International Journal of Scientific Research in Mathematical and Statistical Sciences
Review Paper
Volume-5, Issue-4, pp.419-424, August (2018)
E-ISSN: 2348-4519

Some Common Fixed Point Theorems for Single-Valued and Multi-Valued Mappings in Metric Spaces

Krishnadhan Sarkar[1*], Kalishankar Tiwary[2]

[1]Dep. of Mathematics, Raniganj Girls' College, Raniganj-713358, West Bengal, India
[2]Dep. of Mathematics, Raiganj University, Raiganj-733134, West Bengal, India

**Corresponding Author: sarkarkrishnadhan@gmail.com*

Available online at: www.isroset.org

Accepted 17/Aug/2018, Online 30/Aug/2018

Abstract- In this paper we prove some common fixed point theorems for single-valued and multi-valued mappings, with the help of CLR property and OWC property in metric spaces. In this paper we prove our main theorems in two parts; in part (A) we use OWC property and in part (B) we use CLR property. We also give some examples, corollaries and remarks. These theorems generalize, extend and improve many existing results available in this literature.
2010 AMS Subject Classification: 47H10, 54H25.

***Keywords*:** CLR property, OWC property, Common Fixed Point, Weakly Compatible Mapping

I. Introduction

Ahmed [2] has obtained a common fixed point theorem for multi-valued mappings which actually extends the result of Fisher [11]. Bouhadjera and Djoudi [4] have generalized the results of Ahmed [2] using D-mappings.

In this paper, we prove some common fixed point theorems with the help of CLR property and OWC property which generalize, extend and improve the results of Bouhadjera and Djoudi [4].

Abdou [1] has introduced the notion of occasionally weakly compatible mappings (shortly, OWC property) and the Common limit in the range (shortly, CLR property) for two single-valued and two multi-valued mappings in metric spaces.

II. Preliminaries

Let (X,d) be a metric space and B(X) be the family of all non-empty bounded subset of X. We define two function δ(A,B) and D(A,B) by

$$\delta(A,B)=\sup\{d(a,b): a\in A,\ b\in B\},$$

$$D(A,B)=\inf\{d(a,b): a\in A,\ b\in B\}\ \forall A, B\in B(X).$$

If A contains a single point a, then we will write δ(A,B)= δ({a},B).
If A={a} and B={b}, then we will write δ(A,B)=d(a,b).

It follows from the definition of δ that,

$$\delta(A,B)=\delta(B,A)\geq 0;$$
$$\delta(A,B)\leq \delta(A,C)+\delta(C,B);$$
$$\delta(A,B)=0 \Leftrightarrow A=B=\{a\};$$
$$\delta(A,A)=\text{diam } A,\quad \forall A,B,C\in B(X).$$

Now, CB(X) means the class of all non-empty bounded closed subsets of X.

III. Definitions & Examples

Definition: [4,14]: The mappings F: X→B(X) and f: X→X are δ-compatible if $\lim_{n\to\infty}\delta(Ffx_n,fFx_n)=0$.

Whenever $\{x_n\}$ is a sequence in X such that $fFx_n\in B(X)$, $Fx_n\to\{t\}$ and $fx_n\to t$ for some $t\in X$.

Jungck and Rhoades [13] generalized the above definition as follows:

Definition[4,13]: The mappings F: X→B(X) and f: X→X are weakly compatible if they commute at coincidence points , that is

$$\{t\in X: Ft=\{ft\}\}\subseteq\{t\in X: Fft=fFt\}.$$

It can be easily proved that δ-compatible maps are weakly compatible but the converse is not true.[13]

Definition[1]: Let, (X,d) be a metric space. Two mappings I, J: X→X are said to be occasionally weakly compatible (shortly OWC property) if there exists a point

$u \in X$ such that Iu=Ju and IJu=JIu.

Definition[1]: A single-valued mappings I:X→X and a multi-valued mapping F:X→CB(X) are said to be occasionally weakly compatible (shortly OWC property) if

$IFx \subset FIx$ for some $x \in X$ with $Ix \in Fx$.

Definition[1]: Let, I,J:X→X be single-valued mappings and F,G: X→CB(X) be multi-valued mappings.

(a) A point $x \in X$ is said to be a coincidence point of I and F if $Ix \in Fx$. We denote by C(I,F) the set of all coincidence points of I and F.

(b) A point $x \in X$ is said to be a common fixed point of I,J,F and G if $x=Ix \in Fx$ and $x=Ix=Jx \in Gx$.

Definition[1]: Let, (X,d) be a metric space. Two mappings I,J: X→X are said to satisfy the common limit in the range of I with respect to J (shortly CLR_I property with respect to J) if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jx_n = Iu$ for some $u \in X$.

Example 1: Let, X=[0,∞) be a metric space with usual metric. Define two single-valued mappings I,J: X→X such that $Ix=\frac{x}{k}$ and Jx=kx; $\forall x \in X$ and k is any constant.

Define the sequence $\{x_n\}$ in X by $x_n=\frac{1}{n}$, for each n≥1.

Then we get, $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jx_n = I(0)$

So, the mappings I and J satisfy (CLR_I) property with respect to J.

Definition[1]: Let, (X,d) be a metric space. A single-valued mapping I:X→X and a multi-valued mapping F: X→CB(X) are said to satisfy the common limit in the range of I with respect to F (shortly CLR_I property with respect to F) if there exists a sequence $\{x_n\}$ in X and $A \in CB(X)$ such that $\lim_{n\to\infty} Ix_n = Iu \in A = \lim_{n\to\infty} Sx_n$ for some $u \in X$.

Now, we will give an example of CLR_I property with respect to F.

Example 2: Let, X=[0,∞) be metric space with usual metric. Define a single-valued mapping I: X→X and a multi-valued mapping F:X→CB(X) such that Ix=x+k, Fx=[0,x+k]; $\forall x \in X$ and k is any constant.

Define the sequence $\{x_n\}$ in X by $x_n=\frac{1}{n}$, for each n≥1.

Then we get,

$\lim_{n\to\infty} Ix_n = k = I(0) \in [0,k] = \lim_{n\to\infty} Fx_n$

So, the mappings I and F satisfy (CLR_I) property with respect to F.

IV. Main Results

(A). We prove this main result with the help of OWC property.

Theorem 1:

Let, (X,d) be a metric space. Let, F,G: X→ B(X) be set valued mappings and I,J: X→X be single valued mappings and satisfy the following conditions:

1. $F(X) \subset J(X)$ and $G(X) \subset I(X)$.
2. $\delta^p(Fx,Gy) \le \alpha \max\{d^p(Ix,Jy), \delta^p(Ix,Fx), \delta^p(Jy,Gy)\} + (1-\alpha)[aD^p(Ix,Gy) + bD^p(Jy,Fx)]$

for all x, y in X and p≥1, 0≤α<1, a,b≥0, and α+(1-α)(a+b)<1 (1) The pairs (F,I) and (G,J) are OWC property.

Then, F,G,I,J have a unique common fixed point in X.

Proof:

Since the pairs (F,I) and (G,J) are OWC then there exist u, v in X such that $Iu \in Fu$, $IFu \subset FIu$;

$Jv \in Gv$, $JGv \subset GJv$

So, $IIu \in IFu \subset FIu$. (2)

and

$JJv \in JGv \subset GJv$. (3)

Now, we will prove that Iu=Jv.

If popssible let, Iu ≠Jv, then using the equation (1) we have,

$\delta^p(Fu,Gv) \le \alpha \max\{d^p(Iu,Jv), \delta^p(Iu,Fu), \delta^p(Jv,Gv)\} + (1-\alpha) [aD^p(Iu,Gv) + bD^p(Jv,Fu)]$

$\le \alpha\ \delta^p(Fu,Gv) + (1-\alpha) [a\ \delta^p(Fu,Gv) + b\ \delta^p(Gv,Fu)]$

Or, $\delta^p(Fu,Gv) \le [\alpha+(1-\alpha)(a+b)]\delta^p(Fu,Gv) < \delta^p(Fu,Gv)$ as $\alpha+(1-\alpha)(a+b)<1$

which is a contradiction.

So, Iu=Jv. (4)

Now, we will prove that Iu is a fixed point of I.

If possible let, IIu≠ Iu.

Then, using (1) we get,

$\delta^p(IIu,Iu) = \delta^p(IIu,Jv)$ (by (2))

$\le \delta^p(FIu,Gv)$

$\le \alpha \max\{d^p(IIu,Jv), \delta^p(IIu,FIu), \delta^p(Jv,Gv)\} + (1-\alpha) [aD^p(IIu,Gv) + bD^p(Jv,FIu)]$

$\leq \alpha\delta^p(IIu,Iu) + (1-\alpha)(a+b)\, \delta^p(IIu,Iu)$
$= [\alpha+(1-\alpha)(a+b)]\delta^p(IIu,Iu)$
$<\delta^p(IIu,Iu)$ (as $\alpha+(1-\alpha)(a+b)<1$)
which is a contradiction.
So, $IIu=Iu$. (5)
Now, we will prove $Iu=JIu$. Next,
$\delta^p(Iu,JIu) = \delta^p(Iu,JJv)$ (by (2))
$\leq \delta^p(Fu,GJv)$
$\leq \alpha \max\{d^p(Iu,JJv), \delta^p(Iu,Fu), \delta^p(JJv,GJv)\} + (1-\alpha) [aD^p(Iu,GJv) + bD^p(JJv,Fu)]$
$\leq \alpha\, \delta^p(Iu,JJv) + (1-\alpha)(a+b)\, \delta^p(Iu,JJv)$
$=[\alpha+(1-\alpha)(a+b)]\delta^p(Iu,JIu)$
$<\delta^p(Iu,JIu)$ (as $\alpha+(1-\alpha)(a+b)<1$)
which is a contradiction.
So, $Iu=JIu$. (6)
Then from (3) and (4) we get,
$Iu= IIu= JIu$, (7)
Therefore, Iu is a common fixed point of I and J. (8)
Now, from (2) and (5) we get,
$Iu=IIu \in FIu$. (9)
And, from (4) and (6) we get,
$Iu=JIu=JJv\in GJv=GIu$. (10)
So, from (9) and (10) we get,
Iu is a common fixed point of F and G. (11)
From (8) and (11) we conclude that Iu is a fixed point of F,G,I and J.
Now, from (1) we have ,
$\delta^p(FIu,GIu) \leq \alpha \max\{d^p(IIu,JIu), \delta^p(IIu,FIu), \delta^p(JIu,GIu)\} + (1-\alpha) [aD^p(IIu,GIu) + bD^p(JIu,FIu)]$
$< \alpha \max\{0,0,0\} +(1-\alpha)[a.0+b.0]=0$.
So, $FIu=GIu=\{Iu\}$. (12)
Let, $Iu=z$.
And, we have to prove that z is a unique common fixed point of F,G,I and J.
If possible let, there exist another fixed point $w(\neq z)$ of F,G,J and I.
So, from (1) we have,
$\delta^p(z,w)= \delta^p(Fz,Gw) \leq \alpha \max\{d^p(Iz,Jw), \delta^p(Iz,Fz), \delta^p(Jw,Gw)\}+(1-\alpha)[aD^p(Iz,Gw) + bD^p(Jw,Fz)]$
$=\alpha \max\{d^p(z,w), \delta^p(z,z), \delta^p(w,w)\} + (1-\alpha) [aD^p(z,w) + bD^p(w,z)]$
$\leq[\alpha+(1-\alpha)(a+b)]\delta^p(z,w)$
$<\delta^p(z,w)$ as $\alpha+(1-\alpha)(a+b)<1$
which is a contradiction.
So, z=w i.e., the fixed point z is unique.
Hence, z is a unique common fixed point of F, G, J and I.

Corollary 1: Let, (X,d) be a metric space. Let, F,G: X→ B(X) be set-valued mappings and I,J: X→X be single valued mappings, and satisfy the following conditions:

1. $F(X)\subset J(X)$ and $G(X) \subset I(X)$.
2. $\delta(Fx,Gy) \leq \alpha \max\{d(Ix,Jy), \delta(Ix,Fx), \delta(Jy,Gy)\} + (1-\alpha) [aD(Ix,Gy) + bD(Jy,Fx)]$ for all x,y in X and $p\geq1, 0\leq\alpha<1, a,b\geq0$, and
   $\alpha+(1-\alpha)(a+b)<1$. (13)
3. The pairs (F,I) and (G,J) are OWC property.

Then, F,G,I,J have a unique common fixed point in X.
Proof: Put p=1 in theorem 1, and we will get the result. This corollary is the main theorem of Bouhadjera and Djoudi [4].

Corollary 2: Let, (X,d) be a metric space. Let, F: X→ B(X) be set valued mappings and I,J: X→X be single valued mappings, and satisfy the following conditions:

1. $F(X)\subset J(X)$ and $F(X)\subset I(X)$
2. $\delta^p(Fx,Fy) \leq \alpha \max\{d^p(Ix,Jy), \delta^p(Ix,Fx), \delta^p(Jy,Fy)\} + (1-\alpha) [aD^p(Ix,Fy) + bD^p(Jy,Fx)]$.

   for all x,y in X and $p\geq1$, $0\leq\alpha<1$, $a,b\geq0$, and $\alpha+(1-\alpha)(a+b)<1$. (14)
   The pair (F,I) OWC property.

Then, F,I and J have a unique common fixed point in X.
Proof: Put G=F in Theorem 1, and get the result.

Corollary 3: Let, (X,d) be a metric space. Let, F,G: X→ B(X) be set valued mappings and I: X→X be single valued mappings, and satisfy the following conditions:

1. $F(X)\subset I(X)$ and $G(X) \subset I(X)$.
2. $\delta^p(Fx,Gy) \leq \alpha \max\{d^p(Ix,Iy), \delta^p(Ix,Fx), \delta^p(Iy,Gy)\} + (1-\alpha) [a D^p(Ix,Gy) + b D^p(Iy,Fx)]$

   for all x,y in X and $p\geq1$, $0\leq\alpha<1$, $a,b\geq0$, and $\alpha+(1-\alpha)(a+b)<1$. (15)
   The pairs (F,I) and (G,I) OWC property.

Then, F,G and I have a unique common fixed point in X.
Proof: Put J=I in theorem 1, and get the results.
Remark 3: If we put p=1 in corollary 3, we will get the Corollary 3.2 of Bouhadjera and Djoudi [4]

Corollary 4: Let, (X,d) be a metric space. Let, F: X→ B(X) be set valued mappings and I: X→X be single valued mappings, and satisfies the following conditions:

1. $F(X)\subset I(X)$
2. $\delta^p(Fx,Fy) \leq \alpha \max\{d^p(Ix,Iy), \delta^p(Ix,Fx), \delta^p(Iy,Fy)\} + (1-\alpha) [a D^p(Ix,Fy) + b D^p(Iy,Fx)]$

   for all x,y in X and $p\geq1$, $0\leq\alpha<1$, $a,b\geq0$, and $\alpha+(1-\alpha)(a+b)<1$. (16)
3. (F,I) is OWC property.

Then, F and I have a unique common fixed point in X.
Proof: If we put G=F and J=I in our main theorem then, we will get the result.
Remark 4: If we put p=1, in Corollary 4 , we will get the Corollary 3.1 of Bouhadjera and Djoudi [4]

Corollary 5: Let, (X,d) be a metric space. Let, F_i : X→ B(X), $\forall\, i\in N$ be set valued mappings and I,J: X→X be single valued mappings , and satisfies the following conditions:

1. $F_i(X) \subset J(X)$ and $F_{i+1}(X) \subset I(X)$.
2. $\delta^p(F_ix,F_{i+1}y) \le \alpha \max\{d^p(Ix,Jy),\delta^p(Ix,F_ix),\delta^p(Jy,F_{i+1}y)\}+(1-\alpha)[aD^p(Ix,F_{i+1}y) + bD^p(Jy,F_ix)]$
   for all x,y in X and $p\ge1$, $0\le\alpha<1$, $a,b\ge0$, and $\alpha+(1-\alpha)(a+b)<1$. (17)
3. The pairs (F_i,I) and (F_{i+1},J) OWC property.

Then, F_i,I and J have a unique common fixed point in X, $\forall\, i\in N$.

Remark 5: Corollary 5 is a generalization of Theorem 3.2 and Theorem 3.3 of Bouhadjera and Djoudi [4]

(B) Common fixed point of mappings with the help of CLR_I property.

In this section, we prove common fixed point theorems for two single valued mappings and two multi-valued mappings by CLR_I property.

Definition[1]: Let, (X,d) be a metric space. Let $I,J:X\to X$ and $F,G: X\to CB(X)$. Then I,J,F and G are said to satisfy the common limit in the range of I (shortly, CLR_I property) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X and A, B in CB(X) such that $\lim_{n\to\infty} Fx_n=A$, $\lim_{n\to\infty} Gy_n=B$ and

$\lim_{n\to\infty} Ix_n=\lim_{n\to\infty} Jy_n=Iu \in A \cap B$.

Theorem 2:

Let, (X,d) be a metric space. Let, $F,G: X\to B(X)$ be set valued mappings and $I,J: X\to X$ be single valued mappings, and satisfy the following conditions:

(a) $F(X) \subset J(X)$ and $G(X) \subset I(X)$.

(b) $\delta^p(Fx,Gy) \le \alpha \max\{d^p(Ix,Jy), \delta^p(Ix,Fx), \delta^p(Jy,Gy)\} + (1-\alpha)\,[a\, D^p(Ix,Gy) + b\, D^p(Jy,Fx)]$ for all x,y in X and $p\ge1$, $0\le\alpha<1$, $a,b\ge0$, and $\alpha+(1-\alpha)(a+b)<1$. (1)

The pairs (F,I) and (G,J) satisfy CLR_I property.

If I(X) and J(X) are closed subsets of X, then the following holds:

1. I and F have a coincidence point.
2. J and G have a coincidence point.
3. I and F have a common fixed point if I and F are weakly compatible at v and IIV=Iv for any v $\in C(I,F)$.
4. J and G have a common fixed point if J and G are weakly compatible at v and JJV=Jv for any v $\in C(J,G)$.
5. F,G, I and J have a common fixed point in X if both (3) and (4) hold.

Proof: Since, the pairs (F,I) and (G,J) satisfy CLR_I property then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X and A,B in CB(X) such that

$\lim_{n\to\infty} Fx_n=A$, $\lim_{n\to\infty} Gy_n=B$ and $\lim_{n\to\infty} Ix_n=\lim_{n\to\infty} Jy_n=Iu \in A \cap B$ for some u in X .

Also, as I(X) and J(X) are closed, we have Iu=Iv and Iu=Jw, for some v,w in X.

Now, we will show that $Jw\in Gw$.

If possible let $Jw \notin Gw$.

Then, put $x=x_n$, y=w in equation (1) and get,

$\delta^p(Fx_n,Gw)$
$\le \alpha \max\{d^p(Ix_n,Jw), \delta^p(Ix_n,Fx_n), \delta^p(Jw,Gw)\} + (1-\alpha)$
$[aD^p(Ix_n,Gw) + bD^p(Jw,Fx_n)]$ for all $n\in N$.

Taking $\lim_{n\to\infty}$ in the above inequality we get,

$\delta^p(A,Gw) \le \alpha \max\{d^p(Iu,Jw), \delta^p(Iu,A), \delta^p(Jw,Gw)\} + (1-\alpha)$
$[aD^p(Iu,Gw) + bD^p(Jw,A)]=0$
i.e., $\delta^p(A,Gw)=0$
Now, $\delta^p(Jw,Gw)\le \delta^p(A,Gw)=0$

So, $Jw\in Gw$, (2)

i.e., w is a coincidence point of J and G. (3)

Now, we will show that $Iv\in Fv$.

If possible, let $Iv \notin Fv$.

Then, put x=v, $y=y_n$ in equation (1) and get,

$\delta^p(Fv,Gy_n) \le \alpha \max\{d^p(Iv,Jy_n), \delta^p(Iv,Fx), \delta^p(Jy,Gy_n)\} + (1-\alpha)\,[aD^p(Iy_n,Gy_n) + bD^p(Jy,Fy_n)]$

Taking $\lim_{n\to\infty}$ in the above inequality we get,

$\delta^p(Fv,B) \le \alpha \max\{d^p(Iv,Jw), \delta^p(Iv,Fv), \delta^p(Jw,B)\} + (1-\alpha)$
$[aD^p(Iu,B) + bD^p(Jw,B)]=0$
$\delta^p(Fv,B) \le \alpha\, \delta^p(Iv,Fv)$
$\le \alpha\, \delta^p(B,Fv)<\delta^p(B,Fv)$ as $0\le\alpha<1$.
which is a contradiction.
So, $\delta^p(Fv,B)=0$
Now, $\delta^p(Iv,Fv)\le \delta^p(B,Fv)=0$

So, $Iv\in Fv$. (4)

i.e., v is a coincidence point of I and F. (5)

As I and F are weakly compatible mapping, i.e., IFv=FIv. (6)

And J and G are also weakly compatible mapping,i.e.,JGw=GJw. (7)

Now, we will prove that FIv=Gw.

From (1) we get,

$\delta^p(FIv,Gw) \le \alpha \max\{d^p(IIv,Jw), \delta^p(IIv,FIv), \delta^p(Jw,Gw)\} + (1-\alpha)[aD^p(IIv,Gw) + bD^p(Jw,FIv)]$
$\le [\alpha+(1-\alpha)(a+b)]\ \delta^p(FIv,Gw) <\delta^p(FIv,Gw)$ as $\alpha+(1-\alpha)(a+b)<1$

i.e.,FIv=Gw. (8)

i.e., IIv $\in$ IFv=FIv=Gw. (by (6) and (8)). (9)

Now, we will prove Iv=IIv.

From (1) we get,

$\delta^p(Fv,Gw) \leq \alpha \max\{d^p(Iv,Jw), \delta^p(Iv,Fv), \delta^p(Jw,Gw)\} + (1-\alpha)[aD^p(Iv,Gw) + bD^p(Jw,Fv)]$

$\leq (1-\alpha)(a+b)\,\delta^p(Fv,Gw)$

$<\delta^p(Fv,Gw)$ as $(1-\alpha)(a+b)<1$,

i.e.,$\delta^p(Fv,Gw)=0$. (10)

Now, $\delta^p(Iv,IIv) \leq \delta^p(Fv,Gw)$ (by (4) and (9))

=0 (by (10)),

i.e.,Iv=IIv. (11)

From (9) and (11) we get, Iv= IIv $\in$ IFv=FIv.

Let, Iv=z

i.e., z=Iz$\in$ Fz

i.e., z=Iv is a common fixed point of I and F.

Similarly we can prove that Jw=JJw $\in$ GJw

i.e., Jw is a common fixed point of J and G

Corollary 6: Let, (X,d) be a metric space. Let, F,G: X→ B(X) be set valued mappings and I,J: X→X be single valued mappings, and satisfy the following conditions:

(a) F(X)$\subset$ J(X) and G(X) $\subset$ I(X).

(b) $\delta(Fx,Gy) \leq \alpha \max\{d(Ix,Jy), \delta(Ix,Fx), \delta(Jy,Gy)\} + (1-\alpha)[a\,D(Ix,Gy) + b\,D(Jy,Fx)]$ for all x,y in X and p≥1, 0≤α<1, a,b≥0, and α+(1-α)(a+b)<1

(c) The pairs (F,I) and (G,J) satisfy CLR_I property.

If I(X) and J(X) are closed subsets of X, then the following holds:

1. I and F have a coincidence point.
2. J and G have a coincidence point.
3. I and F have a common fixed point if I and F are weakly compatible at v and IIV=Iv for any v $\in$C(I,F).
4. J and G have a common fixed point if J and G are weakly compatible at v and JJV=Jv for any v $\in$ C(J,G).
5. F,G,I and J have a common fixed point in X if both (3) and (4) hold.

Proof: If we put p=1 in Theorem 2, then we will get the results. This corollary is the main theorem of Bouhadjera and Djoudi [4].

Corollary 7: Let, (X,d) be a metric space. Let, F: X→ B(X) be set valued mappings and I,J: X→X be single valued mappings , and satisfy the following conditions:

(a) F(X)$\subset$ J(X) and F(X) $\subset$ I(X).

(b) $\delta^p(Fx,Fy) \leq \alpha \max\{d^p(Ix,Jy), \delta^p(Ix,Fx), \delta^p(Jy,Fy)\} + (1-\alpha)[a\,D^p(Ix,Fy) + b\,D^p(Jy,Fx)]$ for all x,y in X and p≥1, 0≤α<1, a,b≥0, and α+(1-α)(a+b)<1.

(c) The pairs (F,I) and (F,J) satisfy CLR_I property.

If I(X) and J(X) are closed subsets of X, then the following holds:

(1). I and F have a coincidence point.

(2) J and G have a coincidence point.

(3) I and F have a common fixed point if I and F are weakly compatible at v and IIV=Iv for any v $\in$ C(I,F).

(4) J and G have a common fixed point if J and F are weakly compatible at v and JJV=Jv for any v $\in$C(J,F).

(5) F,I and J have a common fixed point in X if both (3) and (4) are true

Proof: If we put G=F in theorem 2 then we will get the result.

Corollary 8: Let, (X,d) be a metric space. Let, F,G: X→ B(X) be set valued mappings and I: X→X be single valued mapping, and satisfy the following conditions:

(a). F(X)$\subset$ I(X) and G(X) $\subset$ I(X).

(b) $\delta^p(Fx,Gy) \leq \alpha \max\{d^p(Ix,Iy), \delta^p(Ix,Fx), \delta^p(Iy,Gy)\} + (1-\alpha)[a\,D^p(Ix,Gy) + b\,D^p(Iy,Fx)]$ for all x,y in X and p≥1, 0≤α<1, a,b≥0, and α+(1-α)(a+b)<1

(c) The pairs (F,I) and (G,I) satisfy CLR_I property.

If I(X) and J(X) are closed subsets of X, then the following holds:

1. I and Fhave a coincidence point.
2. I and G have a coincidence point.
3. I and F have a common fixed point if I and F are weakly compatible at v and IIV=Iv for any v $\in$C(I,F).
4. J and G have a common fixed point if J and G are weakly compatible at v and IIv=Iv for any v $\in$C(I,G).
5. F,G and I have a common fixed point in X if both (3) and (4) are true

Proof: If we put J=I in our main Theorem 2, then we will get the result.

Corollary 9: Let, (X,d) be a metric space. Let, F: X→ B(X) be set valued mapping and I: X→X be single valued mapping, and satisfy the following conditions:

(a). F(X)$\subset$ I(X)

(b) $\delta^p(Fx,Fy) \leq \alpha \max\{d^p(Ix,Iy), \delta^p(Ix,Fx), \delta^p(Iy,Fy)\}+(1-\alpha)[aD^p(Ix,Fy)+bD^p(Iy,Fx)]$ for all x,y in X and p≥1, 0≤α<1, a,b≥0, and α+(1-α)(a+b)<1.

(c) The pair (F,I) satisfies CLR_I property.

If I(X) is a closed subsets of X, then the following holds:

1. I and Fhave a coincidence point.

2. I and F have a common fixed point if I and F are weakly compatible at v and IIV=Iv for any v $\in$ C(I,F).
3. F and I have a common fixed point in X if both (3) and (4) are true

Proof: If we put G=F and J=I in our main theorem 2, then we will get the result.

Corollary 10: Let, (X,d) be a metric space. Let, F_i : X→ B(X),$\forall$ i$\in$N be set valued mappings and I,J: X→X be single valued mappings , and satisfy the following conditions:

(a) $F_i(X) \subset J(X)$ and $F_{i+1}(X) \subset I(X)$

(b) $\delta^p(F_ix,F_{i+1}y) \leq \alpha \max\{d^p(Ix,Jy), \delta^p(Ix,F_ix), \delta^p(Jy,F_{i+1}y)\} + (1-\alpha)[aD^p(Ix,F_{i+1}y) + bD^p(Jy,F_ix)]$ for all x,y in X and p≥1, 0≤α<1, a,b≥0, and α+(1-α)(a+b)<1.

(c) The pairs (F_i,I) and (F_{i+1},J) satisfy CLR_I property.

If I(X) and J(X) are closed subsets of X, then the following holds:

1. I and F_i have a coincidence point.
2. J and F_{i+1} have a coincidence point.
3. I and F_i have a common fixed point if I and F_i are weakly compatible at v and IIV=Iv for any v $\in$ C(I,F_i).
4. J and F_{i+1} have a common fixed point if J and F_{i+1} are weakly compatible at v and JJV=Jv for any v $\in$ C(J,F_{i+1}).
5. F_i,I and J have a common fixed point in X if both (3) and (4) are true, $\forall$ i$\in$N.

V. Conclusion

In this paper we prove some Common fixed point theorem of single-valued and multi-valued in metric spaces, and this is a generalization of many existing results in this literature. We gave examples, corollaries, remarks which illustrate our main result. There is a huge scope of further works in this topic. Particularly, one can study the common fixed point in metric space by Hausdorff metric in multi-valued mappings or taking more than four mappings in this space. In future, we will extended our work in this topic.

References

[1]. A. A. N. Abdou, Common fixed point results for multi-valued mappings with some examples, J. Nonlinear Sci. appl. 9(2016), 787-798.

[2]. M. A. Ahmed, Common fixed point theorems for weakly compatible mappings, Rocky Mountain J. Math., 33(4) (2003), 1189-1203.

[3]. N.A Assad and S.Sessa, Common fixed points for nonself-maps on compacta, SEA Bull. Math. 16(1992), 1-5.

[4]. H. Bouhadjera and A. Djoudi, Common fixed point theorems for single and set-valued maps without continuity, An. St. Univ. Ovidius Constanta, 16(1) (2008), 49-58.

[5]. N.Chandraet. Al, Coincidences and fixed points of nonexpansive type multi-valued and single-valued maps, Indian J. Pure Appl. Math. 26(1995), 393-401.

[6]. Tong-Huei Chang, Fixed point theorems of contractive type set-valued mappings, Math. Japon. 38(4)(1993), 675-690.

[7]. Y.J.Choet.al , A common fixed point theorem of Meir andKeeler type, Internat. J. Math.Math.Sci. 16(1993), 669-674.

[8]. R. O. Davies and S Sessa, A common fixed point theorem of Gregus type for compatible mappings, Facta Univ. (Nis) Ser. Math. Inform.7(1992), 51-58.

[9]. A. Djoudi and R. Khemis, Fixed points for set and single valued maps without continuity, DemonstratioMathematica, 38(3) (2005), 739-751.

[10]. B. Fisher , Common fixed points of mappings and set-valued mappings, Rostock. Math. Kolloq. 18(1981), 69-77.

[11]. B. Fisher, Common fixed points of mappings and set-valued mappings on a metric spaces, Kyungpook Math. J. 25(1985), 35-42.

[12]. B. Fisher and S Sessa, Two common fixed point theorems for weakly commuting mappings, Period. Math. Hunger. 20(1989), 207-218.

[13]. G. Jungck and B. E. Rhodes, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math., 29(3)(1998), 227-238.

[14] Liu, Li-Shan, Common fixed points of a pair of single-valued mappings and a pair of set-valued mappings , QufuShifanDaxueXuebaoziranKexue Ban, 18(1992), no. 1, 6-10.

[15]. S. Sessa and Y. J. Cho, Compatible mappings and a common fixed point theoremof change type, Publ. Math. Debrecen 43(3-4) (1993), 289-296.

[16]. S. Sessa, M. S. Khan and M. Imdad, Common fixed point theorem with a weak commutativity condition, Glas. Mat. 21(41) (1986), 225-235.

[17]. S. Sessa, B. E. Rhoades and M. S. Khan, On common fixed points of compatible mappings, Internat. J. Math. Math. Sci. 11(1988), 375-392.

[18]. W.Sintunavarat, P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric space, J. Appl. Math. 2011(2011), 14Pages.

[19]. W.Sintunavarat, P. Kumam, Gregus-type common fixed point theorems for tangential multi-valued mappings of integral type in metric spaces, Internat. J. Math. Math. Sci., 2011(2011), 12 pages.

[20] W.Sintunavarat, D. M. Lee, Y. J. Cho, Mizoguchi-Takahashi's type common fixed point theorems without T-weakly commuting condition and invariant approximations, Fixed Point Theory appl., 2014(2014), 10 pages.

[21] K. Tas, M. Telci and B. Fisher, Common fixed point theorems for compatible mappings, Internat. J. Math. Math. Sci. 19(1996), 451-456.

[22]. N. Wairojjana, W. Sintunavarat, P. Kumam, Common tripled fixed point theorems for W-compatible mappings along with the CLR_g- property in abstract metric spaces, J. Inequal. Appl., 2014(2014), 17 pages.