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Research Paper

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Some Common Fixed Point Theorems in 2-Metric Spaces for Contractive Type Conditions

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***Abstract* :** Our aim of this paper is to find some common fixed point theorems in 2-metric spaces by using Nesic type contractive condition, which are generalizations of various known results.

Keywords: *Fixed point, 2-metric space,Contraction, Nesic type.*
AMS Subject Cassification: 47H10, 54H25

I. INTRODUCTION

There have been a number of generalizations of a metric space. One such generalization is 2-metric space. The concept of a 2- metric space was introduced by G$\ddot{a}$hler in [2], having the area of a triangle in R^3 as the inspirational example. Similarly, several fixed point results were obtained for mappings in such spaces. After the introduction of the concept of 2-metric space, many authors have established an analogue of Banach's contraction principle in 2-metric space. Also the researchers like Iseki [3], Lal and Singh [5], Nesic [7] etc. have proved many fixed points in this space.
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II. MATHEMATICAL PRELIMINARIES

Now we will give some basic definitions and well known results that will be needed in the sequel.

Definition 1.1{[3][4]}:
Let X be a non-empty set and R_+ is the set of all non negative reals. Let
d: $X \times X \times X \rightarrow R_+$. If for all x, y , z and u in X. We have
,
(d_1) d(x, y, z) = 0 if at least two of x, y, z are equal,
(d_2) for all $x \neq y$, there exists a point z in X such that $d(x, y, z) \neq 0$,
(d_3) d(x, y, z) = d(x, z , y) = d(y, z, x) and so on.
(d_4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$, $\forall$ x, y, z and u in X

Then d is called a 2- metric on X and the pair (X, d) is called a 2-metric space.

Definition 1.2:
A sequence $\{x_n\}_{n\in N}$ in a 2-metric space (X, d) is called a Cauchy sequence [7] when $d(x_n,x_m,a)\rightarrow 0$ as $n,m \rightarrow \infty$.

Definition 1.3:
A sequence $\{x_n\}_{n\in N}$ in a 2-metric space (X, d) is said to be convergent [7] to an element x in X when $d(x_n,x,a)\rightarrow 0$ as $n\rightarrow\infty$ for all $a\in X$.

Definition 1.4:
A sequence $\{x_n\}_{n\in N}$ in a 2-metric space (X, d) is said a Cauchy sequence if $d(x_n,x_m,a)\rightarrow 0$ as n, $m\rightarrow\infty$ for all $a\in X$.

Definition 1.5:
A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 1.6:
A 2-metric space (X,d) is said to be bounded if there exist a constant K such that
$d(x, y, z) \leq K$ for all x, y, $z \in X$.

Definition 1.7:
A mapping f in 2-metric space (X,d) is said to be orbitally continuous if for all z in X,
$d(f^n x,u,z) \rightarrow 0$ as $n\rightarrow\infty$ implies that

$d(ff^n x,fu,z) \rightarrow 0$ as $n\rightarrow\infty$.

Definition 1.8: A mapping G form a 2- metric space (X,d) into itself is said to be sequentially continuous at a point $x \in X$ if every sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(x_n, x, z) = 0 \text{ for all } z \in X$$
$$\lim_{n\to\infty} d(Gx_n, Gx, z) = 0$$

For every convergent sequence in a 2-metric space is a Cauchy sequence.

Lemma 1.1: Let $\{x_n\}$ be a sequence in complete 2-metric space in X if there exists $h \in [0,1]$ such that

$$d(x_n,x_{n+1}, a) \leq h d(x_{n-1},x_n, a)$$

for all $a \in X$ then $\{x_n\}$ converges to a point in X.

III. MAIN RESULTS

Theorem 1: Let f and g be an orbitally continuous self – map from complete 2-metric space X into itself, f and g satisfies –

$[1+pd(x,y,a)]d(fx , gy ,a) \leq p \max \{ d(x, fx ,a) .d(y, gy ,a) , d(x, gy ,a). d(y, fx ,a)\}$
$+ q \max \{ d(x,y,a), d(x, fx ,a) , d(y, gy ,a) \}$ …………(1.1)

for all x,y and $a \in X$ and $p \geq 0$, $0 < q < 1$, then for each $x \in X$, the sequence $\{ x_n \}$ converges to a common fixed point of f and g.

Proof:

Let $x_0 \in X$ be an arbitrary point and we define $\{ x_n \}$ as $x_1=f(x_0)$, $x_2=g(x_1)$, $x_3=f(x_2)$, $x_4=g(x_3)$, ……………, $x_{2n}=g(x_{2n-1})$, $x_{2n+1}=f(x_{2n})$ …………….(1.2)

Suppose $x_{2n} \neq x_{2n+1}$, for every n=0, 1, 2, ………… then

$[1+pd(x_{2n},x_{2n+1},a)]d(f(x_{2n}) , g(x_{2n+1}) ,a)$
$\leq p \max \{ d(x_{2n}, f(x_{2n}) ,a) .d(x_{2n+1}, g(x_{2n+1}) ,a) , d(x_{2n}, g(x_{2n+1}) ,a). d(x_{2n+1}, f(x_{2n}) ,a)\}$
$+ q \max \{ d(x_{2n},x_{2n+1},a), d(x_{2n}, f(x_{2n}) ,a) , d(x_{2n+1},g(x_{2n+1}) ,a) \}$

which implies-

$[1+pd(x_{2n},x_{2n+1},a)]d(x_{2n+1} , x_{2n+2} ,a)$
$\leq p \max \{ d(x_{2n},x_{2n+1} ,a) .d(x_{2n+1}, x_{2n+2} ,a) , d(x_{2n}, x_{2n+2} ,a).d(x_{2n+1},x_{2n+1} ,a)\}$
$+ q \max \{d(x_{2n},x_{2n+1},a), d(x_{2n}, x_{2n+1} ,a) , d(x_{2n+1}, x_{2n+2} ,a) \}$

I i.e., $d(x_{2n+1} , x_{2n+2} ,a) + pd(x_{2n},x_{2n+1},a).d(x_{2n+1} , x_{2n+2} ,a)$
$\leq p \max \{d(x_{2n}, x_{2n+1} ,a) .d(x_{2n+1}, x_{2n+2} ,a), 0 \}$
$+ q \max \{ d(x_{2n},x_{2n+1},a), d(x_{2n}, x_{2n+1} ,a) , d(x_{2n+1},x_{2n+2} ,a) \}$

i.e, $d(x_{2n+1} , x_{2n+2} ,a) + pd(x_{2n},x_{2n+1},a).d(x_{2n+1} , x_{2n+2} ,a)$
$\leq p\, d(x_{2n}, x_{2n+1} ,a) .d(x_{2n+1}, x_{2n+2} ,a)$
$+ q \max \{ d(x_{2n},x_{2n+1},a), d(x_{2n}, x_{2n+1} ,a) , d(x_{2n+1}, x_{2n+2} ,a) \}$

i.e., $d(x_{2n+1} , x_{2n+2} ,a) \leq q \max \{ d(x_{2n},x_{2n+1},a), d(x_{2n+1}, x_{2n+2} ,a) \}$ ……………………(1.3)

Case 1: If $d(x_{2n+1}, x_{2n+2} ,a)$ is maximum, then

$d(x_{2n+1} , x_{2n+2} ,a) \leq q\, d(x_{2n+1}, x_{2n+2} ,a) \Rightarrow (1- q)\, d(x_{2n+1}, x_{2n+2} ,a) \leq 0$

$\Rightarrow (1-q) \leq 0$ [Since $d(x_{2n+1}, x_{2n+2} ,a) \neq 0$]

$\Rightarrow q >1$ which is contradiction, since $q<1$.

Case 2: So, in (1.3) $d(x_{2n},x_{2n+1},a)$ is maximum.

Now from(1.3) we get

$d(x_{2n+1} , x_{2n+2} ,a) \leq q\, d(x_{2n},x_{2n+1},a)$ ………………(1.4)

Also $q<1 \Rightarrow (1+q) <2 \Rightarrow 1< (2-q) \Rightarrow 1> \frac{1}{(2-q)}$

$\Rightarrow q > \frac{q}{(2-q)}$

Now, from (1.4) we get $d(x_{2n+1} , x_{2n+2} ,a) \leq \frac{q}{(2-q)} d(x_{2n},x_{2n+1},a)$

Therefore, $d(x_{2n+1} , x_{2n+2} ,a) \leq h\, d(x_{2n},x_{2n+1},a)$, putting $h=\frac{q}{(2-q)}$ ………………(1.5)

Now (1.5) hold for all $a \in X$. Hence in view of lemma Nesic type contractive definition, the sequence $\{x_n\}$ converges to some common fixed point $u \in X$, then for all $a \in X$, since f is orbitally continuous, we have

$\lim_{n\to\infty} d(f(f^{2n}(x_0),f(u),a)=0$

$\Rightarrow \lim_{n\to\infty} d(f^{2n+1}(x_0),f(u),a)=0.$

From the definition of 2-metric space,

$d(u,f(u),a) \leq d(u,f(u),f^{2n+1}(x_0)) + d(u,f^{2n+1}(x_0),f(u)) + d(f^{2n+1}(x_0),f(u),a)$

Which tends to zero as $n \to \infty$.

Consequently, $d(u,f(u),a) = 0 \Rightarrow f(u)=u$.

Similarly, for the mapping g, we have

$d(u,g(u),a) = 0 \Rightarrow g(u)=u$.

Therefore $f(u)=g(u)=u$. Hence f and g have common fixed point.

We now prove that this common fixed point is unique. For the uniqueness of u, suppose $v \in X$ be another common fixed point of f and g such that $v \neq u$.

Hence there exists a point $a \in X$ such that $d(u,v,a) \neq 0$ then from (1.1) we have,

$[1+pd(u,v,a)]d(f(u),g(v),a) \leq p \max \{ d(u, f_u ,a) .d(v, g_v ,a) , d(u, g_v ,a). d(v, f_u ,a)\}$
$+ q \max \{ d(u,v,a), d(u, f_u ,a) , d(v, g_v ,a) \}$

$[1+pd(u,v,a)]d(u,v,a) \leq p \max \{ d(u,u,a) .d(v,v,a), d(u,v,a). d(v,u,a)\} + q \max \{ d(u,v,a), d(u,u,a), d(v,v,a) \}$

i.e., $d(u,v,a) \leq q\, d(u,v,a) \Rightarrow d(u,v,a) < 0$, since $q<1$, which is a contradiction.

Hence, $d(u,v,a)=0$ which implies that $u=v$.

Similarly, for the mapping g, we have $d(u,v,a)=0$.

Therefore the sequence $\{ x_n \}$ converges to a common fixed point of f and g.

We now have the following corollary :

Corollary 1.1: Let f and g be an orbitally continuous self –map from complete 2-metric space X into itself, f and g satisfies –

$[1+pd(x,y,a)]d(fx, gy, a) \leq p \max \{ d(x, fx, a) .d(y, gy, a), d(x, gy, a). d(y, fx, a)\}$ ………….(1.6)

for all x,y and a $\in$X and $0< q <1$, then for each $x \in X$, the sequence $\{ x_n \}$ convergence to a common fixed point of f and g.

Proof: If we put p = 0 in theorem 1, the we get our required result.

Theorem 2: Let I and J be an orbitally continuous self-mapping from complete 2-metri space X into itself, I and J satisfies-

$d(Ix,Jy, \leq a_1\{d(x,Ix,z)d(y,Jy,z)+d(x,Jy,z)d(y,Ix,z)\} +a_2\{d(x,y,z)d(y,Ix,z)+d(x,Ix,z)d(y,Jy,z)\}+ a_3d^2(y,Jy,z)$ …………..(2.1)

For all x,y and z $\in$X and $a_1,a_2,a_3 \geq 0$, with $0<a_1+a_2+a_3<1$. Then I and J have a common unique fixed point.

Proof: Fix any $x \in X$. Define $x_0=x$ and let $x_1 \in Ix_0$, $x_2 \in Jx_1$ such that $x_{2n+1}=Ix_{2n}$, $x_{2n+2}=Jx_{2n+1}$.

Now, we have

$d(Ix_{2n},Jx_{2n+1},z)^2 \leq a_1\{d(x_{2n},Ix_{2n},z)d(x_{2n+1},Jx_{2n+1},z) +d(x_{2n},Jx_{2n+1},z)d(x_{2n+1},Ix_{2n},z)\}+ a_2\{d(x_{2n},x_{2n+1},z)d(x_{2n+1},Ix_{2n},z) +d(x_{2n},Ix_{2n},z)d(x_{2n+1},Jx_{2n+1},z)\}+a_3d^2(x_{2n+1},Jx_{2n+1},z)$

$\leq a_1\{d(x_{2n},x_{2n+1},z)d(x_{2n+1},x_{2n+2},z) +d(x_{2n},x_{2n+2},z)d(x_{2n+1},x_{2n+1},z)\}+ a_2\{d(x_{2n},x_{2n+1},z)d(x_{2n+1},x_{2n+1},z)+ d(x_{2n},x_{2n+1},z)d(x_{2n+1},x_{2n+2},z)\}+a_3d^2(x_{2n+1},x_{2n+2},z)$

$\Rightarrow d(x_{2n+1},x_{2n+2},z)^2 \leq a_1d(x_{2n},x_{2n+1},z)d(x_{2n+1},x_{2n+2},z)+a_2d(x_{2n},x_{2n+1},z)d(x_{2n+1},x_{2n+2},z)+a_3d^2(x_{2n+1},x_{2n+2},z)$

i.e, $d(x_{2n+1},x_{2n+2},z) \leq a_1d(x_{2n+1},x_{2n+2},z) +a_2d(x_{2n},x_{2n+1},z) +a_3d(x_{2n+1},x_{2n+2},z)$

i.e, $(1-a_3)\, d(x_{2n+1},x_{2n+2},z) \leq (a_1+a_2)d(x_{2n},x_{2n+1},z)$

i.e, $d(x_{2n+1},x_{2n+2},z) \leq \frac{(a_1+a_2)}{(1-a_3)} d(x_{2n},x_{2n+1},z)$

i.e, $d(x_{2n+1},x_{2n+2},z) \leq h d(x_{2n},x_{2n+1},z)$,

[putting $h=\frac{(a_1+a_2)}{(1-a_3)}$] ……………(2.2)

Therefore the above equation holds for all $z \in X$. Hence in view of lemma[1.1] Nesic type contractive definition, the sequence $\{ x_n \}$ converges to some common fixed point $u \in X$, then for all $z \in X$ we have –

$\lim_{n\to\infty} d((x_{2n}),u,a) =0 \Rightarrow \lim_{n\to\infty} d(I^{2n}(x_0),u,a)=0$

Since f is orbitally continuous, we have $\lim_{n\to\infty} d(I(I^{2n}(x_0),I(u),a)=0$

$\Rightarrow \lim_{n\to\infty} d(I^{2n+1}(x_0),I(u),a)=0$.

Now from the definition of 2-metric space,

$d(u,I(u),a) \leq d(u,I(u),I^{2n+1}(x_0)) + d(u,I^{2n+1}(x_0),I(u)) + d(I^{2n+1}(x_0),I(u),a)$

Which tends to zero as $n \to \infty$.

Consequently, $d(u,I(u),a) = 0 \Rightarrow I(u)=u$.

Similarly, for the mapping J, we have $d(u,J(u),a) = 0 \Rightarrow J(u)=u$.

Therefore $I(u)=J(u)=u$. Hence I and J have common fixed point.

We now prove the uniqueness of the common fixed point. For that case we first show that u is a unique fixed point. We suppose that $v \in X$ be another common fixed point of I and J such that $v \neq u$. Hence there exists a point $z \in X$ such that $d(u,v,z) \neq 0$ then we have,

$d(Iu,Jv,z)^2 \leq a_1\{d(u,Iv,z)d(u,Jv,z)+d(u,Jv,z)d(v,Iu,z)\} + a_2\{d(u,v,z)d(v,Iu,z)+d(u,Iu,z)d(v,Jv,z)\} +a_3d^2(v,Jv,z)$

i.e, $d(u,v,z)^2 \leq a_1\{d(u,v,z)d(u,v,z)+d(u,v,z)d(v,u,z)\} + a_2\{d(u,v,z)d(v,u,z)+d(u,u,z)d(v,v,z)\} +a_3d^2(v,v,z)$

i.e, $d(u,v,z) \leq a_1d(u,v,z) + a_2d(u,v,z)$

i.e, $d(u,v,z) \leq (a_1+ a_2)d(u,v,z)$

$\Rightarrow d(u,v,z) < d(u,v,z)$, ………(2.3)

which is a contradiction. Hence $d(u,v,z)=0$ that implies $u=v$.

∴The sequence $\{x_n\}$ convergences to a common fixed point of I , J.

We have the following corollaries:

Corollary 2.1: Let I be an orbitally continuous self-mapping from complete 2-metri space X into itself, I satisfies the condition-

$d(Ix,Iy,z)^2 \leq a_1\{d(x,Ix,z)d(y,Iy,z)+d(x,Iy,z)d(y,Ix,z)\}+a_2\{d(x,y,z)d(y,Ix,z)+d(x,Ix,z)d(y,Iy,z)\}$ …………..(2.4)

For all x,y and z $\in$X and $a_1,a_2,a_3 \geq 0$, with $0<a_1+a_2 + a_3<1$. Then I has a unique fixed point.

Proof: If we put I=J in theorem 2, then we get our result.

Corollary 2.2: Let I and J be an orbitally continuous self-mapping from complete 2-metri space X into itself, I and J satisfies-

$d(Ix,Jy,$ $\leq$ $a_1\{d(x,Ix,z)d(y,Jy,z)+d(x,Jy,z)d(y,Ix,z)\}$ $+a_2\{d(x,y,z)d(y,Ix,z)+d(x,Ix,z)d(y,Jy,z)\}$ …………..(2.5)

For all x,y and z $\in X$ and $a_1,a_2,a_3 \geq 0$, with $0<a_1+a_2<1$. Then I and J have a common unique fixed point.

Proof: If we put $a_3=0$ in theorem 2, then we have our result.

Corollary 2.3: Let I and J be an orbitally continuous self-mapping from complete 2-metri space X into itself, I and J satisfies-

$d(Ix,Jy,z)^2 \leq a_1\{d(x,Ix,z)d(y,Jy,z)+d(x,Jy,z)d(y,Ix,z)\} + a_3d^2(y,Jy,z)$ …………..(2.6)

For all x,y and z $\in X$ and $a_1,a_3 \geq 0$, with $0<a_1+a_3<1$. Then I and J have a common unique fixed point.

Proof: If we put $a_2=0$ in theorem 2, then we have our result.

Corollary 2.4: Let I and J be an orbitally continuous self-mapping from complete 2-metri space X into itself, I and J satisfies-

$d(Ix,Jy,z)^2 \leq a_2\{d(x,y,z)d(y,Ix,z)+d(x,Ix,z)d(y,Jy,z)\} + a_3d^2(y,Jy,z)$ …………..(2.7)

For all x,y and z $\in X$ and $a_2,a_3 \geq 0$, with $0<a_1+a_2+a_3<1$. Then I and J have a common unique fixed point.

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