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Research Paper

Article on the applications of the Jensen's Inequality in Alternative proofs and Problems Part-2

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Abstract—As it is mentioned in the title that through this article there, we will deal with various problems of the classic Jensen's Inequality more clearly Weighted Jensen's Inequality of both kinds of convex function and concave function. Previously we have done the work on applications of Jensen inequality and their application. Under the title applications, we will mainly discuss two aspects of it which are the application of Jensen inequality in proving well-known inequalities such as Köber's inequality, Jordan's Inequality and Stečkin's inequality and another aspect is the application of Jensen's Inequality in solving and simplifying problems proposals published in different journals on the basis of it there are example problems for readers.

Keywords— Convex Function, Concave Function, AM-GM Inequality, Inequality, Jensen's Inequality,

1. Introduction

Jensen was the first who gave an inequality which is a milestone now in the field in calculus, algebra and in probability theory somehow today in every issue one can easily found some lines in regards of this inequality. If we consider some reputed journal and magazine that contain problems in problem and solution section then we can easily find use of Jensen's Inequality in simplifying given problem or either proving given inequality in the last article of this series Toyesh has proved AM-GM inequality, GM-HM inequality, Cauchy-Schwarz Inequality, Nesbitt's inequality, weighted AM-GM inequality, Power Mean Inequality using Jensen Inequality with realizing about the wideness in the scope of Jensen's Inequality it becomes necessary to write another article and carry on this series based on the Jensen's Inequality respectively.

Journals like SSMJ, Pentagon, Parabola, RMM, AMM, MR, FQ etc. consider well established problems based upon the concept and application of Jensen's Inequality. Before coming on its application, we are used to define this inequality, this time we are introducing Weighted Jensen's Inequality of both kinds. So,
Jensen's Inequality If the function is convex for all $x_1, x_2, \cdots, x_n$ then.

$$\sum_{j=1}^{n} \lambda_j f(x_j) \geq f\left(\sum_{j=1}^{n} \lambda_j x_j\right)$$

Where $\sum_{j=1}^{n} \lambda_j = 1$.
And If the function is concave for all $x_1, x_2, \cdots, x_n$ then

$$\sum_{j=1}^{n} \lambda_j f(x_j) \leq f\left(\sum_{j=1}^{n} \lambda_j x_j\right)$$

Where $\sum_{j=1}^{n} \lambda_j = 1$.
Some mathematicians also prefer to write it as
If the function is convex for all $x_1, x_2, \cdots, x_n$ then

$$\frac{\sum_{j=1}^{n} y_j f(x_j)}{\sum_{j=1}^{n} y_j} \geq f\left(\frac{\sum_{j=1}^{n} y_j x_j}{\sum_{j=1}^{n} y_j}\right)$$

Where $y_1, y_2, \cdots, y_n$ belongs to the positive real numbers.

If the function is convex for all $x_1, x_2, \cdots, x_n$ then

$$\frac{\sum_{j=1}^{n} y_j f(x_j)}{\sum_{j=1}^{n} y_j} \leq f\left(\frac{\sum_{j=1}^{n} y_j x_j}{\sum_{j=1}^{n} y_j}\right)$$

Where $y_1, y_2, \cdots, y_n$ belongs to the positive real numbers

This article contains several sections are as follows Section 1 contains the introduction of Jensen's inequality , Section 2 contain the related work of application of Jensen's inequality, Section 3 contain the some measures of alternative proofs to well know inequality, Section 4 contain the essential steps of solving problems, section 5 explain the problems related to Jensen's inequality, Section 6 Conclusion and further scope it tells about the further on the same topic what readers can try to do.

2. Related Work

Last year Toyesh Prakash Sharma wrote an article with dealing some applications of Jensen's Inequality entitled "Article on the applications of the Jensen's Inequality in Alternative Proofs and Problems" [1] likewise through this article some other alternative proofs are given, and some new example problems and problems are given.

3. Alternative Proofs to Well Know Inequalities

1. **Jordan's Inequality**

   In an article titled "Two generalizations for Jordan's inequality" by Dorin Marghidanu and published in Romanian Mathematical Magazine. With the generalizations he used to prove Jordan's Inequality with the help of Jensen's Inequality here we are providing his proof.

Jordan's Inequality For $x \in \left[0, \frac{\pi}{2}\right]$

$$\frac{2}{\pi} x \leq \sin x < x$$

Proof

From Jensen's inequalioty for concave function We have

$$f[(1-\lambda)x_1 + \lambda x_2] \geq (1-\lambda)f(x_1) + \lambda f(x_2)$$

Suppose $x = (1-\lambda)x_1 + \lambda x_2$

$x = (1-\lambda)x_1 + \lambda x_2$

$\Rightarrow x = x_1 - \lambda x_1 + \lambda x_2$

$\Rightarrow x - x_1 = \lambda(x_2 - x_1)$

$\Rightarrow \lambda = \dfrac{x - x_1}{x_2 - x_1}$

And Hence

$1 - \lambda = \dfrac{x_2 - x}{x_2 - x_1}$

Then inequality becomes

$$f(x) \geq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

Now suppose $f(x) = \sin x$ that is a concave function under the interval $x \in \left[0, \frac{\pi}{2}\right]$ then

$$\sin x \geq \frac{\frac{\pi}{2} - x}{\frac{\pi}{2} - 0} \sin(0) + \frac{x - 0}{\frac{\pi}{2} - 0} \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \sin x \geq \frac{2}{\pi} x$$

And it is well known well popular that $\sin x < x$ for $x \in (0, \pi)$ then with combining both we can say that

$$\frac{2}{\pi} x \leq \sin x < x$$

Hence prove.

2. **Köber's inequality**

In an article titled "Generalizations for Köber's inequality and Stečkin's inequality" by Dorin Marghidanu and published in Romanian Mathematical Magazine. With the generalizations he used to prove Köber's inequality with the help of Jensen's Inequality here we are providing his proof.

Köber's inequality For $x \in \left(0, \frac{\pi}{2}\right)$

$$1 - \frac{2}{\pi} x \leq \cos x < \frac{\pi}{2} - x$$

Proof

From Jensen's inequalioty for concave function We have

We have

$$f[(1-\lambda)x_1 + \lambda x_2] \geq (1-\lambda) f(x_1) + \lambda f(x_2)$$

Suppose $x = (1-\lambda)x_1 + \lambda x_2$

$x = (1-\lambda)x_1 + \lambda x_2$

$\Rightarrow x = x_1 - \lambda x_1 + \lambda x_2$

$\Rightarrow x - x_1 = \lambda(x_2 - x_1)$

$\Rightarrow \lambda = \frac{x - x_1}{x_2 - x_1}$

And Hence

$1 - \lambda = \frac{x_2 - x}{x_2 - x_1}$

Then inequality becomes

$$f(x) \geq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

Now suppose $f(x) = \cos x$ that is a concave function under the interval $x \in \left[0, \frac{\pi}{2}\right]$ then

$$\cos x \geq \frac{\frac{\pi}{2} - x}{\frac{\pi}{2} - 0} \cos(0) + \frac{x - 0}{\frac{\pi}{2} - 0} \cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \cos x \geq 1 - \frac{2}{\pi} x$$

And it is well known well popular that $\cos x < \frac{\pi}{2} - x$ for $x \in \left(0, \frac{\pi}{2}\right)$ then with combining both we can say that

$$1 - \frac{2}{\pi} x \leq \cos x < \frac{\pi}{2} - x$$

Hence prove

3. Stečkin's Inequality

In an article titled "Generalizations for Köber's inequality and Stečkin's inequality" by Dorin Marghidanu and published in Romanian Mathematical Magazine. With the generalizations he used to prove Stečkin's inequality with the help of Jensen's Inequality here we are providing his proof.

Stečkin's Inequality For $x \in \left(0, \frac{\pi}{2}\right)$

$$\frac{4x}{\pi(\pi - 2x)} < \tan x < \frac{\pi x}{\pi - 2x}$$

Proof

From Jordan's inequality, For $x \in \left[0, \frac{\pi}{2}\right]$

$$\frac{2}{\pi} x \leq \sin x < x$$

And from Köber's inequality for For $x \in \left(0, \frac{\pi}{2}\right)$

$$1 - \frac{2}{\pi} x \leq \cos x < \frac{\pi}{2} - x$$

$$\Rightarrow \frac{1}{1 - \frac{2}{\pi} x} \geq \frac{1}{\cos x} > \frac{1}{\frac{\pi}{2} - x}$$

$$\Rightarrow \frac{\pi}{\pi - 2x} \geq \frac{1}{\cos x} > \frac{2}{\pi - 2x}$$

Now with multiplying both

$$\frac{2}{\pi}x\cdot\frac{2}{\pi-2x}\le \sin x\cdot\frac{1}{\cos x}<x\cdot\frac{\pi}{\pi-2x}$$
$$\Rightarrow \frac{4x}{\pi(\pi-2x)}<\tan x<\frac{\pi x}{\pi-2x}$$

Hence prove

4. Example Problems

The below problem is from Mathematical Reflections [2]

1. Let a, b, c, x, y be positive real numbers such that $x + y = 1$. Prove that

$$\sqrt{\frac{a^3}{xa+yb}}+\sqrt{\frac{b^3}{xb+yc}}+\sqrt{\frac{c^3}{xc+ya}}\ge a+b+c$$

Solution

Let $f(t) = 1/\sqrt{t}$ then $f''(t) = 3/4t^2\sqrt{t} > 0\ \ \forall\ t > 0$ so, considered function is convex as a result from Jensen's Inequality $\lambda_1 f(x_1)+\lambda_2 f(x_2)+\lambda_3 f(x_3)\ge f(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)$

Where, $\lambda_1+\lambda_2+\lambda_3=1$. Now let $\lambda_1=\frac{a}{a+b+c}, \lambda_2=\frac{b}{a+b+c}$ and $\lambda_3=\frac{c}{a+b+c}$ then

$$\frac{a}{a+b+c}f(x_1)+\frac{b}{a+b+c}f(x_2)+\frac{c}{a+b+c}f(x_3)\ge f\left(\frac{ax_1+bx_2+cx_3}{a+b+c}\right)$$

$$\Rightarrow \frac{a}{a+b+c}f\left(x+y\frac{b}{a}\right)+\frac{b}{a+b+c}f\left(x+y\frac{c}{b}\right)+\frac{c}{a+b+c}f\left(x+y\frac{a}{c}\right)\ge f\left(\frac{a\left(x+y\frac{b}{a}\right)+b\left(x+y\frac{c}{b}\right)+c\left(x+y\frac{a}{c}\right)}{a+b+c}\right)$$

$$\Rightarrow \frac{a}{a+b+c}\frac{1}{\sqrt{x+y\frac{b}{a}}}+\frac{b}{a+b+c}\frac{1}{\sqrt{x+y\frac{c}{b}}}+\frac{c}{a+b+c}\frac{1}{\sqrt{x+y\frac{a}{c}}}\ge f\left(\frac{ax+by+bx+cy+cx+ay}{a+b+c}\right)=f(x+y)$$

Since $x+y=1$ then $f(x+y)=f(1)=1$ and hence

$$\sqrt{\frac{a^3}{xa+yb}}+\sqrt{\frac{b^3}{xb+yc}}+\sqrt{\frac{c^3}{xc+ya}}\ge a+b+c$$

The below problem is from Pentagon Mathematical Magazine [3]

2. Prove that in an acute ΔABC the following relationship holds:

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}+\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}>6\sqrt{2}$$

Solution

As $\frac{1}{\sin x}$ and $\frac{1}{\cos x}$ are convex functions then using Jensen's inequality for convex function for both one by one

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\ge\frac{3}{\sin\left(\frac{A+B+C}{3}\right)}=\frac{3}{\sin\frac{\pi}{3}}=\frac{6}{\sqrt{3}}$$

And

$$\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\ge\frac{3}{\cos\left(\frac{A+B+C}{3}\right)}=\frac{3}{\cos\frac{\pi}{3}}=6$$

With adding both

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}+\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\ge 6+\frac{6}{\sqrt{3}}=6\left(1+\frac{1}{\sqrt{3}}\right)$$

Since $1+\frac{1}{\sqrt{3}}>\sqrt{2}$

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}+\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}>6\sqrt{2}$$

The below problem is again from Pentagon [4]

3. Prove that if $x, y, z > 0$. then

$$\frac{2x+2y+4z}{4x+4y+3z}+\frac{2x+4y+2z}{4x+3y+4z}+\frac{4x+2y+2z}{3x+4y+4z}\ge\frac{24}{11}$$

Solution

Let a function $f(x) = (2t + 2x)/(4t - x)$ where t is a constant and greater than x. Then the second order derivative of $f(x)$ is $f''(x) = \frac{20t}{(4t-x)^3}$ which is positive for all positive integer x thus considered function is convex in nature so, we can apply Jensen's Inequality here

$$\frac{f(x) + f(y) + f(z)}{3} \geq f\left(\frac{x+y+z}{3}\right)$$

$$\Rightarrow \frac{(2t+2x)}{(4t-x)} + \frac{(2t+2y)}{(4t-y)} + \frac{(2t+2z)}{(4t-z)} \geq 3 \cdot \frac{\left(2t + 2\left(\frac{x+y+z}{3}\right)\right)}{\left(4t - \left(\frac{x+y+z}{3}\right)\right)}$$

Using $t = x + y + z$ gives

$$\frac{2x+2y+4z}{4x+4y+3z} + \frac{2x+4y+2z}{4x+3y+4z} + \frac{4x+2y+2z}{3x+4y+4z} \geq 3 \cdot \frac{\left(2t + \frac{2t}{3}\right)}{\left(4t - \frac{t}{3}\right)}$$

$$\frac{2x+2y+4z}{4x+4y+3z} + \frac{2x+4y+2z}{4x+3y+4z} + \frac{4x+2y+2z}{3x+4y+4z} \geq \frac{24}{11}$$

The below problem is from [5]

4. If $a, b, x, y > 0$ and $n \in N^*$ prove that

$$\frac{(x+y)^n}{2^{(n-1)}} \leq \frac{(ax+by)^n + (bx+ay)^n}{(a+b)^n} \leq x^n + y^n$$

Solution:

Let $f:(0,1) \to R, f(t) = t^n, n \in N^*$ be a convex function, then by Jensen's inequality with weighted $\alpha_1, \alpha_2 \in (0,1), \alpha_1 + \alpha_2 = 1$ we have

$$f(\alpha_1 t_1 + \alpha_2 t_2) \leq \alpha_1 f(t_1) + \alpha_2 f(t_2) \text{ }, \forall\, t_1, t_2 \in (0, \infty).$$

For $\alpha_1 = \alpha_2 = \frac{1}{2}$ we have,

$$f\left(\frac{1}{2}t_1 + \frac{1}{2}t_2\right) \leq \frac{1}{2}f(t_1) + \frac{1}{2}f(t_2) \quad (1)$$

For $\alpha_1 = \frac{a}{a+b}, \alpha_2 = \frac{b}{a+b}$, we have,

$$f\left(\frac{a}{a+b}t_1 + \frac{b}{a+b}t_2\right) \leq \frac{a}{a+b}f(t_1) + \frac{b}{a+b}f(t_2) \quad (2)$$

Now taking t_1 = ax+by , t_2 = bx+ay in (1) , it follows that

$$\frac{1}{2^n}[(ax+by) + (bx+ay)]^n \leq \frac{1}{2}((ax+by)^n + (bx+ay)^n)$$

$$\Leftrightarrow \frac{1}{2^{n-1}}(a+b)^n(x+y)^n \leq (ax+by)^n + (bx+ay)^n$$

$$\Leftrightarrow \frac{(x+y)^n}{2^{n-1}} \leq \frac{(ax+by)^n + (bx+ay)^n}{(a+b)} \quad (3)$$

Taking t_1 = x , t_2 = y in (2), it follows

$$\left(\frac{a}{a+b}x + \frac{b}{a+b}y\right)^n \leq \frac{a}{a+b}x^n + \frac{b}{a+b}y^n; \quad (4)$$

Taking t_1 = y , t_2 = x in (2), it follows

$$\left(\frac{b}{a+b}x + \frac{a}{a+b}y\right)^n \leq \frac{b}{a+b}x^n + \frac{a}{a+b}y^n \quad (5)$$

Adding (4) and (5), we get

$$\frac{(ax+by)^n + (bx+ay)^n}{(a+b)^n} \leq \frac{a}{a+b}x^n + \frac{b}{a+b}y^n + \frac{b}{a+b}x^n + \frac{a}{a+b}y^n$$

$$\Leftrightarrow \frac{(ax+by)^n + (bx+ay)^n}{(a+b)^n} \leq x^n + y^n \quad (6)$$

From (3) and (6) we get the desired inequality.

The given below problem is from [6]

5. Let $f: R^+ \to R^+$ be an increasing, convex function with f(1) = 1, and let x , y , and z be positive real numbers. Prove that for any positive integer n,

$$\left(f\left(\frac{2x}{y+z}\right)\right)^n+\left(f\left(\frac{2y}{z+x}\right)\right)^n+\left(f\left(\frac{2z}{x+y}\right)\right)^n \geq 3$$

Solution: By Nesbitt's inequality,

$$\frac{2x}{y+z}+\frac{2y}{z+x}+\frac{2z}{x+y} \geq 3$$

Then, as the function $g(x) = f(x)^n$ is convex and increasing with g(1) =1, we can use jeansen's inequality to write

$$g\left(\frac{2x}{y+z}\right)+g\left(\frac{2y}{z+x}\right)+g\left(\frac{2z}{x+y}\right) \geq 3g\left(\frac{\frac{2x}{y+z}+\frac{2y}{z+x}+\frac{2z}{x+y}}{3}\right)$$

$$\geq 3g(1) = 3$$

as required.

The below problem is from Romanian Mathematical Magazine [7]

6. If a , b , c > 0 , abc =1, then

$$\frac{a^2+1}{a+1}+\frac{b^2+1}{b+1}+\frac{c^2+1}{c+1} \geq 3$$

Solution: Here $f(x) = \frac{x^2+1}{x+1}$ and as the function f(x) is convex. So we can use Jensen's inequality to write

$$\frac{f(a)+f(b)+f(c)}{3} \geq f\left(\frac{a+b+c}{3}\right)$$

Where, f(a) = $\frac{a^2+1}{a+1}$, f(b) = $\frac{b^2+1}{b+1}$, f(c) = $\frac{c^2+1}{c+1}$

Using AM-GM inequality we can write,

$$\frac{\frac{a^2+1}{a+1}+\frac{b^2+1}{b+1}+\frac{c^2+1}{c+1}}{3} \geq \frac{\left(\frac{a+b+c}{3}\right)^2+1}{\frac{a+b+c}{3}+1}$$

$$\Rightarrow \frac{a^2+1}{a+1}+\frac{b^2+1}{b+1}+\frac{c^2+1}{c+1} \geq 3\left(\frac{(abc)^{\frac{2}{3}}+1}{(abc)^{\frac{1}{3}}+1}\right)$$

$$\Rightarrow \frac{a^2+1}{a+1}+\frac{b^2+1}{b+1}+\frac{c^2+1}{c+1} \geq 3\left(\frac{(1)^{\frac{2}{3}}+1}{(1)^{\frac{1}{3}}+1}\right)$$

$$\Rightarrow \frac{a^2+1}{a+1}+\frac{b^2+1}{b+1}+\frac{c^2+1}{c+1} \geq 3$$

The below problem was proposed by Toyesh Prakash Sharma and was pyblished in AMJ [8]

7. If $1 < a, b, c < 6$ and $a + b + c = 6$. Then show that

$$\frac{\ln a}{b+c}+\frac{\ln b}{c+a}+\frac{\ln c}{a+b} \geq \frac{3}{4}\ln 2$$

Solution:

As $f(x) = \ln x/(6-x)$ is concave function so, from Jensen's Inequality

$$\frac{f(a)+f(b)+f(c)}{3} \geq f\left(\frac{a+b+c}{3}\right)$$

$$\Rightarrow \frac{\ln a}{6-a}+\frac{\ln b}{6-b}+\frac{\ln c}{6-c} \geq \frac{3\ln\left(\frac{a+b+c}{3}\right)}{\left(6-\left(\frac{a+b+c}{3}\right)\right)}$$

$$\Rightarrow \frac{\ln a}{b+c}+\frac{\ln b}{c+a}+\frac{\ln c}{a+b} \geq \frac{3}{4}\ln 2$$

Hence prove.

8. For any positive integers $a, b, c, d > 0$. Show that

$$\left(\frac{a}{b+c+d}\right)^{\frac{a}{b+c+d}}\left(\frac{b}{c+d+a}\right)^{\frac{b}{c+d+a}}\left(\frac{c}{d+a+b}\right)^{\frac{c}{d+a+b}}\left(\frac{d}{a+b+c}\right)^{\frac{d}{a+b+c}} \geq \frac{1}{4}$$

Solution by Proposer

Consider a function $f(x) = x\ln x \Rightarrow f'(x) = 1 = \ln x \Rightarrow f''(x) = 1/x > 0$ for all positive integers then using Jensen's Inequality of convexity due to the convex nature of considered function so,

$$\frac{f(x)+f(y)+f(z)+f(w)}{4} \geq f\left(\frac{x+y+z+w}{4}\right)$$

$$\Rightarrow \frac{x\ln x + y\ln y + z\ln z + w\ln w}{4} \geq \left(\frac{x+y+z+w}{4}\right)\ln\left(\frac{x+y+z+w}{4}\right)$$

$$x^x y^y z^z w^w \geq \left(\frac{x+y+z+w}{4}\right)^{(x+y+z+w)}$$

Suppose $x = \frac{a}{b+c+d}$, $y = \frac{b}{c+d+a}$, $z = \frac{c}{d+a+b}$ and $w = \frac{d}{a+b+c}$ Then,

$$\Rightarrow \left(\frac{a}{b+c+d}\right)^{\frac{a}{b+c+d}}\left(\frac{b}{c+d+a}\right)^{\frac{b}{c+d+a}}\left(\frac{c}{d+a+b}\right)^{\frac{c}{d+a+b}}\left(\frac{d}{a+b+c}\right)^{\frac{d}{a+b+c}}$$

$$\geq \left(\frac{\frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c}}{4}\right)^{\left(\frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c}\right)}$$

Now,

$$\frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c} = \frac{a}{S-a}+\frac{b}{S-b}+\frac{c}{S-c}+\frac{d}{S-d}$$

Where $S = a+b+c+d$ then suppose a function $g(x) = \frac{x}{s-x} \Rightarrow g''(x) = \frac{2s}{(s-x)^3} > 0$ then we can say that the considered function is convex in nature then using Jensen's Inequality we can say that

$$\frac{a}{S-a}+\frac{b}{S-b}+\frac{c}{S-c}+\frac{d}{S-d} \geq 4\frac{\left(\frac{a+b+c+d}{4}\right)}{S-\left(\frac{a+b+c+d}{4}\right)}$$

$$\frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c} \geq 4\frac{\frac{S}{4}}{S-\frac{S}{4}} = \frac{4}{3} > 1$$

So,

$$\left(\frac{a}{b+c+d}\right)^{\frac{a}{b+c+d}}\left(\frac{b}{c+d+a}\right)^{\frac{b}{c+d+a}}\left(\frac{c}{d+a+b}\right)^{\frac{c}{d+a+b}}\left(\frac{d}{a+b+c}\right)^{\frac{d}{a+b+c}} \geq \frac{1}{4}$$

5. Problems

Below problems are given to readers for independent study

1. See in [9]. Let n be positive integer. Show that

$$F_n^{F_n} + L_n^{L_n} \leq 2F_{n+1}^{F_{n+1}}$$

2. See in [10]. For any integer $n \geq 0$, show that

$$\frac{2^{n+1}F_{2n+1}}{2n+1} \geq L_n$$

3. See in [11]. If $n > 0$ and α is the positive root of quadratic equation $x^2 - x - 1 = 0$ then show that the following inequality

$$F_n\alpha^{F_n} + L_n\alpha^{L_n} \leq 2F_{n+1}\alpha^{F_{n+1}}$$

holds. Further, obtain the above inequality using the convexity of a suitable function where the Fibonacci numbers F_n and the Lucas numbers L_n satisfy the conditioned

$F_{n+2} = F_{n+1} + F_n$; $F_0 = 0, F_1 = 1$
$L_{n+2} = L_{n+1} + L_n$; $L_0 = 2, L_1 = 1$

4. See in [12]. Let a, b, c be non-negative numbers such that $\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} = 1$
Prove that

$$\frac{1}{6} \leq a+b \leq \frac{1}{4}$$

5. See in [13]. Prove

$$x^x y^y\left(\Gamma\left(\frac{x+y}{2}\right)\right)^2 \leq \left(\frac{x+y}{2}\right)^{x+y}\Gamma(x)\Gamma(y)$$

for all positive real numbers x and y.
Note: - $\Gamma(x)$, $\Gamma(y)$ and $\Gamma\left(\frac{x+y}{2}\right)$ are gamma function.

6. See in [14]. Let $x_1, x_2, \cdots x_n > 0$ be real numbers and $s = \sum_{i=1}^{n} x_i$. Prove

$$\prod_{i=1}^{n} x_i^{x_i} \geq \left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n}(1+x_i)^{x_i}$$

When does equality occur?

6. Conclusion and Future Scope

As in this article we have deals with two aspects of problems of the Jensen's inequality one is related to the alternative proofs to the well-known inequalities and another is related to the problems proposals published in the journals and magazines but there would also be some space for application of the Jensen's inequality such as someone can find the applications in proving convergence and divergence of the infinite series or even in non-elementary integrals, there could be some applications in physics respectively.

Data Availability
None

Conflict of Interest
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None

Authors' Contributions
Author-1, researched literature, Application and conceived the study.
Author-2, research problem proposals from different journals and solve using Jensen's Inequality.

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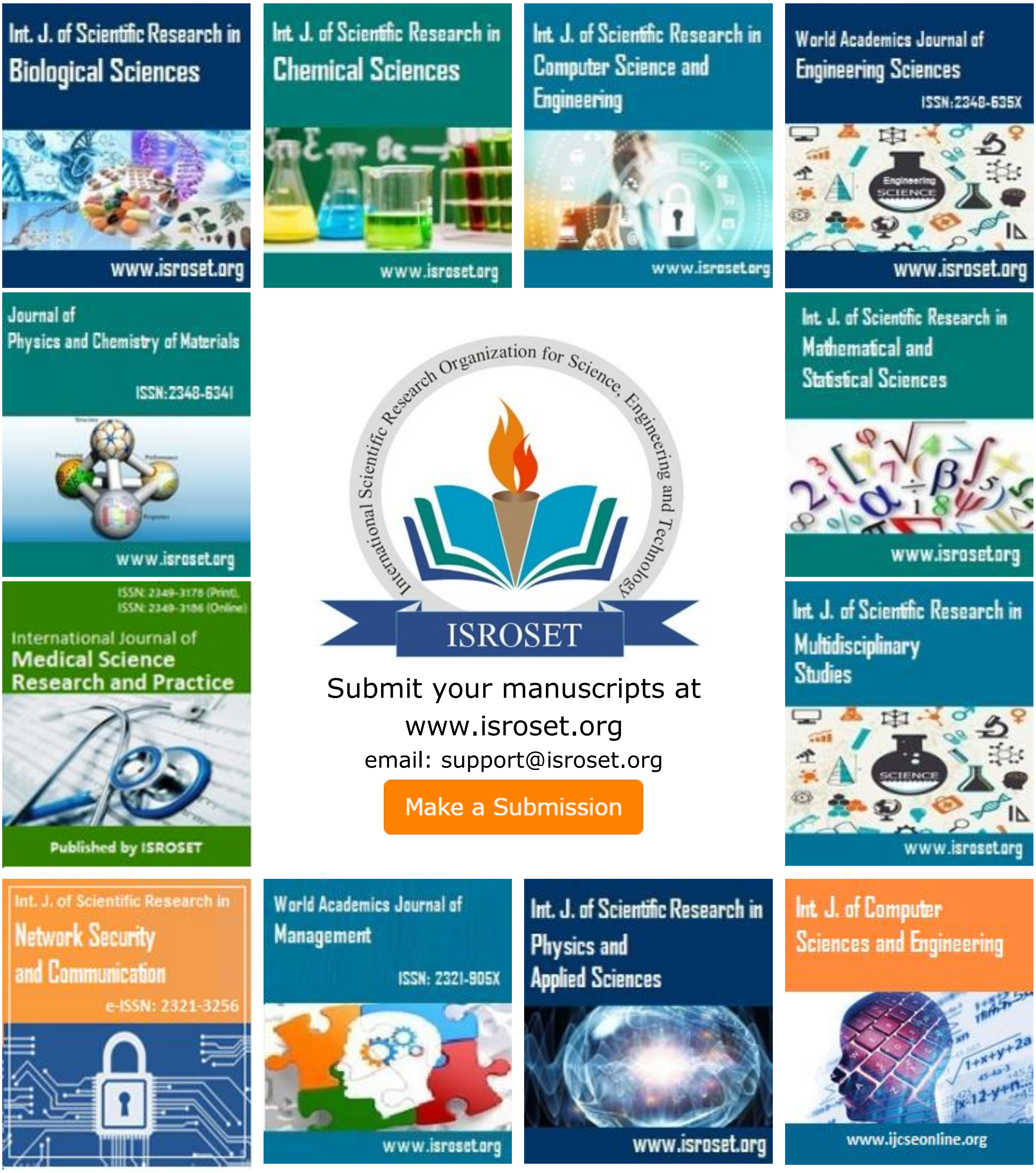

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