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Research Article

On The Applications of Stiefel-Whitney Classes of Real Bott Manifolds

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Abstract— Real Bott manifolds are a type of compact, connected Riemannian manifolds without boundaries, notable as a unique and intriguing category of real toric varieties. This study focuses on identifying a set of topological invariants known as Stiefel-Whitney classes, which are crucial characteristic classes in the context of the set of integers modulo two. We computed the Stiefel-Whitney classes of real Bott manifolds. We applied the method to calculate the Stiefel-Whitney classes of the manifolds. The formula for determining the number of terms in each class was also given.

Keywords— Manifolds, Real Bott manifolds, Stiffel-Whiteney Class

1. Introduction

Let M_n be a flat manifold of dimension n, meaning a compact connected Riemannian manifold without boundary with zero sectional curvature. According to the theorems of [1,2,3] the fundamental group $\pi_1(M_n) = \Gamma$ establishes a short exact sequence:

$$0 \to Z_n \to \Gamma \to^p G \to 0 \qquad (1)$$

where Z_n is a maximal torsion free abelian subgroup of rank n and G is a finite group isomorphic to the holonomy group of M_n. The universal covering of M_n is the Euclidean space R^n and hence Γ is isomorphic to a discrete cocompact subgroup of the isometry group Isom($R^n) = O(n) \times R^n = E(n)$. In that case $p: \Gamma \to G$ is a projection on the first component of the semidirect product $O(n) \times R^n$ and $\pi_1(M_n) = \Gamma$ is a subgroup of $O(n) \times R^n$. Conversely, given a short exact sequence of the form of equation (1), it is known that the group Γ is (isomorphic to) the fundamental group of a flat manifold.

In simple terms, a manifold is an entity that resembles Euclidean space within a small region around each point. For instance, a balloon is a two-dimensional manifold because a small insect with limited vision walking on the balloon's surface would perceive it as flat. This means that in the vicinity of every point on the balloon, there is a neighbourhood that looks like $\mathbb{R}^2$.

2. Related Work

Historically, the concept of characteristic classes was initially introduced by Hopf, Stiefel, and Whitney, although the idea can be traced back to Schubert Calculus and the Italian school of Algebraic Geometry. In general terms, a characteristic class is a tool used to quantify the twisting of a vector bundle. Stiefel-Whitney classes are topological invariants of a real vector bundle that indicate the obstructions to constructing globally independent sets of sections. These classes are indexed from 0 to n, where n is the rank of the vector bundle. If the Stiefel-Whitney class of index i is non-zero, it implies that there cannot be (n-i+1) globally linearly independent sections of the vector bundle. A non-zero nth Stiefel-Whitney class means that every section of the bundle must vanish at some point. Reference [4] provided a purely combinatorial description of Stiefel-Whitney classes for closed flat manifolds with diagonal holonomy representation. Using this description, for any integer d of at least two, they constructed non-spin holonomy groups of integers modulo two with the property that all their finite proper covers possess a spin structure. Additionally, all such covers have trivial Stiefel-Whitney classes. Unlike the case of real Bott manifolds, this demonstrates that for a general closed flat manifold, the existence of a spin structure might not be detectable through its finite proper covers.

3. Theory/Calculation

Let

$$M_n \to^{\Re P^1} M_{n-1} \to^{\Re P^1} . . . , \to^{\Re P^1} M_1 \to^{\Re P^1} M_0 = \{\bullet\} \qquad (2)$$

be a sequence of real projective bundles such that $M_i \to M_{i-1}, i=1,2,...,n$ is a projective bundle of a Whitney sum of a real line bundle L_{i-1} and the trivial line bundle over M_{i-1}. The sequence (2) is called the real Bott tower and the top manifold M_n is called the real Bott manifold, [4,5,6]. Let γ_i be the canonical line bundle over M_i and we set $x_i = w_1(\gamma_i)$ (w_1 is the first Stiefel-Whitney class). Since $H_1(M_{i-1}, \mathbb{Z}_n)$ is additively generated by $x_1, x_2, ..., x_{i-1}$ and L_{i-1} is a line bundle over M_{i-1}, we can uniquely write:

$$w_1(L_{i-1}) = \sum_{i=1}^{i-1} a_{l_i} x_i \qquad (3)$$

where $a_{l_i} \in Z_n$ and $i = 1, 2, ..., n$

From the above we obtain the matrix $A = [a_{l_i}]$ A which is an $(n \times n)$ strictly upper triangular matrix whose diagonal entries are 0 and remaining entries are either 0 or 1. One can observe [7,8,9,10] that the tower (2) is completely determined by the matrix A and therefore we may denote the real Bott manifold M_n by $M_n(A)$. From [5], we can consider $M_n(A)$ as the orbit space

$M(A) = \Re^n / \Gamma(A)$, where $\Gamma(A) \subset E(n)$ is generated by elements

$$S_i = \left(dig[1,...,(-1)^{a_{i,i+1}}], (0,...,0,\frac{1}{2},0,...,0)^T \right)$$

where $(-1)^{a_i,i+1}$ is the $(i+1, i+1)$ position and $\frac{1}{n}$ is the ith coordinate of the last column, $i = 1, 2, ..., n$.

$S_n = \left(I, (0, 0, ..., \frac{1}{2}) \right) \in E(n)$. From [5],

$s_1^2, s_2^2, ..., s_n^2$ commute with each other and generate a free abelian subgroup Z^n. In other words, $M_n(A)$ is a flat manifold with holonomy group Z_2^k of diagonal type where k is the number of non-zero rows of matrix A.

Lemma 1. [5]. The Cohomology ring $H^*(M_n(A), Z_2)$ is generated by degree one element $x_1, ..., x_n$ as a graded ring with n relations $x_j^2 = x_j \sum_{i=1}^{n} a_{ij} x_i$, for $j = 1, ..., n$.

Definition 1. Manifolds: let M be a topological space and let $p \in M$. An $n-$dimensional (local) chart at p is a pair (ϑ, ψ), where ϑ is an open neighborhood of p and ψ is a homeomorphism of ϑ onto some open subset of $\mathbb{R}^n$ i.e., $\psi : \vartheta \to \psi(\vartheta) \leq \mathbb{R}^n$.

Theorem 1. Let A be an $(n \times n)$ the bott matrix. Then, $w_{2k}(M_n(A)) = \sum_{1\le i_1 \le i_2 < ... < i_{2k} \le n} w_{2k}(M_n(A_{i_1 i_2 ..., i_{2k}}))$.

Proof. From Lemma 1, we have that the $2k$ Cohomolgy group of $H^{2k}(M_n(A), Z_2)$ has a basis $B = \{x_{i_1}, x_{i_2}, ..., x_{i_{2k}} : 1 \le i_1 < i_2 < ... < i_{2k} \le n\}$

Moreover, from lemma 1, x_j^2 can be expressed as a linear combination of $x_k x_j$ for $k < j$. Note that this combination always contains an x_j term. Hence, we get that $w_{2k}(M_n(A))$ is a sum of linear elements $w_{2k}(M_n(A)) = \sum_{1\le i_1 \le i_2 < ... < i_{2k} \le n} x_{i_1} x_{i_2} ....x_{i_{2k}}$. Each term $x_{i_1} x_{i_2} ....x_{i_{2k}}$ of this sum is an element from basis B and it is equal to the $2k$ Stiefel-Whitney class of the real bott manifold $M_n(A_{i_1 i_2 ..., i_{2k}})$ so we get:

$w_{2k}(M_n(A)) = \sum_{1\le i_1 \le i_2 < ... < i_{2k} \le n} w_{2k}(M_n(A_{i_1 i_2 ..., i_{2k}}))$. Thus, the $2k-th$ Stiefel-Whitney class of the real Bott manifold $M_n(A)$ is equal to the sum of $2k-th$ Stiefel-Whitney classes of elementary components $M_n(A_{i_1 i_2 ..., i_{2k}})$, $1 \le i_1 < i_2 < ... < i_{2k} \le n$.

4. Results and Discussion

A Bott tower is a smooth complete complex Toric variety which is constructed iteratively as follows: Let $Y_1 = \mathbb{CP}'$. Let L_2 be a holomorphic complex line bundle on $\mathbb{CP}'$. We then let $Y_2 = \mathbb{P}(1 \oplus L_2)$ where 1 is the trivial line bundle on $\mathbb{CP}'$, then Y_2 is a $\mathbb{CP}'$ bundle over $\mathbb{CP}'$. We can iterate this process for $2 \le j \le n$, where at each step L_j is a complex line bundle over Y_{j-1}, and the variety $Y_j = \mathbb{P}(1 \oplus L_j)$ is a $\mathbb{CP}'$ bundle over Y_{j-1}. The variety Y_n thus obtained after $n-$steps is called an $n-$step Bott tower.

Definition 2. Let $\delta = (E, B, P)$ be a real n-dimensional vector bundle over a base space B. there is a class $w(\delta) \in H^*(B, Z_2)$ with the following axioms:

i). $w_0(\delta) = 1$ and $w_i(\delta) = 0$ if $i > \dim \delta$.

ii). The Whitney product formula $w_i(\xi \oplus \delta) = \sum_{j=0}^{i} w_i(\xi) \cup w_{i-j}(\delta)$.

iii). $w_i(\gamma_i) \ne 0$ for the universal line bundle defined on $\mathbb{RP}^\infty$. Furthermore, every $mod2$ characteristic class for $n-$plane bundle can be written uniquely as a polynomial in the Stiefel-Whitney classes $\{w_i,\ldots,w_n\}$ and $w_i(\gamma_i)$ is the generator of the polynomial ring $H^*(\Re P^\infty, Z_2)$.

iv). Naturality. If $\delta=(E,B,P)$ is a real vector bundle and $f:B_1\to B$ is a map, then $w_i(f*\delta)=f*(w_i(\delta))$ which implies that Stiefel-Whitney classes commute with pulling back.

Remark. $w(\delta)=w_0(\delta)+w_1(\delta)+w_2(\delta)+\ldots$ is called the total Stiefel-Whitney class of δ and $w_i(\delta)$ is the ith Stiefel-Whitney class of δ.

Definition 3. For each integer $r\geq 0$, the rth elementary symmetric function in $x_1,x_2,\ldots,x_n$, δ_r is the sum of all products of r distinct variables x_i so that $\delta_0=1$ and $\delta_r=\sum 1\leq i_1<i_2<\ldots<i_r\leq n$, $x_{i_1}\ldots x_{i_r}$ and $\delta_i=0$.

For $i>n$, that is;

$$\delta_0=1,$$

$$\delta_1=x_1+\ldots+x_n$$

$$\delta_2=x_1x_2+x_1x_3+\ldots+x_{n-1}x_n$$

$$\delta_3=x_1x_2x_3+x_1x_2x_4+\ldots+x_{n-2}x_{n-1}x_n$$

:

:

$$\delta n-1=x_2x_3\ldots x_n+x_1x_2+\ldots+x_{n-2}x_n+\ldots+x_2x_3\ldots x_n$$

$$\delta n=x_1x_2\ldots x_n.$$

With the definition above we shall have the following results:

$$\text{For } A=\begin{bmatrix}0&1&1&0&0&0&0\\0&0&1&1&0&0&0\\0&0&0&1&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}$$

We have that:

$$w_4(M(A)=x_1x_2x_3x_4+x_1x_2x_3x_5+x_1x_2x_4x_5+x_1x_3x_4x_5+x_2x_3x_4x_5$$

For the matrix A we have the following

$$A_{1234}=\begin{bmatrix}0&1&1&0&0&0&0\\0&0&1&1&0&0&0\\0&0&0&1&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}$$

$$w_4(M(A_{1234})=x_1x_2x_3x_4$$

$$A_{1235}=\begin{bmatrix}0&1&1&0&0&0&0\\0&0&1&1&0&0&0\\0&0&0&1&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&1&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}$$

$$w_4(M(A_{1235})=x_1x_2x_3x_5$$

$$A_{1245}=\begin{bmatrix}0&1&1&0&0&0&0\\0&0&1&1&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}$$

$$w_4(M(A_{1245})=0$$

$$A_{1345}=\begin{bmatrix}0&1&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&1&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}$$

$$w_4(M(A_{1345})=x_1x_3x_4x_5$$

$$A_{2345} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$w_4(M(A_{2345}) = x_2x_3x_4x_5$$

Thus, we have

$$w_4(M(A_{1234}) + w_4(M(A_{1235}) + w_4(M(A_{1245}) + w_4(M(A_{2345}) = x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5 = w_4(M(A))$$

6. Conclusion and Future Scope

The Stiefel-Whitney classes of the real Bott manifolds are square free homogeneous polynomials (all terms having the same degree). The number of the elementary components of the Bott Matrices, when decomposed, coincides with the number of terms in each Stiefel-Whitney class and is given by the combinatorial formula

$C_i^r = \frac{r!}{i!(r-i)!}$ Where r is the number of non-zero rows (which is the same as the rank of the Bott matrix) and i is the index of the Stiefel-Whitney class. - for a 7×7 Bott matrix, the rank r = 5. We reserve the Stiefel-Whitney classes of the manifolds M_n for $n = 18$ for further scope. Hence, we have table below for rank 5 and 8 respectively.

Table 1. A 7×7 Bott matrix, for rank, r = 5.

$w_i(M_r(A))$	No of terms
w_0	1
w_1	5
w_2	10
w_3	10
w_4	5
w_5	1
w_6	0
w_7	0

The ranks of the Bott Matrices are the same as the dimensions of the real Bott manifolds. A Stiefel-Whitney class of the real Bott manifolds vanishes whenever the class index is greater than the rank of the Bott matrix.

For a 10×10 Bott matrix, the rank is 8. Hence, we have the following table.

Table 2. A 10 × 10 Bott matrix, for rank, r = 8.

$w_i(M_r(A))$	No of terms
w_0	1
w_1	8
w_2	28
w_3	56
w_4	70
w_5	56
w_6	28
w_7	8
w_8	1
w_9	0

Conflict of Interest
No conflicts of interest associated with this study.

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