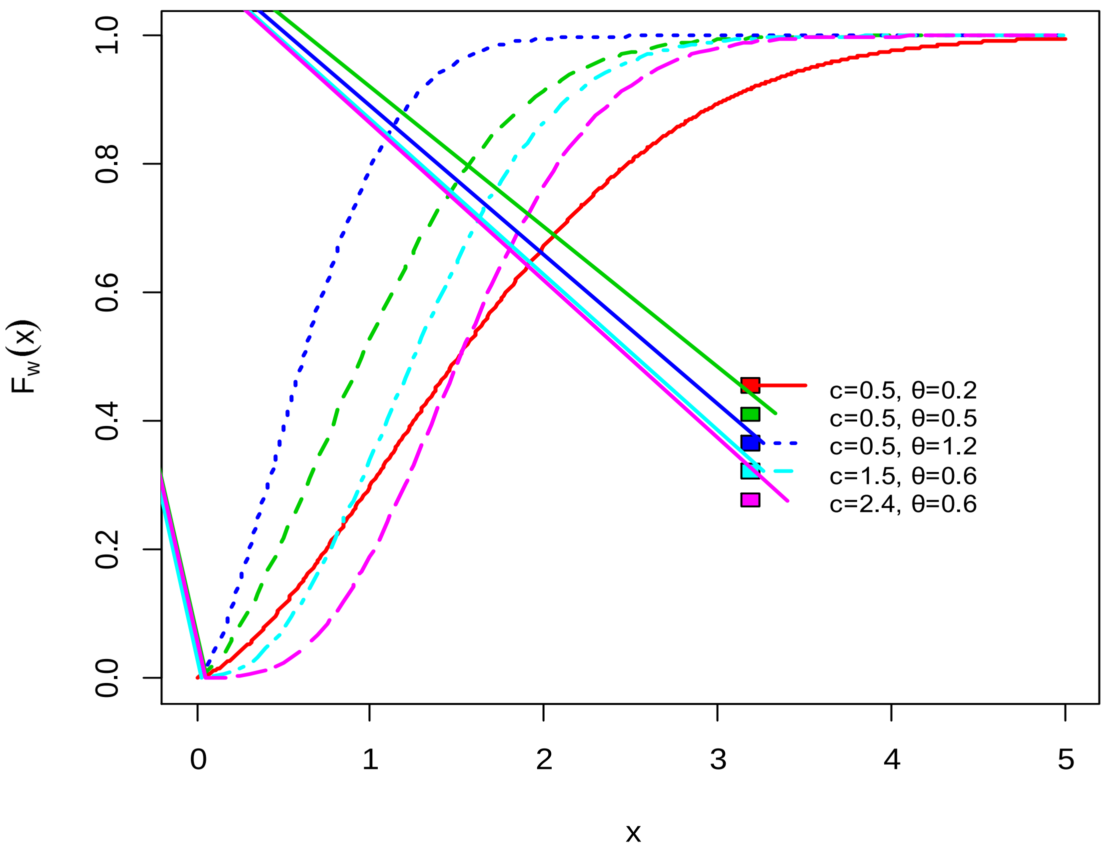


Fig.2: CDF plot of Weighted Quasi Exponential Distribution



III. SPECIAL CASES OF WQED

Case 1: If we put $c=0$, then weighted Quasi Exponential distribution (4) reduces to Quasi Exponential distribution with probability density function as:

$$f(x;\theta,) = \frac{2\theta^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} e^{-\theta x^2} \quad ; \qquad x > 0, \theta > 0 \qquad (6)$$

IV. RELIABILITY ANALYSIS

In this section, we have obtained the reliability, hazard rate, reverse hazard rate and mean residual life function of the proposed weighted Quasi Exponential Distribution.

4.1 Reliability function $R_w(x)$

The reliability function is defined as the probability that a system survives beyond a specified time. It is also referred to as survival or survivor function of the distribution. It can be computed as complement of the cumulative distribution function of the model. The reliability function or the survival function of weighted Quasi Exponential distribution is calculated as:

$$R_w(x) = 1 - F_w(x; c, \theta)$$
$$= 1 - \frac{1}{\Gamma\left(\frac{c+1}{2}\right)} \gamma\left(\frac{c+1}{2}, \theta x^2\right) \quad ; \quad x > 0, c > 0, \theta > 0, \qquad (7)$$

where θ and c are positive parameters and $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$ is a lower incomplete gamma function.

The graphical overview of $R_w(x)$ ***of*** WQED is given in fig.3. As the parameter θ increases, initial fall of the survival curve also tend to increase.

4.2 Hazard Function

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality is given as:

$$h_w(x; c, \theta) = \frac{f_w(x, \theta, c)}{R_w(x, \theta, c)}$$

$$= \frac{2x^c \theta^{\frac{c+1}{2}} e^{-\theta x^2}}{\Gamma\left(\frac{c+1}{2}, \theta x^2\right)} \quad ; \quad x > 0, \ \theta > 0, \ c > 0 \qquad (8)$$

4.3 Reverse Hazard Rate

The reverse hazard rate of the Weighted Quasi Exponential Distribution are respectively given as:

$$h_w(x; \theta, c) = \frac{f_w(x; \theta, c)}{F_w(x; \theta, c)} = \frac{2x^c \theta^{\frac{c+1}{2}} e^{-\theta x^2}}{\gamma\left(\frac{c+1}{2}, \theta x^2\right)} \quad ; \quad x > 0, \ \theta > 0, \ c > 0 \qquad (9)$$

4.4 Mean Residual Life Function

The Mean Residual Life Function of the Weighted Quasi Exponential Distribution is given as:

$$m_w(x; \theta, c) = \frac{1}{S(x; \theta, c)} \int_x^\infty t \, f_c(t; \theta, c) dt$$

$$= \frac{\Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\left(\frac{c+1}{2}\right), \theta x^2\right)} \int_x^\infty \frac{t \, 2\theta^{\frac{c+1}{2}} t^c e^{-\theta t^2} dt}{\Gamma\left(\frac{c+1}{2}\right)}$$

$$Put \ \theta t^2 = x, \qquad dt = \frac{dx}{2x^{\frac{1}{2}} \theta^{\frac{1}{2}}}$$

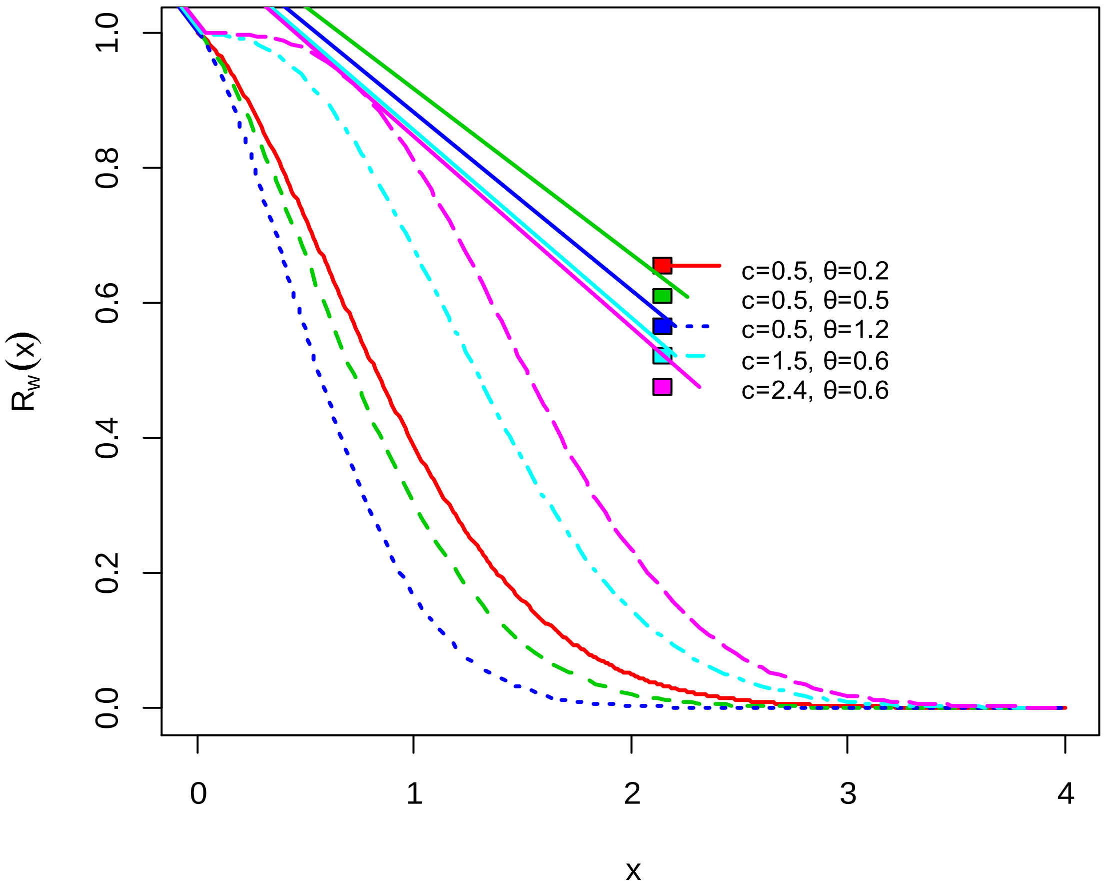
$$As \ t \to x, \ x \to \theta x^2$$

$$As \ t \to \infty, \ x \to \infty$$

After simplification,

$$m_w(x; \theta, c) = \frac{\Gamma\left(\frac{c+2}{2}, \theta x^2\right)}{\theta^{\frac{1}{2}} \Gamma\left(\frac{c+1}{2}, \theta x^2\right)} \quad ; \quad x > 0, \ \theta > 0, \ c > 0 \qquad (10)$$

Fig.3: Reliability Function plot of Weighted Quasi Exponential Distribution



4.5 Harmonic Mean (H.M.)

The Harmonic Mean of the Weighted Quasi Exponential Distribution is given as :

$$H.M. = E\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} f_c(x;\theta,c)\,dx$$

$$= \int_0^\infty \frac{1}{x} \frac{2\,\theta^{\frac{c+1}{2}}\,x^c\,e^{-\theta x^2}\,dx}{\Gamma\left(\frac{c+1}{2}\right)}$$

$$Put\ x^2 = t,\ \ 2xdx = dt,\ \ dx = \frac{dt}{2t^{\frac{1}{2}}}$$

After simplification,

$$H.M. = \frac{\Gamma\left(\frac{c+1}{2}\right)}{\theta^{\frac{1}{2}}\,\Gamma\left(\frac{c}{2}\right)} \qquad ; \qquad x > 0,\ \ \theta > 0,\ \ c > 0 \tag{11}$$

V. STRUCTURAL PROPERTIES OF WQED

In this section various structural properties of WQED has been discussed.

Theorem 5.1 The r^{th} moment about origin of a random variable X following WQED is given by:

$$\mu_r' = \frac{\Gamma\left(\frac{r+c+1}{2}\right)}{\theta^{\frac{r}{2}}\,\Gamma\left(\frac{c+1}{2}\right)} \qquad ; r = 1,2,3, \dots \dots \tag{12}$$

Proof:

$$\mu_r' = E\,(X^r) = \int_0^\infty x^r\, f_c\,(x;\theta,c)$$

$$= \int_0^\infty x^r\, \frac{2\,x^c\,\theta^{\frac{c+1}{2}}\,e^{-\theta x^2}\,dx}{\Gamma\left(\frac{c+1}{2}\right)}$$

$$Put\ x^2 = t,\ \ 2xdx = dt,\ \ dx = \frac{dt}{2\,t^{\frac{1}{2}}}$$

After simplification,

$$\mu_r' = \frac{\Gamma\left(\frac{r+c+1}{2}\right)}{\theta^{\frac{r}{2}}\,\Gamma\left(\frac{c+1}{2}\right)}$$

Moments: First four moments about origin are given as:

$$\mu_1' = \frac{\Gamma\left(\frac{c+2}{2}\right)}{\theta^{\frac{1}{2}}\,\Gamma\left(\frac{c+1}{2}\right)} \tag{13}$$

$$\mu_2' = \frac{\Gamma\left(\frac{c+3}{2}\right)}{\theta\,\Gamma\left(\frac{c+1}{2}\right)} \tag{14}$$

$$\mu_3' = \frac{\Gamma\left(\frac{c+4}{2}\right)}{\theta^{\frac{3}{2}}\,\Gamma\left(\frac{c+1}{2}\right)} \tag{15}$$

$$\mu_4' = \frac{\Gamma\left(\frac{c+5}{2}\right)}{\theta^2\,\Gamma\left(\frac{c+1}{2}\right)} \tag{16}$$

Variance:

$$\sigma^2 = \frac{\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{c+3}{2}\right) - \left\{\Gamma\left(\frac{c+2}{2}\right)\right\}^2}{\theta\left\{\Gamma\left(\frac{c+1}{2}\right)\right\}^2} \tag{17}$$

Variation, Skewness and Kurtosis: Coefficient of variation, Skewness and Kurtosis are given by:

$$C.V. = \frac{\left\{\Gamma\left(\frac{c+3}{2}\right)\Gamma\left(\frac{c+1}{2}\right)\right\}^{\frac{1}{2}}}{\Gamma\left(\frac{c+2}{2}\right)} \tag{18}$$

$$\gamma_1 = \frac{\Gamma\left(\frac{c+4}{2}\right)\left\{\Gamma\left(\frac{c+1}{2}\right)\right\}^2 - 3\,\Gamma\left(\frac{c+2}{2}\right)\Gamma\left(\frac{c+3}{2}\right)\Gamma\left(\frac{c+1}{2}\right) + 54\left\{\Gamma\left(\frac{c+2}{2}\right)\right\}^3}{\left\{\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{c+3}{2}\right) - \left\{\Gamma\left(\frac{c+2}{2}\right)\right\}^2\right\}^{\frac{3}{2}}\left\{\Gamma\left(\frac{c+1}{2}\right)\right\}^{\frac{3}{2}}} \tag{19}$$

$$\gamma_2 = \theta^{\frac{1}{2}}\frac{\left\{\Gamma\left(\frac{c+5}{2}\right)\left\{\Gamma\left(\frac{c+1}{2}\right)\right\}^3 - 4\,\Gamma\left(\frac{c+2}{2}\right)\left\{\Gamma\left(\frac{c+1}{2}\right)\right\}^2 + 6\,\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{c+3}{2}\right)\left\{\Gamma\left(\frac{c+2}{2}\right)\right\}^3 + 3\left\{\Gamma\left(\frac{c+2}{2}\right)\right\}^4\right\}}{\left\{\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(\frac{c+3}{2}\right) - \left\{\Gamma\left(\frac{c+2}{2}\right)\right\}^2\right\}^2} \tag{20}$$

Theorem 5.2 The moment generating function and characteristic function of a random variable *X* following WQED are given by:

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\Gamma\left(\frac{r+c+1}{2}\right)}{\theta^{\frac{r}{2}}\,\Gamma\left(\frac{c+1}{2}\right)} \tag{21}$$

$$\psi_x(t) = \sum_{r=0}^{\infty}\frac{(\iota t)^r}{r!}\frac{\Gamma\left(\frac{r+c+1}{2}\right)}{\theta^{\frac{r}{2}}\,\Gamma\left(\frac{c+1}{2}\right)} \tag{22}$$

Proof: From the definition of MGF, we have

$$M_x(t) = E[e^{tx}]$$

$$M_x(t) = \int_0^{\infty} e^{tx} f_w(x;\theta,c)dx$$

$$M_x\ (t) = \int_0^\infty e^{tx}\ \frac{2x^c\ \theta^{\frac{c+1}{2}} \exp(-\theta x^2)}{\Gamma\left(\frac{c+1}{2}\right)} dx$$

$$M_x\ (t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\ \frac{2\ \theta^{\frac{c+1}{2}}\ \int_0^\infty x^{r+c} \exp(-\theta x^2)\ dx}{\Gamma\left(\frac{c+1}{2}\right)}$$

$$M_x\ (t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\ \frac{\Gamma\left(\frac{r+c+1}{2}\right)}{\theta^{\frac{r}{2}}\,\Gamma\left(\frac{c+1}{2}\right)}$$

Also, we have

$$\phi_x(t) = M_x(it)$$

Therefore,

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{(\iota t)^r}{r!}\ \frac{\Gamma\left(\frac{r+c+1}{2}\right)}{\theta^{\frac{r}{2}}\,\Gamma\left(\frac{c+1}{2}\right)}$$

VI. STOCHASTIC ORDERINGS

Stochastic Ordering of positive continuous random variable is an important tool for judging the comparative behavior. A random variable *X* is said to be smaller than a random variable *Y* in the

(i) $Stochastic\ order\ (X \leq_{st} Y)\ if\ F_X\ (x) \geq\ F_Y\ (x)\ for\ all\ x$

(ii) $Hazard\ rate\ order\ (X\ \leq_{hr}\ Y)\ if\ h_X\ (x)\ \geq\ h_Y\ (x)\ for\ all\ x$

(iii) $Mean\ residual\ life\ order\ \ (X\ \leq_{mrl} Y)\ if\ m_X\ (x) \leq\ m_Y\ (x)\ for\ all\ x$

(iv) $Likelihood\ ratio\ order\ \ (X\ \leq_{lr})\ if\ \ \frac{f_X\,(x)}{f_Y\,(x)}\ \ decreases\ in\ x.$

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions.

$$X\ \leq_{lr} Y\ \Rightarrow X\ \leq_{hr} Y\ \Rightarrow X\ \leq_{mrl} Y$$

$$\Downarrow \qquad (23)$$

$$X\ \leq_{st} Y$$

The WQED is ordered with respect to the strongest "likelihood ratio" ordering as shown in the following theorem:

Theorem 6.1: Let $X \sim WQED\ (\theta_1, c_1) and\ Y \sim WQED\ (\theta_2, c_2).\ If\ \theta_1 =\ \theta_2\ and\ c_1\ \geq\ c_2\ (or\ if\ c_1 =\ c_2\ and\ \theta_1\ \geq\ \theta_2), then\ X\ \leq_{lr}\ Y\ and\ hence\ X\ \leq_{hr}\ Y,\ X\ \leq_{mrl}\ and\ X\ \leq_{st} Y.$

Proof: We have

$$\frac{f_X\ (x;\theta_1,c_1)}{f_Y\ (x;\theta_2,c_2)} = \frac{2\ x^{c_1}\ \theta_1^{\frac{c_1+1}{2}}\ e^{-\theta_1 x^2}}{2\ x^{c_2}\ \theta_2^{\frac{c_2+1}{2}}\ e^{-\theta_2 x^2}}\quad \frac{\Gamma\left(\frac{c_2+1}{2}\right)}{\Gamma\left(\frac{c_1+1}{2}\right)} \qquad ; \qquad x > 0, \theta > 0, c > 0$$

Now

$$log\frac{f_X\ (x;\theta_1,c_1)}{f_Y\ (x;\theta_2,c_2)} = logx^{c_1-c_2} + \left(\frac{c_1+1}{2}\right)\ log\ \theta_1 - \left(\frac{c_2+1}{2}\right)\ log\ \theta_2 + \log\left(\Gamma\left(\frac{c_2+1}{2}\right)\right) - \log\left(\Gamma\left(\frac{c_1+1}{2}\right)\right) - \theta_1 x^2 + \theta_2 x^2$$

Thus

$$\frac{d}{dx}\log\frac{f_X(x;\theta_1,c_1)}{f_Y(x;\theta_2,c_2)} = \frac{c_1 - c_2}{x} - 2x\ (\theta_1 - \theta_2) \tag{24}$$

$\boldsymbol{Case\ (i)}$: $If\ \theta_1 = \theta_2\ and\ c_2 \geq c_1\ then$

$$\frac{d}{dx}\log\frac{f_X(x;\theta_1,c_1)}{f_Y(x;\theta_2,c_2)} = -\left(\frac{c_2 - c_1}{x}\right) < 0$$

$\boldsymbol{Case\ (ii)}$: $If\ c_1 = c_2\ and\ \theta_1 \geq \theta_2\ then$

$$\frac{d}{dx}\log\frac{f_X(x;\theta_1,c_1)}{f_Y(x;\theta_2,c_2)} = -2x\ (\theta_1 - \theta_2) < 0$$

$This\ means\ that\ X\ \leq_{lr}\ Y\ and\ hence\ X\ \leq_{hr}\ Y,\ X\ \leq_{mrl}\ and\ X\ \leq_{st}\ Y.$

This theorem shows the flexibility of WQED over QED and Exponential Distribution.

VII. ORDER STATISTICS

Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots\ldots X_{(n)}$ be the ordered statistics of the random sample $X_1, X_2, X_3, \ldots\ldots X_n$ drawn from the continuous distribution with cumulative distribution function $F_X(x)$ and probability density function $f_X(x)$, then the probability density function of rth order statistics $X_{(r)}$ is given by:

$$f_{x(r)}\ (x;\theta) = \frac{n!}{(r-1)!\,(n-r)!}\ f(x)[F(x)]^{r-1}[1-F(x)]^{n-r} \quad ; r = 1,2,3,\ldots.,n \tag{25}$$

Using the equations (4) and (5), the probability density function of rth order statistics of Weighted Quasi Exponential Distribution is given by

$$f_{w(r)}\ (x;\theta,c) = \frac{n!}{(r-1)!(n-r)!}\frac{2\,x^c\,\theta^{\frac{c+1}{2}}\,e^{-\theta x^2}}{\Gamma\left(\frac{c+1}{2}\right)}\left[\frac{\gamma\left(\frac{c+1}{2},\theta x^2\right)}{\Gamma\left(\frac{c+1}{2}\right)}\right]^{r-1}\left[1-\frac{\gamma\left(\frac{c+1}{2},\theta x^2\right)}{\Gamma\left(\frac{c+1}{2}\right)}\right]^{n-r} \tag{26}$$

Then the PDF of first order $X_{(1)}$ of Weighted Quasi Exponential Distribution is given by:

$$f_{w(1)}\,(x;\theta,c) = \frac{2\,n\,x^c\,\theta^{\frac{c+1}{2}}\,e^{-\theta x^2}}{\Gamma\left(\frac{c+1}{2}\right)}\left[1-\frac{\gamma\left(\frac{c+1}{2},\theta x^2\right)}{\Gamma\left(\frac{c+1}{2}\right)}\right]^{n-1} \tag{27}$$

And the PDF of nth order $X_{(n)}$ of Weighted Quasi Exponential Distribution is given by:

$$f_{w(n)}\,(x;\theta,c) = \frac{2\,n\,x^c\,\theta^{\frac{c+1}{2}}\,e^{-\theta x^2}}{\Gamma\left(\frac{c+1}{2}\right)}\left[\frac{\gamma\left(\frac{c+1}{2},\theta x^2\right)}{\Gamma\left(\frac{c+1}{2}\right)}\right]^{n-1} \tag{28}$$

VIII. PARAMETER ESTIMATION OF WQED

In this section we discuss the parameter estimation of WQED using maximum likelihood method. Let $x_1, x_2, \ldots\ldots x_n$ be a random sample of size n from WQED and let f_x be the observed frequency in the sample corresponding to $X = x(x = 1,2,\ldots\ldots,n)$ such that $\sum_{x=1}^{n} f_x = n$, where n is the largest observed value having non-zero frequency.

The likelihood function, L of the WQED is given by:

$$L\,(x|\theta,c) = \prod_{i=1}^{n}\left[\frac{2\,x^c\,\theta^{\frac{c+1}{2}}\,e^{-\theta x^2}}{\Gamma\left(\frac{c+1}{2}\right)}\right]$$

$$= \frac{2^n\,\prod_{i=1}^{n} x_i^c\;\theta^{n\left(\frac{c+1}{2}\right)}\exp\left(-\theta\sum_{i=1}^{n} x_i^2\right)}{\left(\Gamma\left(\frac{c+1}{2}\right)\right)^n} \tag{29}$$

The log likelihood function is given by:

$$\log L\,(x|\theta,c) = nlog2 - nlog\left(\Gamma\left(\frac{c+1}{2}\right)\right) + n\left(\frac{c+1}{2}\right)l + c\sum_{i=1}^{n} logx_i - \theta\sum_{i=1}^{n} x_i^2 \tag{30}$$

Differentiating log likelihood function partially w.r.t. θ, c and equating to zero, we will have the following system of equations.

$$\frac{\partial logL}{\partial\theta}\,(x|\theta,c) = \frac{n(c+1)}{2\theta} - \sum_{i=1}^{n} x_i^2 = 0 \tag{31}$$

$$\frac{\partial logL}{\partial c}\,(x|\theta,c) = \sum_{i=1}^{n} logx_i + \frac{n}{2}\,log\theta - n\left[log\left(\frac{c+1}{2}\right) - \frac{1}{2\left(\frac{c+1}{2}\right)}\right] = 0 \tag{32}$$

The two equations (31) and (32) do not seem to be solved directly.
However, the Fisher's Scoring Method can be applied to solve these equations, we have

$$\frac{\partial^2}{\partial\theta^2}\log L\,(x|\theta,c) = -\frac{n(c+1)}{2\theta^2} \tag{33}$$

$$\frac{\partial^2}{\partial\theta\partial c}\log L\,(x|\theta,c) = \frac{n}{2\theta} \tag{34}$$

$$\frac{\partial^2}{\partial c^2}\log L\,(x|\theta,c) = -\frac{2n}{c+1} - \frac{n}{(c+1)^2} \tag{35}$$

$The\ following\ equations\ for\ \hat{\theta}\ and\ \hat{c}\ can\ be\ solved$

$$\begin{bmatrix} \frac{\partial^2}{\partial\theta^2} logL(x|\theta,c) & \frac{\partial^2}{\partial\theta\partial c} logL(x|\theta,c) \\ \frac{\partial^2}{\partial\theta\partial c} logL(x|\theta,c) & \frac{\partial^2}{\partial c^2} logL(x|\theta,c) \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{c}=c_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{c}-c_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial\theta} logL \\ \frac{\partial}{\partial c} logL \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{c}=c_0}}, \tag{36}$$

where θ_0 and c_0 are the initial values of θ and c respectively. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{c}$ are obtained.

8.1 Simulation Study of ML estimators

In this section, we study the performance of Maximum Likelihood (ML) estimators for different sample sizes (n=25, 75, 100, 200, 300). Using R Studio statistical software, we have employed the inverse CDF technique for data generation from WQED. The process was repeated 1000 times for calculation of bias, variance and Mean Square Error (MSE). In table 1 and table 2, for six random parameter combinations of WQED, decreasing trend is being observed in average bias, variance and MSE as we increase the sample size. Hence, we conclude that the performance of ML estimators is quite well and consistent in case of WQED.

Table 1: Simulation Study of ML estimators for WQED

Parameter	n	c=0.4, θ =0.1			c=0.8, θ =0.5		
		Bias	Variance	MSE	Bias	Variance	MSE
c	25	0.29508387	0.10477509	0.19184958	0.29365550	0.39627280	0.48250633
θ		0.03311198	0.00252820	0.00362460	0.12208990	0.05615163	0.07105757
c	75	0.05154067	0.03898041	0.04163685	-0.01009297	0.06135634	0.06145821
θ		0.00519562	0.00020277	0.00022976	0.01599402	0.01560709	0.01586290
c	100	0.06157836	0.02652511	0.03031700	0.03029941	0.03854376	0.03946182
θ		0.00887857	0.00046085	0.00053968	0.02615699	0.00680154	0.00748573
c	150	0.04626861	0.01709377	0.01923456	0.03711938	0.02395483	0.02533268
θ		0.00587462	0.00020866	0.00024317	0.04335404	0.00264317	0.00452274
c	300	0.01851527	0.01093768	0.01128050	0.06242868	0.01851739	0.02241473
θ		0.00435778	0.00013273	0.00015172	0.00898394	0.00259007	0.00267079
Parameter	n	c=1.0, θ =1.0			c=1.5, θ =1.4		
		Bias	Variance	MSE	Bias	Variance	MSE
c	25	0.1847895	0.2917726	0.3259198	0.2913450	0.2935412	0.3784231
θ		0.1608719	0.1098745	0.1357543	0.1909398	0.1789303	0.2153883
c	75	0.1584497	0.1489303	0.1740366	0.1827054	0.2620161	0.2953974
θ		0.0914149	0.0338958	0.0422525	0.1404241	0.1063482	0.1260671
c	100	0.0880106	0.0862309	0.0939767	0.1306202	0.1448995	0.1619611

θ		0.0632672	0.0320663	0.0360690	0.1043162	0.0470996	0.0579815
c	150	-0.0095045	0.0188924	0.0189827	0.0684008	0.0756592	0.0803379
θ		-0.0039635	0.0127324	0.0127481	0.0572045	0.0421825	0.0454549
c	300	-0.0194109	0.0143424	0.0147192	0.0619261	0.0424973	0.0463321
θ		-0.0011606	0.0039006	0.0039019	0.0269206	0.0207587	0.0214834
Parameter	n	c=2.0, θ =1.8			c=2.5, θ =2.2		
		Bias	Variance	MSE	Bias	Variance	MSE
c	25	0.3216602	0.5656419	0.6691072	0.3046158	1.1592629	1.2520537
θ		0.3068082	0.3810906	0.4752219	0.2745611	0.5426086	0.6179924
c	75	0.1108177	0.2080930	0.2203736	0.4046974	0.4506761	0.6144560
θ		0.0711911	0.1091186	0.1141868	0.3074740	0.2376478	0.3321880
c	100	-0.0499511	0.0928117	0.0953068	0.2655176	0.1887181	0.2592177
θ		0.0370241	0.0447384	0.0461092	0.2389096	0.1410869	0.1981647
c	150	-0.0448039	0.1027789	0.1047863	0.1163078	0.1233218	0.1368493
θ		-0.0175745	0.0715160	0.0718249	0.0801877	0.0667703	0.0732004
c	300	0.0218581	0.0632248	0.0637025	0.0182998	0.0664720	0.0668069
θ		0.0366628	0.0261835	0.0275276	0.0090304	0.0299644	0.0300459

IX. MODEL COMPARISON BASED ON SIMULATED DATA FROM WQED

In order to compare the Weighted Model with the base model on the basis of simulated data. We proceed by simulating a data from WQED using inverse CDF technique. The data generation is based on two sets of parameter combinations with sample sizes (n=10, 25, 75,150,300). It is clear from the table 2 and table 3, that weighted parameter plays a highly significant role for large samples. Even though in small samples, the AIC, AICC, BIC and Negative Loglikelihood values are also minimum in case of Weighted model but the likelihood ratio test reveals that the role of weighted parameter exhibits a highly significant role in case of large samples only. LR statistic for testing H_0 versus H_1 is $\psi = 2(L(\hat{\Theta}) - L(\hat{\Theta}_0))$, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under H_1 and H_0. The statistic ψ is asymptotically ($as\ n \to \infty$) distributed as χ^2_k, with k degrees of freedom which is equal to the difference in dimensionality of $\hat{\Theta}$ and $\hat{\Theta}_0$. H_0 will be rejected if the LR-test p-value is <0.01 (or LR Statistic value >6.635) at 99% confidence level.

Table 2: Model Comparison Based On Simulated Data From WQED.

$c = 0.5, \theta = 0.6$				Parameter Estimates		Likelihood Ratio Statistic
Criterion	Weighted Distribution	Base Distribution	Sample Size (n)	WQED	QED	
-logL	7.969889	9.058179	10	$\hat{c} = 0.867\,(0.731)$ $\hat{\theta} = 0.651\,(0.332)$	$\hat{\theta} = 0.35\,(0.155)$	2.17
AIC	19.93978	20.11636				
AICC	23.93978	24.11636				
BIC	20.54495	20.41894				
-logL	14.99104	19.52808	25	$\hat{c} = 1.32\,(0.581)$ $\hat{\theta} = 1.04\,(0.326)$	$\hat{\theta} = 0.448\,(0.127)$	9.07
AIC	33.98208	41.05616				
AICC	35.12494	42.19902				
BIC	36.41983	42.27503				
-logL	54.62365	61.74654	75	$\hat{c} = 0.781\,(0.253)$ $\hat{\theta} = 0.733(0.138)$	$\hat{\theta} = 0.411(0.067)$	14.24

AIC	113.2473	125.4931				
AICC	113.5853	125.8311				
BIC	117.8823	127.8106				
-logL	108.1393	123.5538	150	$\hat{c}=0.829\,(0.184)$ $\hat{\theta}=0.752(0.09)$	$\hat{\theta}=0.411(0.047)$	30.83
AIC	220.2786	249.1075				
AICC	220.443	249.2719				
BIC	226.2999	252.1182				
-logL	239.7377	262.1220	300	$\hat{c}=0.655\,(0.117)$ $\hat{\theta}=0.615\,(0.058)$	$\hat{\theta}=0.371\,(0.03)$	44.768
AIC	483.4755	526.2439				
AICC	483.5566	526.3250				
BIC	490.883	529.9477				

Table 3: Model Comparison Based On Simulated Data From WQED.

$c=1.5, \theta=1.2$				Parameter Estimates		Likelihood Ratio Statistic
Criterion	Weighted Distribution	Base Distribution	Sample Size (n)	WQED	QED	
-logL	2.511932	6.37295	10	$\hat{c}=2.87\,(1.606)$ $\hat{\theta}=2.31\,(1.09)$	$\hat{\theta}=0.596\,(0.266)$	7.722
AIC	9.023863	14.7459				
AICC	13.02386	18.7459				
BIC	9.629033	15.0484				
-logL	11.11552	21.3645	25	$\hat{c}=3.09\,(0.1.07)$ $\hat{\theta}=1.582(0.471)$	$\hat{\theta}=0.386\,(0.109)$	20.49
AIC	26.23104	44.7291				
AICC	27.37390	45.8719				
BIC	28.66879	45.9480				
-logL	30.35412	51.4450	75	$\hat{c}=2.01\,(0.44)$ $\hat{\theta}=1.63(0.287)$	$\hat{\theta}=0.541\,(0.088)$	42.18
AIC	64.70824	104.890				
AICC	65.04626	105.228				
BIC	69.34321	107.207				
-logL	83.86831	113.0094	150	$\hat{c}=1.407\,(0.241)$ $\hat{\theta}=1.14(0.14)$	$\hat{\theta}=0.473(0.055)$	58.28
AIC	171.7366	228.0187				
AICC	171.9010	228.1831				
BIC	177.7579	231.0294				
-logL	154.5960	223.6294	300	$\hat{c}=1.64\,(0.19)$ $\hat{\theta}=1.27\,(0.11)$	$\hat{\theta}=0.481\,(0.04)$	138.06
AIC	313.1921	449.2588				
AICC	313.2732	449.3399				
BIC	320.5996	452.9626				

X. APPLICATIONS OF WEIGHTED QUASI EXPONENTIAL DISTRIBUTION (WQED)

The goodness of fit of the WQED in comparison with QED has been explained with a real life data set given in table 4. The following data represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross & Clark (1975).

Table 4: Lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross & Clark (1975).

1.1	1.4	1.3	1.7	1.9	1.8	1.6
2.2	1.7	2.7	4.1	1.8	1.5	1.2
1.4	3	1.7	2.3	1.6	2	

Estimates of the unknown parameters is carried out in R software along with calculation of model comparison criterion values like AIC, AICC and BIC values. In order to compare the two models using the AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion). The better distribution corresponds to lesser AIC, AICC and BIC values. The generic formulas for calculation of AIC, AICC and BIC are

$$\text{AIC} = 2k-2\log L \qquad \text{AICC} = \text{AIC}+\frac{2k(k+1)}{n-k-1} \text{ and BIC} = k\log n-2\log L,$$

where k is the number of parameters in the statistical model, n is the sample size and -log$L$ is the maximized value of the log-likelihood function under the considered model.

From table 5, it is observed that weighted quasi exponential distribution have the lesser AIC, AICC, -log$L$ and BIC values as compared to quasi exponential distribution, which witness that WQED fits better than QED for data given in table 4. Kolmogorov Smirnov p-value is greater than 0.05 for weighted quasi exponential distribution whereas, it is less than 0.05 in case of quasi exponential distribution and exponential distribution which signifies that WQED fits statistically well the data set given in table 4 whereas QED and ED does not fit the data well. Hence we can conclude that the weighted quasi exponential distribution leads to a better fit than the quasi exponential distribution and exponential distribution in case of data given in table 4.

Table 5: ML estimates, -logL, AIC, AICC, BIC, KS-distance, KS p-values for fitted WQAD and QAD for data set 1.

Distribution	Weighted Quasi Exponential Distribution	Quasi Exponential Distribution	Exponential Distribution
-logL	19.17005	28.57925	32.83708
AIC	42.3401	59.1585	67.67416
AICC	43.8401	60.6585	69.17416
BIC	44.33156	60.15423	68.66989
KS-Distance	0.19674	0.52618	0.43951
P-value	0.4211	0.000031	0.000882
ML Estimates	$\hat{c} = 3.70$ (1.32) $\hat{\theta} = 0.58$ (0.19)	$\hat{\theta} = 0.1225$ (0.038)	$\hat{\theta} = 0.5263$ (0.1177)

XI. CONCLUSION

A new lifetime distribution is introduced using weighting technique. Statistical properties of the proposed model are studied and application in handling dataset representing Lifetime's relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross & Clark (1975) is analyzed.

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