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Research Article

Hybrid of Semi-Analytical Methods for the Solution of Fractional Fredholm Integro-Differential Equations

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Abstract— This seminar introduces a hybrid approach combining the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM) to enhance the performance of ADM (Ansari & Ahmad, 2023) in solving fractional integro-differential equations. The primary objective of the research is to eliminate the use of Lagrange's multiplier in ADM and the Adomian polynomial in VIM. A new scheme is proposed, offering series solutions for both integer and non-integer orders. The proposed method demonstrates superiority over its counterparts, and several examples are presented to validate its effectiveness.

Keywords— Fractional integro-differential equation, Adomian decomposition method, Variational Iteration method

1. Introduction

Fractional Fredholm integro-differential equations represent a crucial class of mathematical models with broad applications across scientific and engineering fields [1]. These equations, characterized by non-local and non-integer-order derivatives, present significant challenges in deriving analytical solutions, which calls for the development of novel methodologies. Researchers have recognized the need for versatile and accurate solution strategies, leading to the exploration of various techniques. This study focuses on the creation of a hybrid method that combines the Variational Iteration Method (VIM) and the Modified Decomposition Method (MDM) to address the complexities associated with fractional Fredholm integro-differential equations.

The importance of this study lies in its potential to advance the field of applied mathematics and computational science by addressing the inherent challenges of solving fractional Fredholm integro-differential equations. These equations are pivotal in modeling complex systems in areas such as fluid dynamics, control theory, and biological processes, where traditional integer-order models fall short in capturing the nuances of real-world phenomena. Despite their widespread applications, the difficulty in obtaining analytical solutions for these equations has limited their practical utilization. By developing a hybrid method that integrates the strengths of the Variational Iteration Method (VIM) and the Modified Decomposition Method (MDM), this study not only enhances the accuracy and efficiency of existing solution techniques but also provides a robust framework for future research. The proposed approach has the potential to bridge the gap between theoretical advancements and their practical applications, thus contributing significantly to the advancement of mathematical modeling and computational methods.

2. Related Work

The Variational Iteration Method (VIM) has gained popularity due to its effectiveness in solving differential equations, as it iteratively optimizes functionals to approach solutions [2, 3, 4]. Likewise, the Modified Adomian Decomposition Method (MADM) has proven successful in solving linear Fredholm integro-differential equations by systematically decomposing terms [5, 6]. Both methods have made substantial contributions to mathematical modeling and are widely applied across various scientific disciplines. However, existing methods for solving fractional Fredholm integro-differential equations have certain limitations, prompting the development of hybrid approaches [7, 5, 8, 9]. This research contributes to the growing interest in combining the strengths of different techniques to improve accuracy and efficiency. By integrating VIM and MADM, this study aims to provide an innovative approach to tackling the challenges of fractional Fredholm integro-differential equations, advancing the field of applied mathematics and computational modeling.

3. Theory/Calculation

3.1 The Hybrid Technique for Nonlinear Fractional Fredholm Integro-Differential Equations.

This section provides a comprehensive analysis of the nonlinear fractional integro-differential equation and the hybrid methodology proposed for its resolution.

$$D^{\alpha}u(x) = \left. \begin{array}{l} p(t)u(x) + g(x) + \\ \lambda \int_0^1 k(x,t)F[u(t)]dt \end{array} \right\} \quad (1)$$

and

$$\left. \begin{array}{l} D^{\alpha}u(x) = p(t)u(x) + g(x) \\ + \lambda \int_0^x k(x,t)F[u(t)]dt \end{array} \right\} \quad (2)$$

for $x \epsilon [0,1]$, with the initial conditions $u^i = \delta_i$, $i = 0, 1, 2, \ldots$

Where g, p and k are known functions, u(x) is the unknown function, and D^{α} represents the Caputo fractional differential operator of order α\alphaα. Equations (1) and (2) emerge in the mathematical modeling of diverse physical phenomena, including heat conduction in materials, as well as conduction, convection, and radiation problems [10].

Using the hybrid method, the correction functional is formulated by integrating the Variational Iteration Method (VIM) with the Adomian Decomposition Method (ADM) iterates as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{\infty} (D^{\alpha}u(x) - p(t)u(x) - g(x) - \lambda \int_0^1 (x,t)F(u(t))dt \right] dt \quad \ldots(3)$$

The correction functional is now derived using the Adomian recursive relations, expressed as follows:

$$u_{n+1}(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{\infty} \left(D^{\alpha}[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^1 (x,t)F(u_k(t))dt \right) \right] dt \quad \ldots(4)$$

$n \geq 0$.

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \ldots(5)$$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \left(D^{\alpha}[u_0(x)] - p(t)u_0(x) - g(t) - \lambda \int_0^1 (x,t)F(u_0(t))dt \right) \right] dt \quad \ldots(6)$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{1} \left(D^{\alpha}[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^1 (x,t)F(u_k(t))dt \right) \right] dt \quad \ldots(7)$$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{2} \left(D^{\alpha}[u_k(x)] \; p(t)u_k(x) - g(t) - \lambda \int_0^1 (x,t)F(u_k(t))dt \right) \right] dt \quad \ldots(8)$$

$$u_n(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{n-1} \left(D^{\alpha}[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^1 (x,t)F(u_k(t))dt \right) \right] dt \quad \ldots(9)$$

And the series solution is given as

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + u_1(x) + u_2(x) + u_3 \ldots \quad \ldots(10)$$

3.2 The Hybrid Technique for Nonlinear Fractional Volterra Integro-Differential Equations

To demonstrate the implementation of the hybrid approach, we examine Equation (2) subject to the specified initial conditions. Within the hybrid framework, the correction functional is initially constructed using the Variational Iteration Method (VIM) as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{\infty} (D^{\alpha}u(x) - p(t)u(x) - g(x) - \lambda \int_0^t (x,r)F(u(r))dr \right] dt \quad \ldots(11)$$

The correctional function is now represented using Adomian's recursive relations as follows:

$$u_{n+1}(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{\infty} \left(D^{\alpha}[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^t (x,r)F(u_k(r))dr \right) \right] dt \quad \ldots(12)$$

$n \geq 0$.

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \ldots(13)$$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \left(D^{\alpha}[u_0(x)] - p(t)u_0(x) - g(t) - \lambda \int_0^t (x,r)F(u_0(r))dr \right) \right] dt \quad \ldots(14)$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{1} \left(D^{\alpha}[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^t (x,r)F(u_k(r))dr \right) \right] dt \quad \ldots(15)$$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{2} \left(D^{\alpha}[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^t (x,r)F(u_k(r))dr \right) \right] dt \quad \ldots(16)$$

…

$$u_n(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{n-1} \left(D^{\alpha}[u_k(x)] - p(t)u_k(x) - g(t) - \lambda \int_0^t (x,r)F(u_k(r))dr \right) \right] dt \quad \ldots(17)$$

And the series solution is given as

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \ldots \quad \ldots(18)$$

3.3 The Hybrid Approach for Solving Linear Fractional Fredholm Integro-Differential Equations

This section examines the linear fractional integro-differential equation, where $F[u(t)]$ in Eqn. (1) becomes $u(t)$.

According to the hybrid method, we apply I^{α} to both sides of the Eqn. (1), we obtain;

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^{\alpha}\left[p(t)\sum_{k=0}^{\infty} u_k(t) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{m-1} u_k(t)\,dt\right] \quad \ldots(19)$$

Where

$$\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

It is derived from the specified initial condition.

Based on the above equation, the iterates are computed using the following recursive procedure:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

$$u_{n+1}(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{\infty} u_k(t) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{m-1} u_k(t)\,dt\right] \quad \ldots(20)$$

$n \geq 0.$

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

$$u_1(x) = I^{\alpha}\left[p(t)u_0(t) + g(x) + \lambda\int_0^1 (x,t)u_0(t)dt\right] \quad \ldots(21)$$

$$u_2(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{1} u_k(x) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{1} u_k(t)\,dt\right] \quad \ldots(22)$$

$$u_3(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{2} u_k(x) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{2} u_k(t)\,dt\right] \quad \ldots(23)$$

…

$$u_n(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{n-1} u_k(x) + g(x) + \lambda\int_0^1 (x,t)\sum_{k=0}^{n-1} u_k(t)\,dt\right] \quad \ldots(24)$$

And the series solution is given as

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots \quad \ldots(25)$$

3.4 The Hybrid Approach for Solving Linear Fractional Volterra Integro-Differential Equations

Finally, we apply the hybrid method to the linear fractional Volterra integro-differential equation. Specifically, we analyze Eq. (1) with $F(u(t)) = u(t)$, and with given initial conditions. According to the hybrid method, we first construct the correction functional by applying the integral operator I^{α} to both sides of Eq. (2) owing to the VIM and ADM iterates as follows:

Using the properties of fractional integrals and derivatives, we obtain:

$$D^{\alpha}f(x) = I^{m-\alpha}\left(\frac{d^m}{dx^m}f(x)\right) \quad \ldots(26)$$

$$I^{\alpha}D^{\alpha}u(x) = u(x) - \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \ldots(27)$$

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^{\alpha}\left[p(t)\sum_{k=0}^{\infty} u_k(t) + g(x) + \lambda\int_0^x (x,t)\sum_{k=0}^{m-1} u_k(t)\,dt\right] \quad \ldots(28)$$

Where

$$\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

Is obtained from the given initial condition

From the above equation the iterate are determined by the following recursive way

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \ldots(29)$$

$$u_{n+1}(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{\infty} u_k(t) + g(x) + \lambda\int_0^x (x,t)\sum_{k=0}^{m-1} u_k(t)\,dt\right] \quad \ldots(30)$$

$n \geq 0.$

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}$$

$$u_1(x) = I^{\alpha}\left[p(t)u_0(t) + g(x) + \lambda\int_0^x (x,t)u_0(t)dt\right] \quad \ldots(31)$$

$$u_2(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{1} u_k(x) + g(x) + \lambda\int_0^x (x,t)\sum_{k=0}^{1} u_k(t)\,dt\right] \quad \ldots(32)$$

$$u_3(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{2} u_k(x) + g(x) + \lambda\int_0^x (x,t)\sum_{k=0}^{2} u_k(t)\,dt\right] \quad \ldots(33)$$

…

$$u_n(x) = I^{\alpha}\left[p(t)\sum_{k=0}^{n-1} u_k(x) + g(x) + \lambda\int_0^x (x,t)\sum_{k=0}^{n-1} u_k(t)\,dt\right] \quad \ldots(34)$$

And the series solution is given as

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots \quad \ldots (35)$$

4. Experimental Method/Procedure/Design

4.1 Linear and Nonlinear FFIDE

To demonstrate the effectiveness of the hybrid method in solving fractional-order integro-differential equations, this section focuses on both linear and nonlinear first-order FFIDEs.

Example 1: We examine a linear fractional FIDE as discussed in [(Ansari & Ahmad, 2023); (Wang et al., 2021)].

$$D^\alpha u(x) = 1 - \frac{x}{3} + \int_0^1 x\,t u(t)dt\,, \quad x, \alpha \epsilon [0,1]. \quad \ldots (36)$$

With exact solution

$$u(x) = x$$

In view of Eqn. (19), the Eqn. (36) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^\alpha\left[1 - \frac{x}{3} + \int_0^1 xt\sum_{k=0}^{m-1} u_k(t)\,dt\right] \quad \ldots (37)$$

Now, we rewrite Eqn. (37) in Adomian recursive relations as follows:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = 0 + I^\alpha[1] = \frac{x^\alpha}{\Gamma(1+\alpha)} \quad \ldots (38)$$

$$u_{n+1}(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt\sum_{k=0}^{m-1} u_k(t)\,dt\right] \quad \ldots (39)$$

$n \geq 0.$

$$u_1(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xtu_0(t)dt\right] = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right)dt\right] = 0 \quad \ldots (40)$$

$$= I^\alpha\left[-\frac{x}{3} + \frac{x}{(\alpha+2)\Gamma(1+\alpha)}\right] = 0$$

$$u_2(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt\sum_{k=0}^{1} u_k(t)\,dt\right] = 0 \quad \ldots (41)$$

$$u_3(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt\sum_{k=0}^{2} u_k(t)\,dt\right] = 0$$

…

$$u_{n+1}(x) = 0\,, \quad n > 0$$

So that the analytic solution is

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) \cong \frac{x^\alpha}{\Gamma(1+\alpha)} + 0 + 0 + \cdots = \frac{x^\alpha}{\Gamma(1+\alpha)} \quad \ldots (43)$$

TABLE 1: Numerical Results for Different Cases α (Example 1)

X	EXACT	HYBRID (α=1)	HYBRID (α=0.9)	HYBRID (α=0.8)	HYBRID (α=0.7)
0	0	0	0	0	0
0.1	0.1	0.1	0.13089729	0.170165429	0.219588076
0.2	0.2	0.2	0.24426298	0.296275221	0.356721882
0.3	0.3	0.3	0.351835603	0.409796587	0.473798446
0.4	0.4	0.4	0.455810839	0.515845121	0.579496408
0.5	0.5	0.5	0.557190444	0.616662213	0.677466395
0.6	0.6	0.6	0.656548451	0.7134973	0.769687847
0.7	0.7	0.7	0.754256205	0.807141664	0.857387963
0.8	0.8	0.8	0.850573101	0.89813852	0.941394691
0.9	0.9	0.9	0.945690256	0.986882258	1.022300353
1	1	1	1.039754134	1.073671274	1.100547405

Table 1 shows the numerical solution for various values ofα. It is observed that the results obtained for α=1 coincides with the exact solution, whereas for other values of α, the approximate solutions are in good agreement with the exact solution.

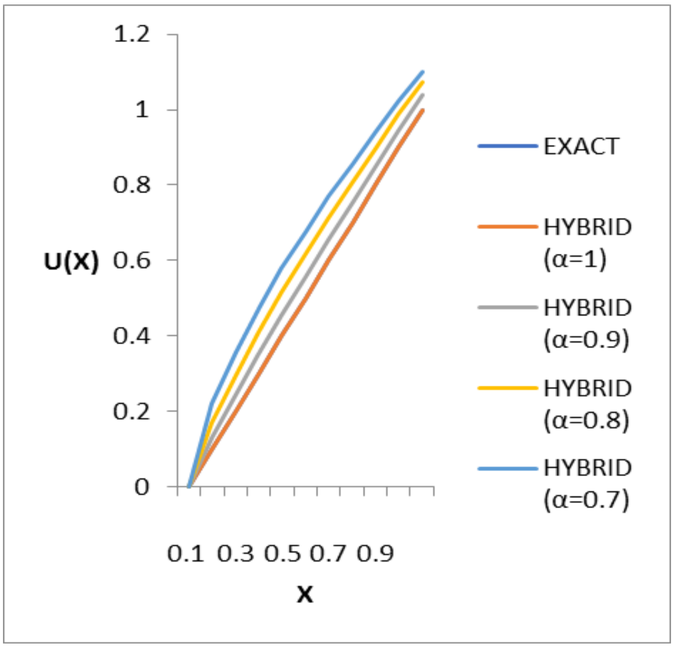

FIGURE 1: The plots of approximate solutions for fractional order α corresponding to example 1 are shown. From this figure, it is obvious that when α=1, the exact and approximate solution coincide.

Example 2: Here, we consider the nonlinear fractional FIDE. (Wang et al., 2021) and (Okai J. O, 2020) … (42)

$$D^\alpha u(x) = 1 - \frac{x}{4} + \int_0^1 x\,t[u(t)]^2 dt \quad \ldots (48)$$

For $0 < \alpha \leq 1$ and with initial condition

$$u(0) = 0.$$

The exact solution

$u(x) = x$

In view of Eqn. (33), the Eqn. (48) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^\alpha\left[1 - \frac{x}{4} + \int_0^1 xt\sum_{k=0}^{m-1}[u_k(t)]^2 dt\right] \quad \ldots(49)$$

Now, we rewrite Eqn. (49) in Adomian recursive relations as follows:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = 0 + I^\alpha[1] = \frac{x^\alpha}{\Gamma(1+\alpha)} \quad \ldots(50)$$

$$u_{n+1}(x) = I^\alpha\left[-\frac{x}{4} + \int_0^1 xt\sum_{k=0}^{m-1}[u_k(t)]^2 dt\right] \quad \ldots(51)$$

$n \geq 0.$

$$u_1(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt[u_0(t)]^2 dt\right] = 0 \quad \ldots(52)$$

$$u_2(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt\sum_{k=0}^{1}[u_k(t)]^2 dt\right] = 0 \quad \ldots(53)$$

$$u_3(x) = I^\alpha\left[-\frac{x}{3} + \int_0^1 xt\sum_{k=0}^{2}[u_k(t)]^2 dt\right] = 0 \quad \ldots(54)$$

…

$$u_{n+1}(x) = 0\ , \qquad n > 0$$

So that the analytic solution is

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) \cong \frac{x^\alpha}{\Gamma(1+\alpha)} + 0 + 0 + \cdots = \frac{x^\alpha}{\Gamma(1+\alpha)} \quad \ldots(55)$$

TABLE 2: The numerical results for various α (Example 2)

X	EXACT	HYBRID (α=1)	HYBRID (α=0.3)	HYBRID (α=0.6)	HYBRID (α=0.9)
0	0	0	0	0	0
0.1	0.1	0.1	0.558444121	0.281124038	0.13089729
0.2	0.2	0.2	0.687525359	0.426104362	0.24426298
0.3	0.3	0.3	0.776454658	0.543463943	0.351835603
0.4	0.4	0.4	0.846443005	0.64585344	0.455810839
0.5	0.5	0.5	0.905046148	0.738380103	0.557190444
0.6	0.6	0.6	0.955927814	0.823737302	0.656548451
0.7	0.7	0.7	1.001173014	0.903559604	0.754256205
0.8	0.8	0.8	1.042093577	0.978930759	0.850573101
0.9	0.9	0.9	1.079574147	1.050614737	0.945690256
1	1	1	1.114242509	1.119174954	1.039754134

The results for different values of α are depicted in Table 2. It is also observed the method produced the exact solution for the fractional order α=1, whereas the accuracy is decreased with decreasing the value of α.

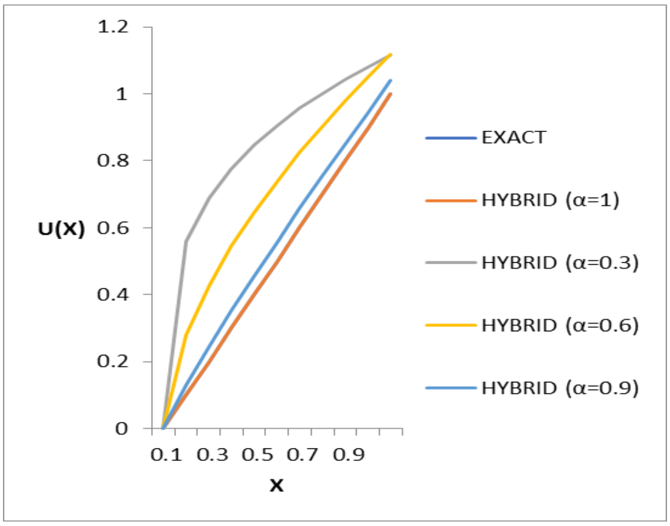

FIGURE 2: The plots of approximate solutions for fractional order α corresponding to example 2 are shown. From this figure, it is obvious that when α=1, the exact and approximate solution coincides.

TABLE 3: Comparison between the exact solution and solution obtain through the Hybrid method, LDM, LT and the ADM for Example 2 (α=1)

X	EXACT	HYBRID	LDM (Yang, 2013)	LT (Wang et al.,2021)	ADM (Okai J. O, 2020)
0	0	0	0	0	0
0.1	0.1	0.1	0.09875	0.0999	0.0996875
0.2	0.2	0.2	0.19500	0.1998	0.1987500
0.3	0.3	0.3	0.28875	0.2997	0.2971875
0.4	0.4	0.4	0.38000	0.3996	0.3950000
0.5	0.5	0.5	0.46875	0.4995	0.4921875
0.6	0.6	0.6	0.55500	0.5994	0.5887500
0.7	0.7	0.7	0.63875	0.6993	0.6846875
0.8	0.8	0.8	0.72000	0.7992	0.7800000
0.9	0.9	0.9	0.79875	0.8991	0.8746875
1	1	1	0.8750	0.999	0.9687500

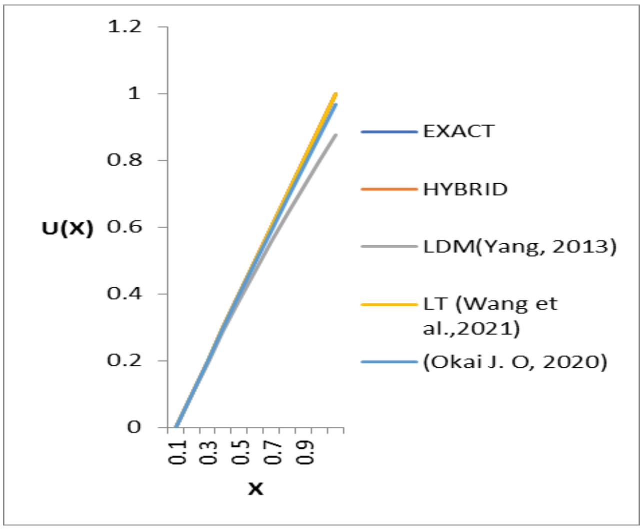

FIGURE 3: The plots of approximate solutions for fractional order α=1 corresponding to example 2 are shown for the Hybrid method, LDM, LT, ADM and the exact solution. From this figure, it is obvious that the Hybrid method is in a good agreement with the exact solution.

4.2 Linear and Nonlinear FVIDE.

In order to show the effectiveness of the Hybrid method for solving integro-differential equations of fractional order, in this section, we consider linear and nonlinear first order FVIDE.

Example 3: Here, we consider the linear fractional VIDE (Yang, 2013), (Ilejimi D.O, 2019) and (Rawashdeh, 2006):

$$D^{3/4}u(x) = \frac{6x^{9/4}}{\Gamma(13/4)} + \left(\frac{-x^2e^x}{5}\right)u(x) + \int_0^x e^x\, tu(t)dt \qquad \dots(56)$$

With the initial condition

$$u(0) = 0 \qquad \dots(57)$$

and the exact solution is $u(x) = x^3$.

In view of Eqn. (25), the Eqn. (56) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^{3/4}\left[\frac{6x^{9/4}}{\Gamma(13/4)} + \left(\frac{-x^2e^x}{5}\right)u(x) + \int_0^x e^x\, tu(t)dt\right] \qquad \dots(58)$$

Now, we rewrite Eqn. (58) in Adomian recursive relations as follows:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = 0 + I^{3/4}\left[\frac{6x^{9/4}}{\Gamma(13/4)}\right] = x^3 \qquad \dots(59)$$

$$u_{n+1}(x) = I^{3/4}\left[\left(\frac{-x^2e^x}{5}\right)\sum_{k=0}^{m-1} u_k(x) + \int_0^x e^x t\sum_{k=0}^{m-1} u_k(t)\, dt\right] \qquad \dots(60)$$

$n \geq 0$.

$$u_1(x) = I^{3/4}\left[\left(\frac{-x^2e^x}{5}\right)u_0(x) + \int_0^x e^x tu_0(x)dt\right] = 0 \qquad \dots(61)$$

$$u_2(x) = I^{3/4}\left[\left(\frac{-x^2e^x}{5}\right)\sum_{k=0}^{1} u_k(x) + \int_0^x e^x t\sum_{k=0}^{1} u_k(t)\, dt\right] = 0 \qquad \dots(62)$$

$$u_{n+1}(x) = 0 \;, \qquad n > 1$$

So that the analytic solution is

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) \cong x^3$$

TABLE 4: Comparison between the exact solution and solution obtain through the Hybrid method, Collocation method and the Picards method for Example 3 ($\alpha = 3/4$)

X	EXACT	HYBRID	Collocation method	Picard's
0	0	0	0	0
0.1	0.001	0.001	0.0008	0.00092
0.2	0.008	0.008	0.0064	0.00736
0.3	0.027	0.027	0.0216	0.02484
0.4	0.064	0.064	0.0512	0.05888
0.5	0.125	0.125	0.1	0.115
0.6	0.216	0.216	0.1728	0.19872
0.7	0.343	0.343	0.2744	0.31556
0.8	0.512	0.512	0.4096	0.47104
0.9	0.729	0.729	0.5832	0.67068
1	1	1	0.8	0.92

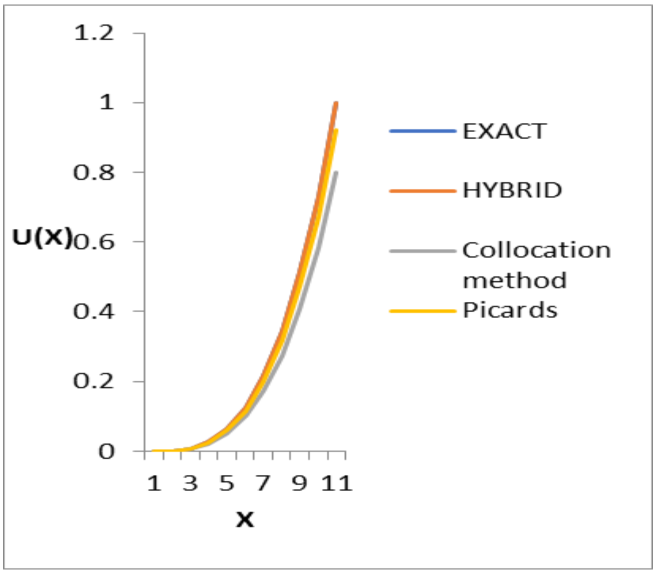

FIGURE 4: The plots of approximate solutions the Hybrid method, Collocation method, the Picards method and the exact solution for Example 3 are shown. From this figure, it is obvious that the Hybrid method is in a good agreement with the exact solution.

Example 4: Here, we consider the nonlinear FVIDE(Yang, 2013) and (Ilejimi D.O, 2019)

$$D^{6/5}u(x) = \frac{5x^{4/5}}{2\Gamma(4/5)} - \frac{x^9}{252} + \int_0^x (x-t)^2\,[u(t)]^3dt \qquad \dots(64)$$

With the initial condition

$$u(0) = u'(0) = 0$$

and the exact solution is $u(x) = x^2$.

In view of Eqn. (25), the Eqn. (64) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + I^{6/5}\left[\frac{5x^{4/5}}{2\Gamma(4/5)} - \frac{x^9}{252} + \int_0^x (x-t)^2[u(t)]^3dt\right] \qquad \dots(66)$$

Now, we rewrite Eqn. (66) in Adomian recursive relations as follows: $\dots(63)$

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = 0 + I^{6/5}\left[\frac{5x^{4/5}}{2\Gamma(4/5)}\right] = x^2 \qquad \dots(67)$$

$$u_{n+1}(x) = I^{6/5}\left[-\frac{x^9}{252} + \int_0^x (x-t)^2\sum_{k=0}^{m-1}[u_k(t)]^3\, dt\right] \qquad \dots(68)$$

$n \geq 0$.

$$u_1(x) = I^{6/5}\left[-\frac{x^9}{252} + \int_0^x (x-t)^2[u_0(t)]^3\, dt\right] = 0 \qquad \dots(69)$$

$$u_2(x) = I^{6/5}\left[-\frac{x^9}{252} + \int_0^x (x - t)^2 \sum_{k=0}^{1} [u_k(t)]^3 \, dt\right] = 0 \qquad \dots(70)$$

$$u_{n+1}(x) = 0 \ , \qquad n > 1$$

So that the analytic solution is

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) \cong x^2 \qquad \dots(71)$$

TABLE 5: Comparison between the exact solution and solution obtain through the Hybrid method, Collocation method and the Picards method for Example 4 ($\alpha = 6/5$)

X	EXACT	HYBRID	COLLOCATION METHOD	Picard's method
0	0	0	0	0
0.1	0.001	0.001	0.0099	0.0093
0.2	0.008	0.008	0.0396	0.0372
0.3	0.027	0.027	0.0891	0.0837
0.4	0.064	0.064	0.1584	0.1488
0.5	0.125	0.125	0.2475	0.2325
0.6	0.216	0.216	0.3564	0.3348
0.7	0.343	0.343	0.4851	0.4557
0.8	0.512	0.512	0.6336	0.5952
0.9	0.729	0.729	0.8019	0.7533
1	1	1	0.99	0.93

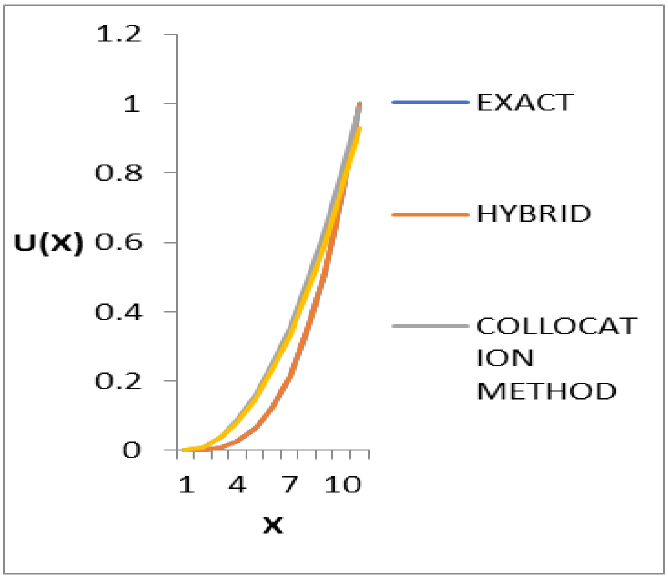

FIGURE 5: The plots of approximate solutions the Hybrid method, Collocation method, the Picards method and the exact solution for Example 4 are shown. From this figure, it is obvious that the Hybrid method is in a good agreement with the exact solution.

5. Results and Discussion

Based on the numerical analysis performed, we observed that the Hybrid method converges well with the exact solution when **α=1 (Example** 1 and 2, see Table 1 and 2), the comparison shows that as **α** →1, the approximate solution tends to x, which is the exact solution of the two equations. The computational procedures are easy to understand and gives a rapidly convergence series solution when compared with LDM, LT, ADM. In Example 3 and 4 the hybrid yields the exact solution in just two iterations (See Table 4 and 5).

The hybrid method demonstrates a significant improvement over established techniques such as the Local Decomposition Method (LDM), Laplace Transform Method (LT), and Adomian Decomposition Method (ADM) in solving fractional Fredholm integro-differential equations. While these traditional methods often require numerous iterations and complex computational processes to achieve acceptable accuracy, the hybrid method achieves a rapidly converging series solution with remarkable efficiency. Its ability to deliver high-precision results with fewer iterations reduces computational time and resource usage, making it an attractive choice for practical applications. This advantage is particularly critical when dealing with complex equations that demand significant computational effort or when time-sensitive solutions are required.

A noteworthy demonstration of the hybrid method's efficiency is observed in Examples 3 and 4 (Tables 4 and 5), where the exact solutions are obtained in just two iterations. This rapid convergence not only highlights the robustness of the hybrid approach but also sets it apart as a method capable of handling a wide range of fractional equations with varying levels of complexity. By minimizing computational overhead while maintaining accuracy, the hybrid method addresses the limitations of existing techniques and bridges the gap between theoretical advancements and their practical implementation. Its ability to consistently produce accurate and efficient solutions positions it as a valuable tool for researchers and practitioners, thereby making a significant contribution to the field of mathematical modeling and applied sciences.

6. Conclusion and Future Scope

From the present work, the following conclusions may be drawn:

1. In these examples, we solved linear fractional integro-differential equations, the Hybrid method gives more accurate results than other methods studied.
2. The work is completely void of linearization.
3. In solving nonlinear fractional integro-differential equations, the Hybrid method gives more accurate results than the collocation method and Adomian decomposition method.
4. From the given nonlinear Examples, it is obvious that the work was carried out without the Adomian's polynomials and the Langrage's multiplier.
5. The results obtained are in good agreement with that of exact solution, and therefore as α approach the integer order under consideration the Hybrid method tends to the exact solution, the Hybrid method described is precise and accurate.

Also, from this work, we can recommend the following open problems for future work:

1. Studying the behaviour and solution of real-life problems, which may be modeled as fractional integro-differential equations.
2. Studying fractional integro-differential equations resulting from other types of fractional integro-differential operators, linear or nonlinear.
3. Study the singular fractional integro-differential equations with the aid of the Hybrid method.
4. Using the Hybrid method for solving other equations such as the heat equations (PDE).

Data Availability

Not applicable.

Conflict of Interest

All authors declare that they do not have any conflict of interest.

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Authors' Contributions

All authors reviewed and edited the manuscript and approved the final version of the manuscript

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