International Journal of Scientific Research in Physics and Applied Sciences
Vol.11, Issue.5, pp.01-18, October 2023
E-ISSN: 2348-3423
Available online at: www.isroset.org

Research Paper

Zero-Mass Scalar Field with Interacting and Non-interacting Two Fluids in *f* (*R*, *T*) Gravity

Kalpana Pawar[1], N.T. Katre[2*]

[1]Department of Mathematics, Shivaji Science College, Nagpur, Dist. Nagpur (M.S.), India
[2]Department of Mathematics, Nabira Mahavidyalaya, Katol, Dist. Nagpur (M.S.), India

**Corresponding Author: narendra.katre@rediffmail.com*

Received: 13/Aug/2023; **Accepted**: 15/Sept/2023; **Published**: 31/Oct/2023. | **DOI**: https://doi.org/10.26438/ijsrpas//v11i5.118

Abstract—This paper deals with the investigation of an accelerated expansion of a spatially homogeneous and isotropic flat Friedman-Lemaitre-Robertson-Walker (FLRW) universe in presence of zero-mass scalar fields associated with non-interacting and interacting barotropic fluid and dark energy in the framework of $f(R, T)$ gravity. The exact solutions to the field equations have been obtained in two cases: power law and exponential law of volumetric expansion. Some physical and geometrical properties have been investigated for both power and exponential law models in non-interacting and interacting cases; in particular, the energy conditions and density parameters. The physical stability of the derived cosmological models has also been examined. We find that the models with exponential volumetric expansion are open, have accelerating expansion and physically stable; while, the models with power law volumetric expansion are open in both accelerating and decelerating cases, but physically stable and unstable in decelerating and accelerating case respectively.

Keywords—FLRW space-time, $f(R, T)$ gravity, Dark Energy, Zero-mass scalar field, Interacting and Non-interacting, Physical stability.

1. Introduction

In recent years, the discovery of an accelerated cosmic expansion has led to the advancement in modern cosmology. The cosmological observations including high redshift supernova experiment [1-5], Wilkinson Microwave Anisotropy Probe (WMAP) experiment [6, 7], fluctuation of cosmic microwave background radiation (CMBR) [8, 9] and large scale structure (LSS) [10, 11] have produced large theoretical and observational evidence for an accelerated expansion of the universe. But still the reason for an accelerated expansion of the universe is not fully confirmed. It is said that the Einstein's general theory of relativity (GTR) does not explain the modern scenario of accelerated cosmic expansion. Thus, a number of theories alternative to GTR have been proposed by the researchers in order to investigate the cause of accelerated cosmic expansion. Astrophysical observations suggest that there is some kind of repulsive force in the universe which is pushing the cosmic objects farther apart in the space, and that this accelerated cosmic expansion is driven by mysterious dark energy (DE) with a large negative pressure [12-18]. The observational evidences of DE [1-7] suggest that nearly two-third and one-fourth parts of the universe's mass consist of DE and dark matter (DM) respectively, and the remaining consists of baryonic matter. The DE part of the universe is usually characterized by the dynamically variable quantity called as an equation of state (EoS) parameter, denoted by ω, and it is equal to the ratio of spatially homogeneous pressure to the energy density of DE. In order to explain the accelerated cosmic expansion, two methods have been suggested in the literature; one is to investigate different DE candidates and the second is to modify GTR. A number of probable DE candidates have been proposed in recent years. The cosmological constant Λ [15, 19, 20] is the simplest among all the DE candidates which is characterized by $\omega = -1$. The most commonly used primary DE candidates are scalar field models, such as time varying quintessence model [21-25] characterized by $-1 < \omega < -1/3$ in which DE density decreases over time as $\rho \propto a(t)^{-3(1+\omega)}$ ($a(t)$ is the scale factor) [26, 27], and k-essence [28-30], and phantom energy with $\omega < -1$ [31, 32]. Some other candidates of DE are quintom [33, 34], tachyon [35, 36], chameleon [37], holographic dark energy (HDE) [38-41], Ricci DE [42], new age graphic DE [43, 44], Chaplygin gas [45], extended Chaplygin gas [46, 47] and the generalized Chaplygin gas [48, 49], etc.

Another way to explain the accelerated expansion of the universe can be found in the modification of GTR. In this context, many modified theories of gravity have been developed in the last few years by modifying Einstein-Hilbert (E-H) action of GTR. Some of the popular modified gravity theories are the $f(R)$ theory proposed by Buchdahl [50], $f(T)$ theory proposed by Ferraro and Fiorini [51], and Bengochea and Ferraro [52], and $f(R, T)$ theory proposed by Harko et al.[53]. Also, there are a number of modified theories with varying cosmological implications such as $f(G)$ theory [54-56], $f(R, G)$ theory [57, 58], $f(T, B)$ theory [59], $f(Q)$ theory [60, 61], etc. Recently, a new extension of $f(Q)$ known as the $f(Q, T)$ theory has been proposed by Yixin et al. [62, 63]. Many researchers have investigated an accelerated cosmic expansion by studying various aspects of different cosmological models in these modified gravity theories by taking different kind of matter sources and different volumetric expansion laws. In examining an unsolved issue of the unification of gravitational and quantum theories, the basic obstacle is the "zero-mass scalar field". The theories of gravitation that describe zero-mass scalar fields associated with gravitational field have attracted considerable attention in the recent time.

Inspired by the discussions and explorations related to the issue of accelerating cosmic expansion made by the researchers in past using modified gravity theories, in this paper we consider a spatially homogeneous and isotropic flat FLRW space-time in the presence of zero-mass scalar fields associated with non-interacting and interacting barotropic fluid and dark energy in the framework of $f(R, T)$ gravity. We study the models by applying power law and exponential law of volumetric expansion. This paper is organized as follows: Section 2 contains the brief review of the work done by some researchers in the framework of some modified theories of gravitation which is related to the study carried out in this paper; $f(R, T)$ formalism in brief has been provided in Section 3; Section 4 is devoted to the metric and corresponding field equations in $f(R, T)$ theory. The solutions to the field equations are obtained in section 5. Section 6 deals with the construction of non-interacting and interacting two-fluid cosmological models by using power law and exponential law of volumetric expansion. Also, some physical and geometrical properties of the models along with their graphical behavior have been investigated in this section. In section 7, the physical stability of the derived models is examined. Lastly, Section 8 summarizes the conclusions.

2. Related Work

The 'Big-Bang' of the universe at an initial epoch can be avoided by introducing a zero-mass scalar field [64]. In this regard, some researchers [65, 66] have explored cosmological models with zero-mass scalar field. Recently, Chirde and Shekh [66] have studied the isotropic background for interacting two fluid scenario coupled with zero-mass scalar field in the framework of $f(R)$ Gravity; Pawar et al. [67] have studied accelerating expansion of the universe consisting of two fluids coupled with zero-mass scalar field in $f(R)$ gravity. Houndjo and Piattella [68], Samanta [69], Singh and Singh [70] and many other researchers have done remarkable work on DE models in the framework of $f(R, T)$ gravity. Houndjo [71], discussed the transition of matter dominated phase to an accelerated phase by reconstructing $f(R, T)$ gravity as $f(R, T) = f_1(R) + f_2(T)$. Due to the fact that the astrophysical data indicates that the total energy of the universe is occupied by DE, DM and baryonic matter, many researchers have studied the scenario of interacting and non-interacting two-fluid models in different theories of gravitation. In particular, Amirhashchi et al. [72, 73], have studied interacting and non-interacting two-fluid DE models with time dependent deceleration parameter in FRW universe in GTR. They have also studied two-fluid viscous dark energy models in non-flat and isotropic FRW universe [74]. Pradhan et al. [75], have discussed the scenario of two- fluid DE models with constant deceleration parameter in FRW universe. Saha et al. [76], have revisited the two-fluid DE model in FRW universe discussed earlier by Amirhashchi et al. Adhav et al. [77] have studied the anisotropic Bianchi type-I universe with DM and HDE. Rao et al. [78] have investigated the evolution of DE parameter within five dimensional Kaluza-Klein DE model of the universe in the framework of Saez-Ballester theory of gravitation.

3. $f(R, T)$ Formalism

The action of $f(R, T)$ gravity obtained by Harko et al. [53] from Einstein-Hilbert variational principle and using the system of units where $8\pi G = 1 = c$, is given by

$$S = \frac{1}{2}\int [f(R,T) + 2L_m]\sqrt{-g}\, d^4x \;, \quad (1)$$

where $f(R, T)$ is an arbitrary function of the Ricci scalar R and of the trace T of the trace-energy tensor of the matter, L_m is the matter Lagrangian density, and g is the determinant of the metric tensor g_{ij}.

The stress-energy tensor T_{ij} of the matter source is given by

$$T_{ij} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\, L_m\right)}{\delta g^{ij}} \;, \quad (2)$$

and its trace is $T = g^{ij}T_{ij}$.

By considering the matter Lagrangian density L_m depends only on the metric tensor components g_{ij} and not on its derivatives, equation (2) reduces to

$$T_{ij} = g_{ij}L_m - 2\frac{\partial L_m}{\partial g^{ij}} \;. \quad (3)$$

The field equations of $f(R, T)$ gravity obtained by varying the action (1) with respect to the metric tensor g_{ij} are given by

$$f_R(R,T)R_{ij} - \frac{1}{2}f(R,T)g_{ij} + g_{ij}\left(\nabla^i\nabla_i f_R(R,T)\right)$$

$$-\nabla_i\nabla_j f_R(R,T)=T_{ij}-f_T(R,T)(T_{ij}+\Theta_{ij}), \quad (4)$$

where $f_R(R,T)=\frac{\partial f(R,T)}{\partial R}$, $f_T(R,T)=\frac{\partial f(R,T)}{\partial T}$, ∇_i is a covariant derivative, and Θ_{ij} is defined as

$$\Theta_{ij}=-2T_{ij}-p\,g_{ij} . \quad (5)$$

Harko et al. [53] obtained some particular classes of $f(R,T)$ gravity models as given below:

$f(R,T)=R+2f(T)$, $f(R,T)=f_1(R)+f_2(T)$ and $f(R,T)=f_1(R)+f_2(R)\,f_3(T)$.

Generally, the field equations depend on the physical nature of the matter field, through the tensor Θ_{ij}. Hence, depending on the nature of matter source, several theoretical models of $f(R, T)$ gravity can be obtained. Reddy et al. [79, 80], Singh and Singh [81] and Sahoo et al. [82] have assumed $f(R, T) = R + 2f(T)$ in their studies of cosmological models. Yadav [83] and Brahma et al. [84] have assumed $f(R, T) = f_1(R) + f_2(T)$ with $f_1(R)=\mu R$, $f_2(T)=\mu T$ while studying the models with matter source as string cloud and perfect fluid, respectively. In this paper, we assume the matter source as barotropic fluid and dark energy with zero-mass scalar field, and

$$f(R,T)=f_1(R)+f_2(T)=\alpha R+\alpha T , \quad (6)$$

where α is an arbitrary parameter.

Using (5) and (6) in (4), we get the field equations in the form:

$$R_{ij}-\frac{1}{2}R\,g_{ij}=\left(1+\frac{1}{\alpha}\right)T_{ij}+\left(p+\frac{T}{2}\right)g_{ij} . \quad (7)$$

4. Metric and Field Equations

We consider the spatially homogeneous and isotropic flat FLRW metric in the form:

$$ds^2=dt^2-a^2(t)\left(dx^2+dy^2+dz^2\right), \quad (8)$$

where $a(t)$ is the metric potential or the scale factor of the universe.

The Ricci Scalar R for the universe (8) is obtained as

$$R=-6\left(\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\right) . \quad (9)$$

The overhead dots represent the differentiations with respect to time t.

The stress-energy tensor due to barotropic fluid, DE with zero-mass scalar fields is taken as

$$T_{ij}=(\rho+p)u_i u_j-p\,g_{ij}+\left[\psi,_i\psi,_j-\frac{1}{2}g_{ij}\psi,_k\psi^{'k}\right], \quad (10)$$

where $u^i=(0,0,0,1)$ is the four velocity vector in the co-moving coordinates satisfying $u^i u_i=1$. p, ρ and ψ are the isotropic pressure, energy density and zero-mass scalar field, respectively.

Here $p=p_m+p_\Lambda$ and $\rho=\rho_m+\rho_\Lambda$, where p_m and ρ_m are the pressure and energy density of barotropic fluid, respectively; p_Λ and ρ_Λ are the pressure and energy density of DE, respectively. Also, $p_m=\omega_m\rho_m$ and $p_\Lambda=\omega_\Lambda\rho_\Lambda$, where ω_m and ω_Λ are the EoS parameters of barotropic fluid and DE respectively.

The scalar field ψ satisfies the equation,

$$\psi i_{;i}=0 . \quad (11)$$

In the co-moving coordinate system, from (10), we have

$$T_1^1=T_2^2=T_3^3=-p-\frac{1}{2}\dot\psi^2,\quad T_4^4=\rho+\frac{1}{2}\dot\psi^2,$$

$$T_j^i=0 \text{ for } i\neq j , \quad (12)$$

and hence the trace is

$$T=(\rho-3p)-\dot\psi^2 . \quad (13)$$

The field equations (7) and (11) with the help of (9), (12) and (13) for the metric (8) reduce to

$$2\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}=-\left(\frac{3}{2}+\frac{1}{\alpha}\right)p+\frac{1}{2}\rho-\left(1+\frac{1}{2\alpha}\right)\dot\psi^2, \quad (14)$$

$$3\frac{\dot a^2}{a^2}=\left(\frac{3}{2}+\frac{1}{\alpha}\right)\rho-\frac{1}{2}p+\frac{1}{2\alpha}\dot\psi^2 , \quad (15)$$

and

$$\ddot\psi+3\frac{\dot a}{a}\dot\psi=0 . \quad (16)$$

We now define some parameters which are important in cosmological observations. The average scale factor (a), spatial volume (V), the mean Hubble parameter (H), mean isotropy parameter (A_m), expansion scalar (θ), shear scalar (σ), deceleration parameter (q), density parameter (Ω), speed of sound (ϑ_s) are defined respectively as

$$a=\sqrt[3]{V},\ V=a^3,\ H=\frac{\dot V}{3V}=\frac{1}{3}(H_1+H_2+H_3)=\frac{\dot a}{a} ,$$

$$A_m=(1/3)\sum_{i=1}^{3}[(H_i-H)/H]^2,\ \theta=u^i_{;i}=3H ,$$

$$\sigma^2=(3/2)A_m H^2,\quad q=-a\,\ddot a/\dot a^2=(1/H)^{\dot{}}-1,$$

$$\Omega_\Lambda=\rho/3H^2,\quad \text{and}\quad \vartheta_s=\dot p/\dot\rho .$$

5. Solution of the Field Equations

By integrating Equation (16), we obtain

$$\dot{\psi} = h a^{-3} , \tag{17}$$

where *h* is a constant of integration.

The field equations (14) and (15), with the help of an equation (17), yield

$$p = \left\{ -\frac{\alpha}{(\alpha+1)(2\alpha+1)} \left[\frac{\dot{a}^2}{a^2} + (3\alpha+2)\frac{\ddot{a}}{a} \right] - \frac{(3\alpha+1)}{2(2\alpha+1)} h^2 a^{-6} \right\}, \tag{18}$$

$$\rho = \left\{ \frac{\alpha}{(\alpha+1)(2\alpha+1)} \left[(4\alpha+3)\frac{\dot{a}^2}{a^2} - \alpha \frac{\ddot{a}}{a} \right] - \frac{\alpha+1}{2(2\alpha+1)} h^2 a^{-6} \right\}. \tag{19}$$

The energy conservation equation ($T^{ij}{}_{;j} = 0$) for the matter, yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 . \tag{20}$$

We assume that the EoS parameter of the barotropic matter to be a constant [72-76] as,

$$\omega_m = \frac{p_m}{\rho_m} = \text{Constant}, \tag{21}$$

while ω_Λ has been considered to be a function of time *t*.

Now, we obtain the pressure and density of the two-fluid with zero-mass scalar field in non-interacting and interacting scenarios.

First, we assume that two-fluids do not interact with each other. Thus, the energy conservation equation (20) leads to the separate equations for barotropic matter and DE [72-76] as,

$$\dot{\rho}_m + 3\frac{\dot{a}}{a}(\rho_m + p_m) = 0 , \tag{22}$$

$$\dot{\rho}_\Lambda + 3\frac{\dot{a}}{a}(\rho_\Lambda + p_\Lambda) = 0 . \tag{23}$$

As equation (22) contains ω_m, a constant, therefore it is integrable; while equation (23) contain ω_Λ which is time dependent parameter and hence not integrable. Equation (22), on integration, leads to

$$\rho_m = \rho_0 \, a^{-3(1+\omega_m)} , \tag{24}$$

where ρ_0 is an integrating constant.

Next, we consider the interaction between barotropic matter and DE components. In this case, the energy densities will no longer satisfy independent conservation laws; and we can write the equations as

$$\dot{\rho}_m + 3(\dot{a}/a)(\rho_m + p_m) = Q , \tag{25}$$

$$\dot{\rho}_\Lambda + 3(\dot{a}/a)(\rho_\Lambda + p_\Lambda) = -Q . \tag{26}$$

The quantity Q expresses the strength of interaction between DE components. Following Amirhashchi et al. [72-76], Adhav et al. [77] and Rao et al. [78], we consider the interaction term Q in the form

$$Q = 3H\delta\rho_m , \tag{27}$$

where δ is the coupling coefficient, can be considered as constant. Solving (25) with the use of (27), we obtain

$$\rho_m = \rho_0 \, a^{-3(1+\omega_m-\delta)} , \tag{28}$$

where ρ_0 is a constant of integration.

In the next section, we construct two cosmological models by considering two different forms of volumetric expansion laws, and discuss their behavior for non-interacting and interacting cases.

6. Cosmological Models

We construct the cosmological models by considering two forms of volumetric expansion laws: (i) Power law; and (ii) Exponential law.

6.1 Model with power law volumetric expansion

We consider the power law for volumetric expansion in the form:

$$V = c_1 t^{3m} , \tag{29}$$

where c_1 and m are constants. Hence, the average scale factor is

$$a = c_1^{1/3} t^m . \tag{30}$$

The metric (8) with the scale factor given in equation (30) assumes the form

$$ds^2 = dt^2 - c_1^{2/3} t^{2m}\left(dx^2 + dy^2 + dz^2\right) . \tag{31}$$

For this model the cosmological parameters defined in section 4 are obtained as follows:

$$H = \frac{m}{t} , \quad \theta = \frac{3m}{t} , \quad q = \frac{1-m}{m} , \quad A_m = 0, \quad \sigma^2 = 0 .$$

It is observed that the spatial volume is zero at $t = 0$ and it is an increasing function of time. The Hubble's parameter and expansion scalar are decreasing functions of cosmic time t; these are very large initially and tend to zero as $t \to \infty$. Thus, the model has initial singularity, i.e., the universe starts evolving with a big-bang, and it expands with decreasing rate. The mean anisotropy parameter and shear scalar are zero, indicating the model is shear free and isotropic throughout the evolution. The deceleration parameter is constant and its value depends on m; it is negative for $m > 1$ and $m < 0$, positive for $0 < m < 1$, and zero for $m = 1$. Thus, $m > 1$ and $m < 0$, $0 < m < 1$, and $m = 1$ correspond to the universe's accelerating expansion, decelerating expansion and expansion with constant rate, respectively.

6.1.1 Non-interacting two-fluid model

Using equation (30) in the equations (18), (19) and (24), we obtain

$$\rho_m = \rho_0 \, t^{-3m(1+\omega_m)} \, c_1^{-(1+\omega_m)} , \tag{32}$$

$$\rho_\Lambda = \left\{ \frac{\alpha m\left[3m(\alpha+1)+\alpha\right]}{(\alpha+1)(2\alpha+1)} t^{-2} - \frac{(\alpha+1)h^2}{2(2\alpha+1)c_1^2} t^{-6m} - \frac{\rho_0}{c_1^{(1+\omega_m)}} t^{-3m(1+\omega_m)} \right\}, \tag{33}$$

$$p_\Lambda = \left\{ -\frac{\alpha m\left[3m(\alpha+1)-(3\alpha+2)\right]}{(\alpha+1)(2\alpha+1)} t^{-2} - \frac{(3\alpha+1)h^2}{2(2\alpha+1)c_1^2} t^{-6m} - \frac{\rho_0\,\omega_m}{c_1^{(1+\omega_m)}} t^{-3m(1+\omega_m)} \right\}. \tag{34}$$

The EoS parameter of dark energy, $\omega_\Lambda = p_\Lambda / \rho_\Lambda$, is obtained in terms of cosmic time t as

$$\omega_\Lambda = \left[\frac{ -\frac{\alpha m\left[3m(\alpha+1)-(3\alpha+2)\right]}{(\alpha+1)(2\alpha+1)} t^{-2} - \frac{(3\alpha+1)h^2}{2(2\alpha+1)c_1^2} t^{-6m} - \frac{\rho_0\,\omega_m}{c_1^{(1+\omega_m)}} t^{-3m(1+\omega_m)} }{ \frac{\alpha m\left[3m(\alpha+1)+\alpha\right]}{(\alpha+1)(2\alpha+1)} t^{-2} - \frac{(\alpha+1)h^2}{2(2\alpha+1)c_1^2} t^{-6m} - \frac{\rho_0}{c_1^{(1+\omega_m)}} t^{-3m(1+\omega_m)} } \right]. \tag{35}$$

The metric (31) together with (32) to (35) represents zero-mass scalar fields coupled with two-fluid non-interacting DE model in *f* (*R*, *T*) gravity.

Since the values of m, such as $m>1$, $0<m<1$, and $m=1$ correspond to the universe's accelerating expansion, decelerating expansion and expansion with constant rate, respectively; we have shown the graphical behavior of some cosmological parameters of the constructed model at $m=0.4$ (for the case: $0<m<1$), $m=1$ and $m=1.5$ (for the case $m>1$).

From equations (32), (33) and (34), it is observed that the quantities ρ_Λ, ρ_m, p_Λ and p_m becomes infinite as $t \to 0$. This shows that the model has initial singularity. The graphical behavior of DE density ρ_Λ, DE pressure ρ_Λ, EoS parameter ω_Λ for DE, matter density ρ_m and matter pressure p_m versus cosmic time t are shown in Figure 1, 2, 3, 4 and 5 respectively.

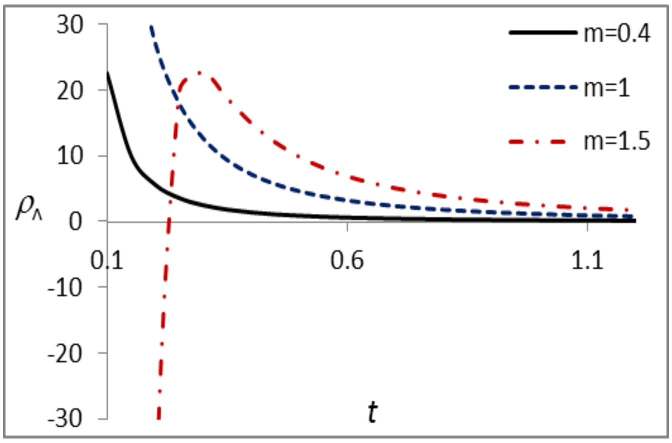

Figure 1: The plot of ρ_Λ vs. *t* for $\alpha=\rho_0=h=1,\ c_1=100$ and $\omega_m=0.5$ in non-interacting two-fluid model with power law volumetric expansion.

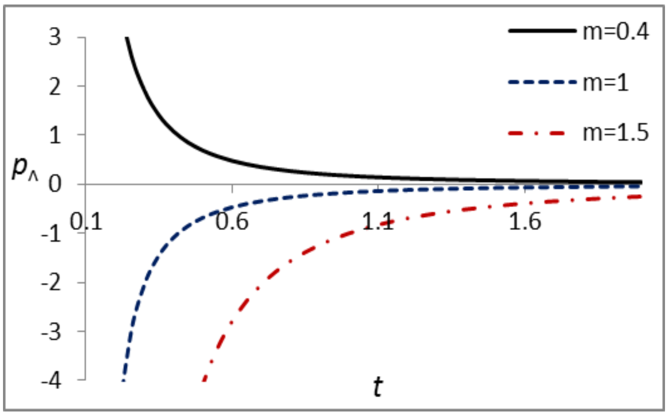

Figure 2: The plot of p_Λ vs. *t* for $\alpha=\rho_0=h=1,\ c_1=100$ and $\omega_m=0.5$ in non-interacting two-fluid model with power law volumetric expansion.

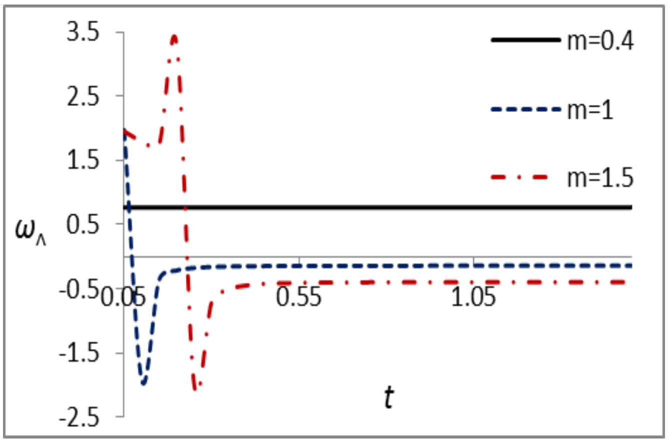

Figure 3: The plot of ω_Λ vs. *t* for $\alpha=\rho_0=h=1,\ c_1=100$ and $\omega_m=0.5$ in non-interacting two-fluid model with power law volumetric expansion.

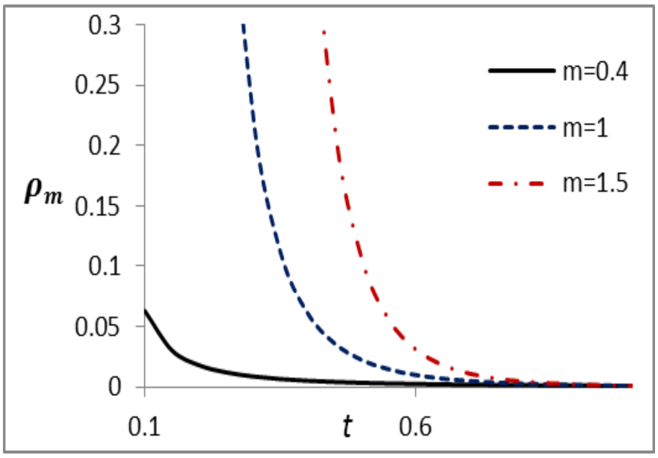

Figure 4: The plot of ρ_m vs. t for $\rho_0=1,\ c_1=100$ and $\omega_m=0.5$ in non-interacting two-fluid model with power law volumetric expansion.

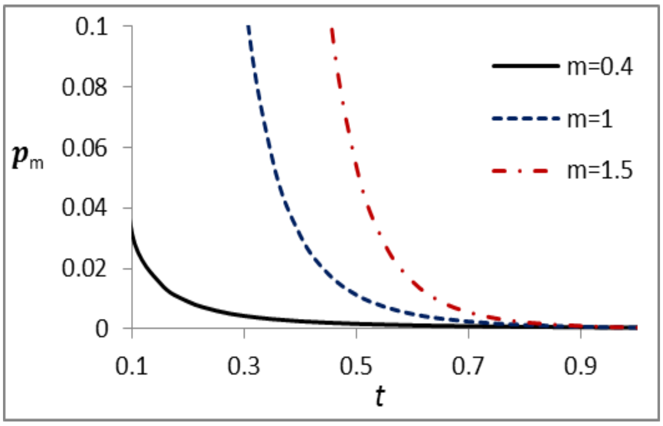

Figure 5: The plot of p_m vs. t for $\rho_0=1,\ c_1=100$ and $\omega_m=0.5$ in non-interacting two-fluid model with power law volumetric expansion.

It is observed that, for $0<m<1$, the universe has infinitely large energy density and it declines with time and tends to zero for large t. Thus the universe is free from big rip. In case of $m\geq 1$, the DE density has large negative value for very short period of time at early stage, and then it increases rapidly and becomes positive, and then behaves as the decreasing function of time such that it tends to zero for large *t*.

For $0<m<1$, the DE pressure ρ_Λ is a positive decreasing function of time, while it is negative throughout and increasing for $m\geq 1$. In both the cases, it finally tends to zero as expected for an expanding universe.

For $0<m<1$, the EoS parameter of DE has a constant positive value 0.78 (approx.) showing the matter dominance of the universe. For $m\geq 1$, at the initial stage, $\omega_\Lambda>1$ indicates the matter dominated era of the early universe. It decreases rapidly, becomes negative and crosses -1 in the early stage of evolution, i.e., from $\omega_\Lambda>-1$ (quintessence region) to $\omega_\Lambda<-1$ (phantom region), which is a Quintom DE scenario stated by Zhang [85]. Later on it tends to the same constant negative value lying in (-1, 0), i.e., it remains present in the quintessence region throughout the passage of time, which is acceptable as per the SNe Ia observational data.

Both density ρ_m and pressure p_m of matter are the positive decreasing functions of time, and these converge to zero for large t as expected.

Energy conditions:

Energy conditions play an important role in comprehending the geometric structure of the universe. The weak energy conditions (WEC), dominant energy conditions (DEC) and strong energy conditions (SEC) stated in [72-76] are respectively,

(i) $\rho_{(eff)}\geq 0$, $(\rho-p)_{(eff)}\geq 0$, (ii) $(\rho+p)_{(eff)}\geq 0$, and (iii) $(\rho+3p)_{(eff)}\geq 0$.

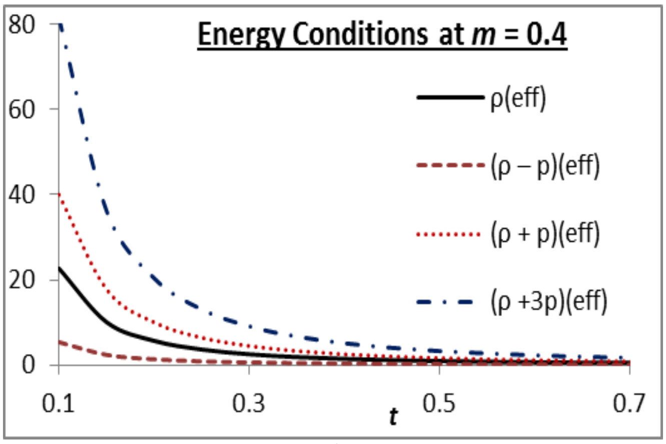

Figure 6(a): The plot of energy conditions vs. t for $\alpha=\rho_0=h=1,\ c_1=100$, $\omega_m=0.5$ and $m=0.4$ in non-interacting two-fluid model with power law volumetric expansion.

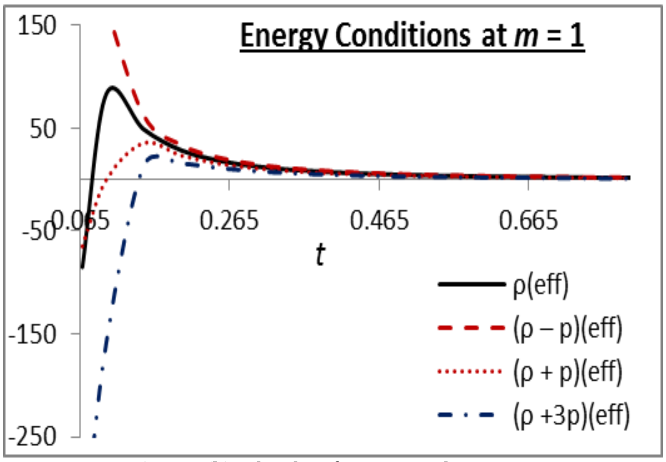

Figure 6(b): The plot of energy conditions vs. t for $\alpha=\rho_0=h=1,\ c_1=100$, $\omega_m=0.5$ and $m=1$ in non-interacting two-fluid model with power law volumetric expansion.

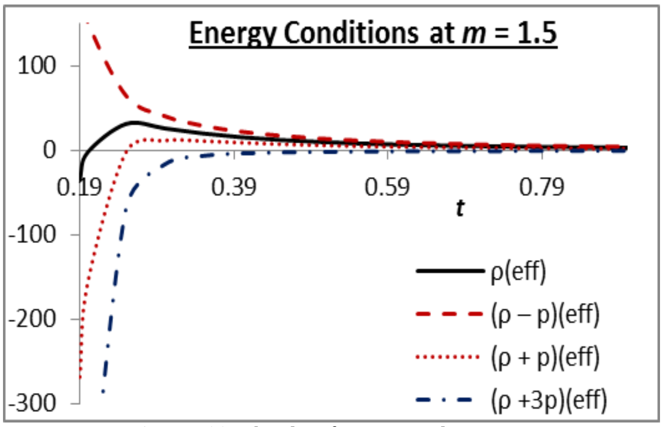

Figure 6(c): The plot of energy conditions vs. t for $\alpha = \rho_0 = h = 1,\ c_1 = 100$, $\omega_m = 0.5$ and $m = 1.5$ in non-interacting two-fluid model with power law volumetric expansion.

The L.H.S. of these energy conditions are depicted in the Figures 6(a), 6(b) and 6(c). For $0 < m < 1$ (decelerating case), all energy conditions are satisfied throughout the evolution of the universe. For $m = 1$ (case of expansion with constant rate), the conditions are not satisfied at the early stage of evolution whereas later on all the conditions are satisfied. For $m > 1$ (case of accelerating expansion), the conditions are not satisfied at the early stage of evolution whereas at later time all the conditions are satisfied except SEC and it is due to an accelerating cosmic expansion. Thus, our results are in fair resemblance with the relevant observations found in the literature.

Density parameters:

The density parameters $\Omega_m = \rho_m / 3H^2$ and $\Omega_\Lambda = \rho_\Lambda / 3H^2$ corresponding to the barotropic matter and DE components respectively, are obtained as

$$\Omega_m = (\rho_0 / 3m^2) c_1^{-(1+\omega_m)}\ t^{-3m(1+\omega_m)+2}, \tag{36}$$

$$\Omega_\Lambda = \frac{1}{3m^2}\left\{ \frac{\alpha m\left[3m(\alpha+1)+\alpha\right]}{(\alpha+1)(2\alpha+1)} - \frac{(\alpha+1)h^2}{2(2\alpha+1)c_1^2\, t^{6m-2}} - \frac{\rho_0}{c_1^{(1+\omega_m)}}\, t^{-3m(1+\omega_m)+2} \right\}. \tag{37}$$

Then the overall density parameter given by $\Omega = \Omega_m + \Omega_\Lambda$ is

$$\Omega = \frac{\alpha\left[3m(\alpha+1)+\alpha\right]}{3m(\alpha+1)(2\alpha+1)} - \frac{(\alpha+1)h^2}{6m^2(2\alpha+1)c_1^2\, t^{6m-2}}. \tag{38}$$

The behavior of density parameters is demonstrated in the Figures 7(a), 7(b) and 7(c) for different values of m.

In case of $m = 0.4$ (i.e., $0 < m < 1$), the DE component dominates the evolution. In case of $m \geq 1$, it is observed that the ordinary matter dominates the universe in the early stage of evolution, but later, DE component dominates the evolution of the universe. In all the cases, the value of overall density parameter Ω is lying between 0.35 and 0.5 (i.e., $\Omega < 1$), showing the universe is open, which is not strictly compatible with the present-day observations of a flat universe.

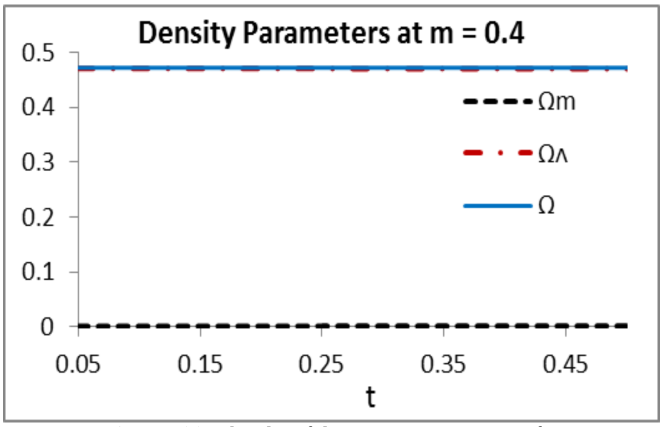

Figure 7(a): The plot of density parameters Vs. t for $\alpha = \rho_0 = h = 1,\ c_1 = 100$, $\omega_m = 0.5$ and $\boldsymbol{m} = 0.4$ in non-interacting two-fluid model with power law volumetric expansion.

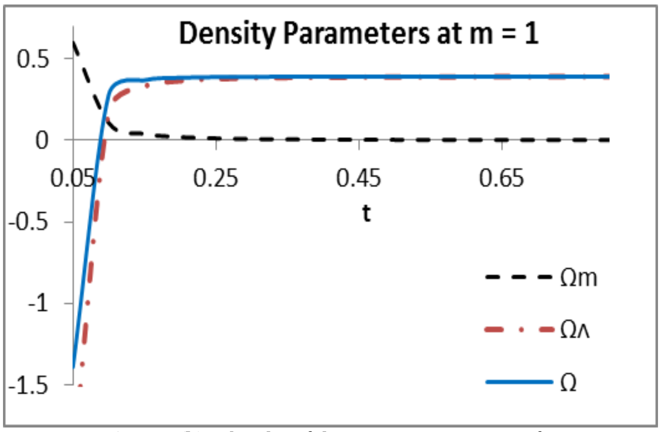

Figure 7(b): The plot of density parameters Vs. t for $\alpha = \rho_0 = h = 1,\ c_1 = 100$, $\omega_m = 0.5$ and $\boldsymbol{m} = 1$ in non-interacting two-fluid model with power law volumetric expansion.

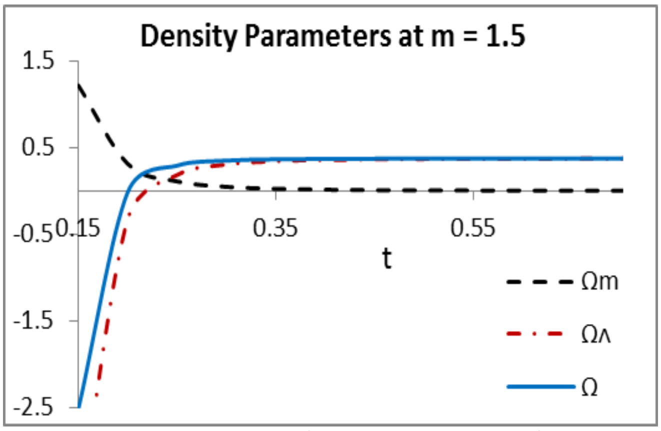

Figure 7(c): The plot of density parameters Vs. t for $\alpha = \rho_0 = h = 1,\ c_1 = 100$, $\omega_m = 0.5$ and $\boldsymbol{m} = 1.5$ in non-interacting two-fluid model with power law volumetric expansion.

6.1.2 Interacting two-fluid model

Using equation (30) in the equations (18), (19) and (28), we obtain

$$\rho_m = \rho_0 t^{-3m(1+\omega_m-\delta)} c_1^{-(1+\omega_m-\delta)}, \tag{39}$$

$$\rho_\Lambda = \left\{ \frac{\alpha m\left[3m(\alpha+1)+\alpha\right]}{(\alpha+1)(2\alpha+1)} t^{-2} - \frac{(\alpha+1)h^2}{2(2\alpha+1)c_1^2} t^{-6m} - \frac{\rho_0}{c_1^{(1+\omega_m-\delta)}} t^{-3m(1+\omega_m-\delta)} \right\}, \tag{40}$$

$$p_\Lambda = \left\{ -\frac{\alpha m\left[3m(\alpha+1)-(3\alpha+2)\right]}{(\alpha+1)(2\alpha+1)} t^{-2} - \frac{(3\alpha+1)h^2}{2(2\alpha+1)c_1^2} t^{-6m} - \frac{\rho_0\,\omega_m}{c_1^{(1+\omega_m-\delta)}} t^{-3m(1+\omega_m-\delta)} \right\}. \tag{41}$$

The EoS parameter of dark energy is obtained in terms of cosmic time t as

$$\omega_\Lambda = \left[\frac{ -\frac{\alpha m\left[3m(\alpha+1)-(3\alpha+2)\right]}{(\alpha+1)(2\alpha+1)} t^{-2} - \frac{(3\alpha+1)h^2}{2(2\alpha+1)c_1^2} t^{-6m} - \frac{\rho_0\,\omega_m}{c_1^{(1+\omega_m-\delta)}} t^{-3m(1+\omega_m-\delta)} }{ \frac{\alpha m\left[3m(\alpha+1)+\alpha\right]}{(\alpha+1)(2\alpha+1)} t^{-2} - \frac{(\alpha+1)h^2}{2(2\alpha+1)c_1^2} t^{-6m} - \frac{\rho_0}{c_1^{(1+\omega_m-\delta)}} t^{-3m(1+\omega_m-\delta)} } \right]. \tag{42}$$

The metric (31) together with (39) to (42) represents zero-mass scalar fields coupled with two-fluid interacting DE model in $f(R,T)$ gravity.

The graphical behavior of energy density and pressure of barotropic matter and DE, and EoS parameter of DE, in interacting two-fluid case is shown in the figures 8 to 12. Figures depict the same behavior of these parameters as in non-interacting case with slight increase/decrease in their values. There is a slight increase in the values of ρ_Λ and p_Λ. The value of ω_Λ is slightly decreased in case of $0<m<1$, while slightly increased in case of $m \geq 1$ but tends to the same constant value lying in (-1, 0), i.e., it remains present in the quintessence region for later time. Both ρ_m and p_m are the positive decreasing functions of time with considerable amount of decrease in values than in non-interacting case, and these converge to zero for large t as expected.

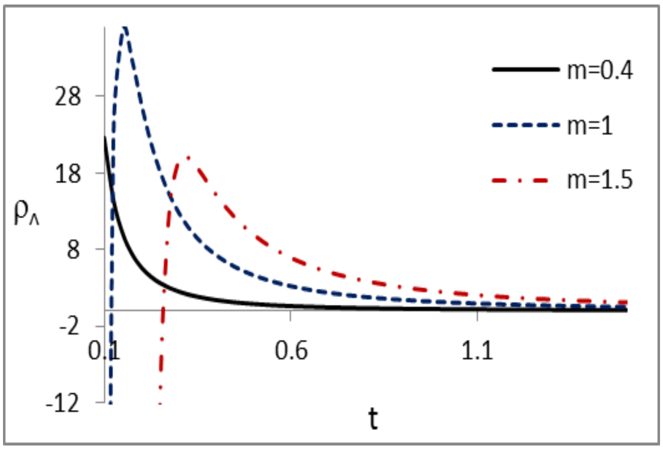

Figure 8: The plot of ρ_Λ vs. *t* for $\alpha = \rho_0 = h = 1,\ c_1 = 100,\ \delta = 0.8$ and $\omega_m = 0.5$ in interacting two-fluid model with power law volumetric expansion.

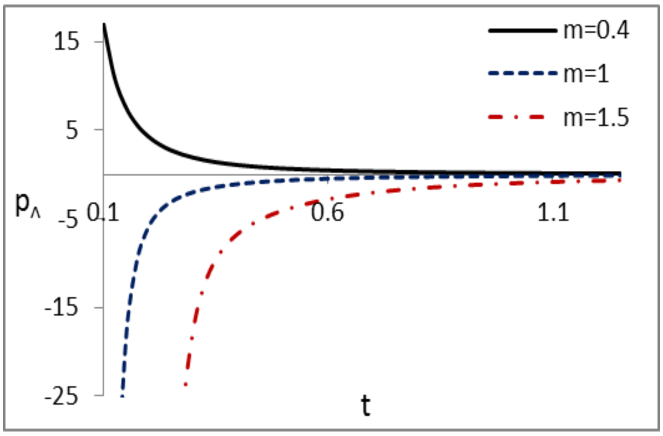

Figure 9: The plot of p_Λ vs. *t* for $\alpha = \rho_0 = h = 1,\ c_1 = 100,\ \delta = 0.8$ and $\omega_m = 0.5$ in interacting two-fluid model with power law volumetric expansion.

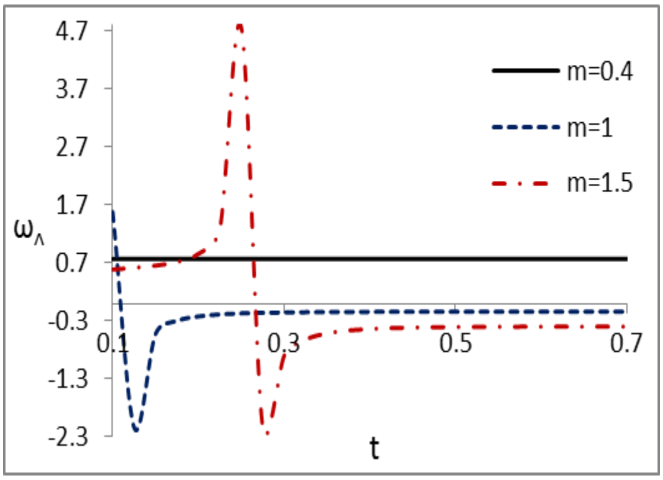

Figure 10: The plot of ω_Λ vs. *t* for $\alpha = \rho_0 = h = 1,\ c_1 = 100,\ \delta = 0.8$ and $\omega_m = 0.5$ in interacting two- fluid model with power law volumetric expansion.

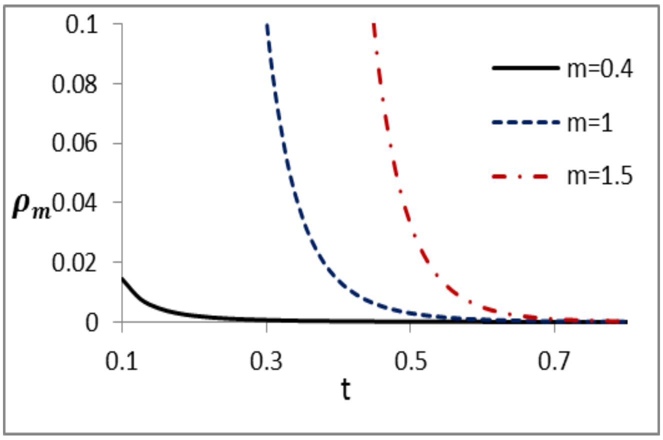

Figure 11: The plot of ρ_m vs. t for $\rho_0 = 1,\ c_1 = 100,\ \delta = 0.8$ and $\omega_m = 0.5$ in interacting two-fluid model with power law volumetric expansion.

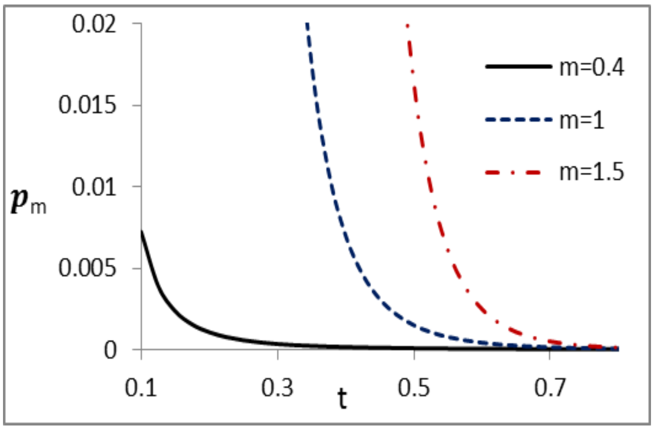

Figure 12: The plot of p_m vs. t for $\rho_0 = 1,\ c_1 = 100,\ \delta = 0.8$ and $\omega_m = 0.5$ in interacting two-fluid model with power law volumetric expansion.

The density parameters Ω_m and Ω_Λ corresponding to the barotropic matter and DE components respectively, are obtained as

$$\Omega_m = (\rho_0 / 3m^2) c_1^{-(1+\omega_m-\delta)}\ t^{-3m(1+\omega_m-\delta)+2}, \tag{43}$$

$$\Omega_\Lambda = \frac{1}{3m^2}\left\{ \frac{\alpha m\left[3m(\alpha+1)+\alpha\right]}{(\alpha+1)(2\alpha+1)} - \frac{(\alpha+1)h^2}{2(2\alpha+1)c_1^2\, t^{6m-2}} - \frac{\rho_0}{c_1^{(1+\omega_m-\delta)}} t^{-3m(1+\omega_m-\delta)+2} \right\}. \tag{44}$$

Adding (43) and (44), we get an overall density parameter Ω in case of interacting fluids, which is the same as obtained in non-interacting case, and hence have the same properties as discussed previously.

Also, the expressions for $\rho_{(eff)}$, $(\rho - p)_{(eff)}$, $(\rho + p)_{(eff)}$ and $(\rho + 3p)_{(eff)}$ in case of interacting fluids, are the same as in non-interacting case, and hence the energy conditions are same.

6.2 Model with exponential volumetric expansion

We consider the law for exponential volumetric expansion in the form:

$$V = c_2 e^{3kt}, \tag{45}$$

where c_2 and k are constants. Hence, the average scale factor is

$$a = c_2^{1/3} e^{kt}. \tag{46}$$

The metric (8) with the scale factor given in equation (46) assumes the form

$$ds^2 = dt^2 - c_2^{2/3} e^{2kt}\left(dx^2 + dy^2 + dz^2\right). \tag{47}$$

For this model the cosmological parameters defined in section 3 are obtained as follows:

$$H = k,\quad \theta = 3k,\quad q = -1,\ A_m = 0,\quad \sigma^2 = 0.$$

It is observed that the spatial volume is constant (c_2) at $t = 0$, and it is the increasing function of time. Thus, the model has no initial singularity, and the universe starts expanding with some nonzero fixed volume, and it expands exponentially with the time. The Hubble's parameter and expansion scalar are constant throughout the evolution of the universe. The mean anisotropy parameter and shear scalar are zero, indicating the model is shear free and isotropic throughout the evolution of the universe. The constant negative value of deceleration parameter ($q = -1$) indicates an accelerating expansion of the universe.

6.2.1 Non-interacting two-fluid model

Using equation (46) in the equations (18), (19) and (24), we obtain

$$\rho_m = \frac{\rho_0}{C_2^{(1+\omega_m)}} e^{-3k(1+\omega_m)t}, \tag{48}$$

$$\rho_\Lambda = \frac{3\alpha k^2}{2\alpha+1} - \frac{(\alpha+1)h^2 e^{-6kt}}{2(2\alpha+1)c_2^2} - \frac{\rho_0\, e^{-3k(1+\omega_m)t}}{c_2^{1+\omega_m}}, \tag{49}$$

$$p_\Lambda = -\frac{3\alpha k^2}{2\alpha+1} - \frac{(3\alpha+1)h^2 e^{-6kt}}{2(2\alpha+1)c_2^2} - \frac{\rho_0\, \omega_m e^{-3k(1+\omega_m)t}}{c_2^{1+\omega_m}}. \tag{50}$$

The EoS parameter of dark energy, $\omega_\Lambda = p_\Lambda / \rho_\Lambda$, is obtained in terms of cosmic time t as

$$\omega_\Lambda = \left[\frac{ -\dfrac{3\alpha k^2}{2\alpha+1} - \dfrac{(3\alpha+1)h^2 e^{-6kt}}{2(2\alpha+1)c_2^2} - \dfrac{\rho_0\, \omega_m e^{-3k(1+\omega_m)t}}{c_2^{1+\omega_m}} }{ \dfrac{3\alpha k^2}{2\alpha+1} - \dfrac{(\alpha+1)h^2 e^{-6kt}}{2(2\alpha+1)c_2^2} - \dfrac{\rho_0 e^{-3k(1+\omega_m)t}}{c_2^{1+\omega_m}} } \right] \tag{51}$$

The metric (47) together with (48) to (51) represents zero-mass scalar fields coupled with two-fluid non-interacting DE model in $f(R, T)$ gravity. The model is free from any kind of singularities.

The variation of energy density and pressure of DE and barotropic matter, and EoS parameter of DE, with cosmic time t is shown in the Figures (13) to (17). Both ρ_m and p_m are positive decreasing functions of time, which tends to zero for large time. The energy density (ρ_Λ) of DE is constant in the initial stage; it increases very slowly for very small period of time and then achieve a constant positive value throughout the expansion of universe. The pressure (p_Λ) of the dark fluid has a constant negative value initially; it increases very slowly for a very small period of time and becomes constant (negative) throughout the evolution of the universe. It is observed that the EoS parameter (ω_Λ) of dark energy has a fixed negative value less than (-1), initially; it increases with time and approaches towards -1, but does not cross the phantom divide or cosmological constant ($\omega_\Lambda = -1$) region. This is found compatible with the cosmological tests based on present data, including SNe Ia data as well as CMB anisotropy and mass power spectrum. Thus our derived model represents early stage evolution as well as the present universe.

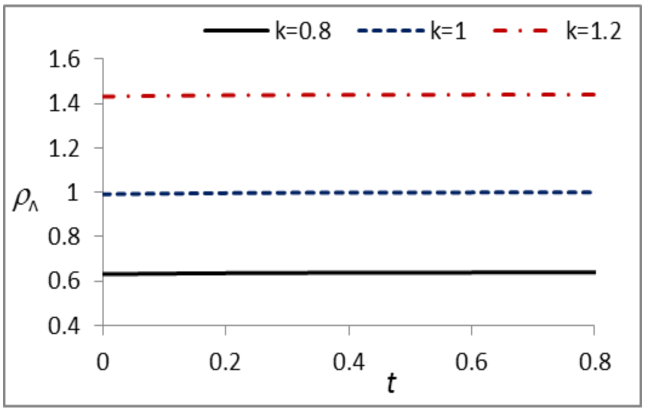

Figure 13: The plot of ρ_Λ vs. t for $\alpha = \rho_0 = h = 1,\ c_2 = 25$ and $\omega_m = 0.5$ in non-interacting two-fluid model with exponential volumetric expansion.

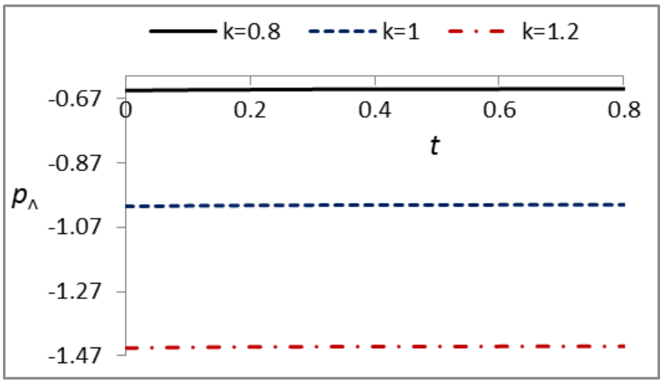

Figure 14: The plot of p_Λ vs. t for $\alpha = \rho_0 = h = 1,\ c_2 = 25$ and $\omega_m = 0.5$ in non-interacting two-fluid model with exponential volumetric expansion.

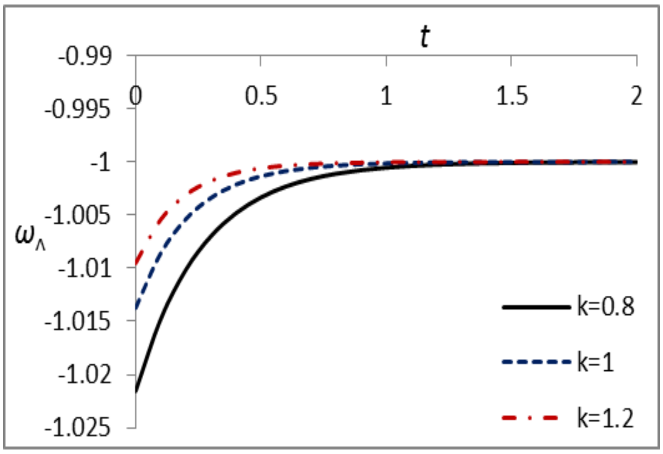

Figure 15: The plot of ω_Λ vs. t for $\alpha = \rho_0 = h = 1,\ c_2 = 25$ and $\omega_m = 0.5$ in non-interacting two-fluid model with exponential volumetric expansion.

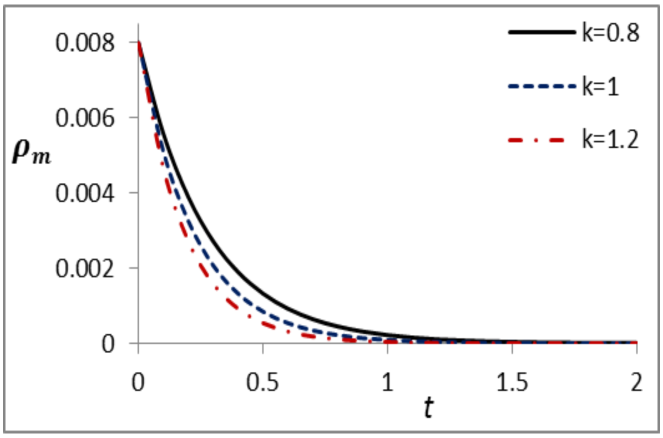

Figure 16: The plot of ρ_m vs. t for $\rho_0 = 1,\ c_2 = 25$ and $\omega_m = 0.5$ in non-interacting two-fluid model with exponential volumetric expansion.

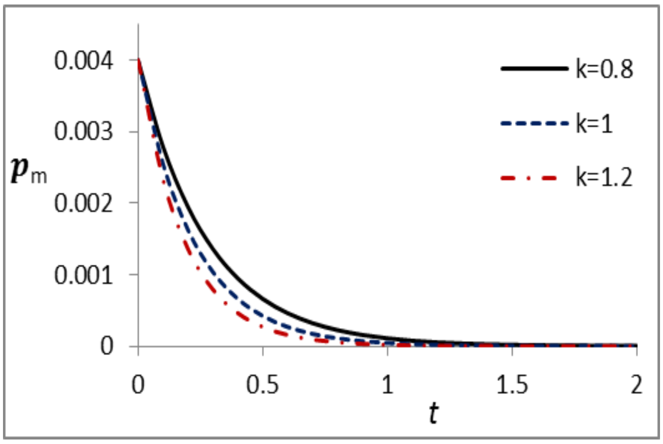

Figure 17: The plot of p_m vs. t for $\rho_0 = 1,\ c_2 = 25$ and $\omega_m = 0.5$ in non-interacting two-fluid model with exponential volumetric expansion.

Energy conditions:

The variation of the L.H.S. of energy conditions versus cosmic time t for non-interacting two-fluid model with exponential expansion is shown in the Figure 18. It is observed that only the weak energy conditions ($\rho_{eff} \geq 0$, $(\rho - p)_{eff} \geq 0$) are satisfied in this model.

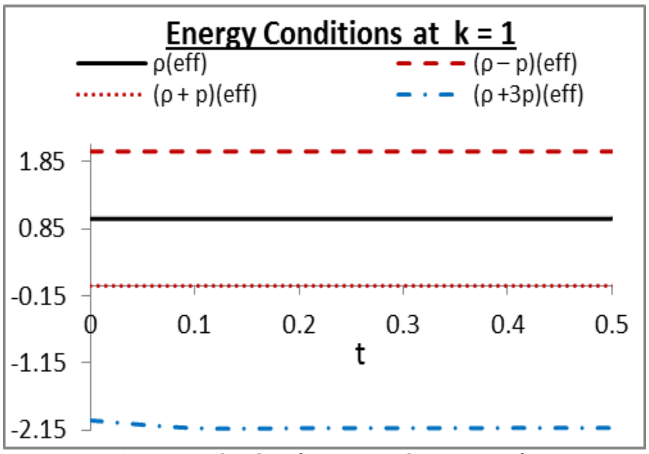

Figure 18: The plot of energy conditions Vs. t for $\alpha = \rho_0 = h = 1,\ c_2 = 25$ and $\omega_m = 0.5$ in non-interacting two-fluid model with exponential volumetric expansion.

Density parameters:

The expressions for density parameter $\Omega_m = \rho_m / 3H^2$ of barotropic matter and density parameter $\Omega_\Lambda = \rho_\Lambda / 3H^2$ of DE components are obtained as follows:

$$\Omega_m = \frac{\rho_0}{3k^2 C_2^{1+\omega_m}} e^{-3k(1+\omega_m)t}, \qquad (52)$$

$$\Omega_\Lambda = \frac{\alpha}{2\alpha+1} - \frac{(\alpha+1)h^2 e^{-6kt}}{6(2\alpha+1)c_2^2 k^2} - \frac{\rho_0 e^{-3k(1+\omega_m)t}}{3k^2 C_2^{1+\omega_m}}. \qquad (53)$$

Then the overall density parameter given by $\Omega = \Omega_m + \Omega_\Lambda$ is

$$\Omega = \frac{\alpha}{2\alpha+1} - \frac{(\alpha+1)h^2}{6(2\alpha+1)c_2^2 k^2 e^{6kt}}. \qquad (54)$$

It is found that the nature of density parameters is same for all $k > 0$, we have shown it graphically for $k = 1$. It is observed that the DE energy density (Ω_Λ) dominates the evolution of universe throughout the time, which may be the probable cause for accelerating expansion of the present universe. The value of the total density parameter is less than 1, and hence our derived model predicts an open universe. This is not strictly compatible with the observational results as the present day universe is very close to the flat universe.

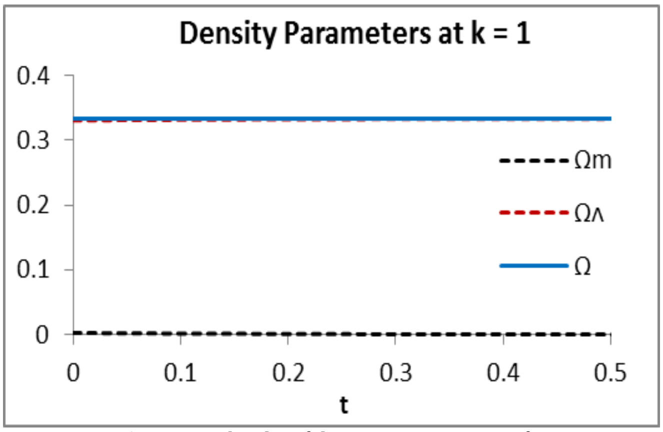

Figure 19: The plot of density parameters vs. t for $\alpha = \rho_0 = h = 1,\ c_2 = 25$ and $\omega_m = 0.5$ in non-interacting two-fluid model with exponential volumetric expansion.

6.2.2 Interacting two-fluid model

Using equation (46) in the equations (18), (19) and (28), we obtain

$$\rho_m = \frac{\rho_0}{c_2^{(1+\omega_m-\delta)}} e^{-3k(1+\omega_m-\delta)t}, \qquad (55)$$

$$\rho_\Lambda = \frac{3\alpha k^2}{(2\alpha+1)} - \frac{(\alpha+1)h^2 e^{-6kt}}{2(2\alpha+1)c_2^2} - \frac{\rho_0 e^{-3k(1+\omega_m-\delta)t}}{c_2^{(1+\omega_m-\delta)}}, \qquad (56)$$

$$p_\Lambda = -\frac{3\alpha k^2}{(2\alpha+1)} - \frac{(3\alpha+1)h^2 e^{-6kt}}{2(2\alpha+1)c_2^2} - \frac{\rho_0 \omega_m e^{-3k(1+\omega_m-\delta)t}}{c_2^{(1+\omega_m-\delta)}} \qquad (57)$$

The EoS parameter for dark energy is obtained in terms of cosmic time t as

$$\omega_\Lambda = \left[\frac{\dfrac{-3\alpha k^2}{(2\alpha+1)} - \dfrac{(3\alpha+1)h^2 e^{-6kt}}{2(2\alpha+1)c_2^2} - \dfrac{\rho_0 \omega_m e^{-3k(1+\omega_m-\delta)t}}{c_2^{(1+\omega_m-\delta)}}}{\dfrac{3\alpha k^2}{(2\alpha+1)} - \dfrac{(\alpha+1)h^2 e^{-6kt}}{2(2\alpha+1)c_2^2} - \dfrac{\rho_0 e^{-3k(1+\omega_m-\delta)t}}{c_2^{(1+\omega_m-\delta)}}}\right] \qquad (58)$$

The metric (47) together with (55) to (58) represents zero-mass scalar fields coupled with two-fluid interacting DE model in $f(R, T)$ gravity.

Figures (20) to (24) depict the time variation of energy density and pressure of DE and barotropic matter, and EoS parameter of DE, in interacting two-fluid scenario of the model with exponential volumetric expansion. From the figures, it is observed that the above stated parameters behave in same way as in the case of non-interacting two-fluids, but with small increase/decrease in their values. The values of ρ_Λ, p_Λ and ω_Λ in an interacting case are slightly decreased than in non-interacting case, while the values of ρ_m and p_m in an interacting case are slightly increased than in non-interacting case. The graph of EoS parameter shows that the interacting model remains present in the phantom region throughout the evolution of universe.

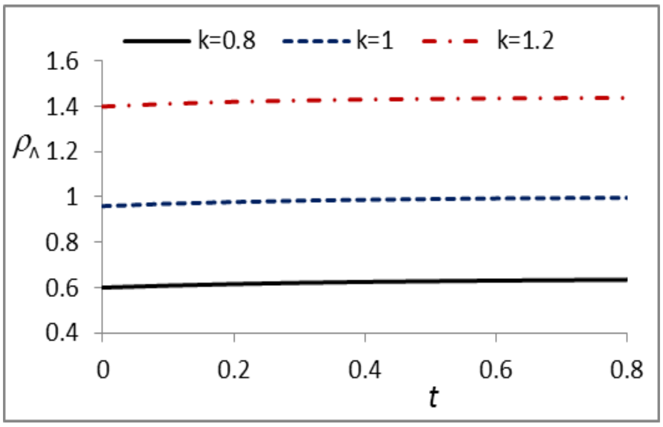

Figure 20: The plot of ρ_Λ vs. t for $\alpha=\rho_0=h=1,\ c_2=25,$ and $\omega_m=\delta=0.5$ in interacting two-fluid model with exponential volumetric expansion.

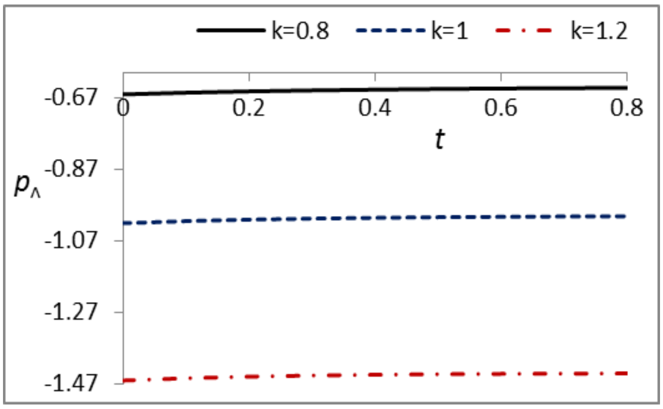

Figure 21: The plot of p_Λ vs. t for $\alpha=\rho_0=h=1,\ c_2=25,$ and $\omega_m=\delta=0.5$ in interacting two-fluid model with exponential volumetric expansion.

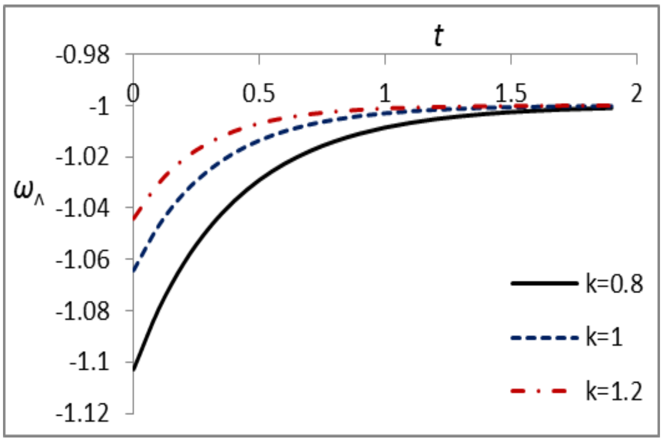

Figure 22: The plot of ω_Λ vs. t for $\alpha=\rho_0=h=1,\ c_2=25,$ and $\delta=\omega_m=0.5$ in interacting two-fluid model with exponential volumetric expansion.

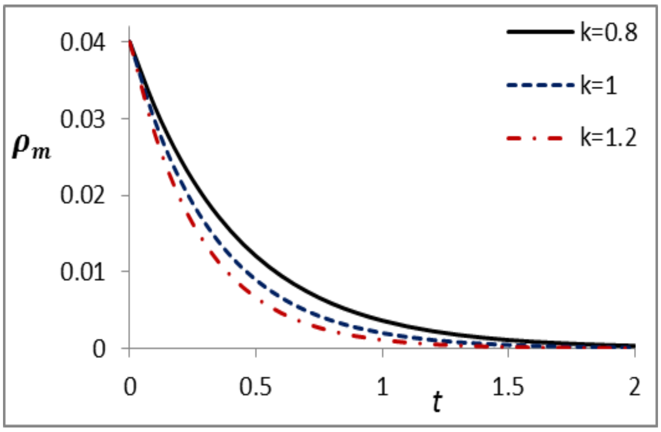

Figure 23: The plot of ρ_m vs. t for $\rho_0=1,\ c_2=25,$ and $\omega_m=\delta=0.5$ in interacting two-fluid model with exponential volumetric expansion.

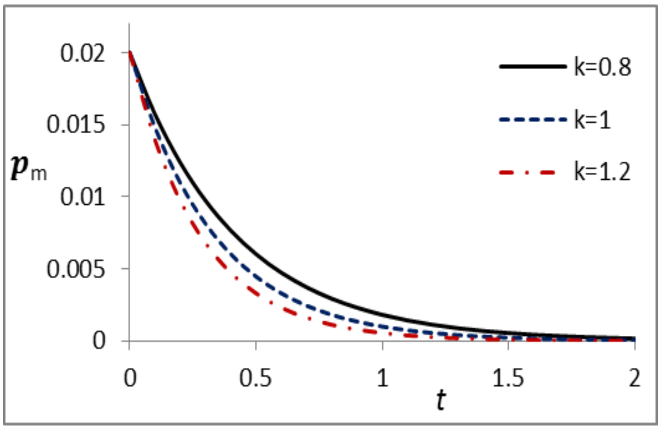

Figure 24: The plot of p_m vs. t for $\rho_0=1,\ c_2=25,$ and $\omega_m=\delta=0.5$ in interacting two-fluid model with exponential volumetric expansion.

The density parameters $\Omega_m=\rho_m/3H^2$ and $\Omega_\Lambda=\rho_\Lambda/3H^2$ corresponding to the barotropic matter and DE components respectively, are obtained as

$$\Omega_m=\frac{\rho_0\, e^{-3k(1+\omega_m-\delta)t}}{3k^2\, c_2^{(1+\omega_m-\delta)}}\,, \tag{59}$$

and

$$\Omega_\Lambda=\frac{\alpha}{2\alpha+1}-\frac{(\alpha+1)h^2e^{-6kt}}{6(2\alpha+1)k^2c_2^2}-\frac{\rho_0\, e^{-3k(1+\omega_m-\delta)t}}{3k^2c_2^{(1+\omega_m-\delta)}}\,. \tag{60}$$

Then by adding (59) and (60), we obtain the overall density parameter Ω, which is the same as obtained in non-interacting case, and hence has the same properties as discussed in the non-interacting case.

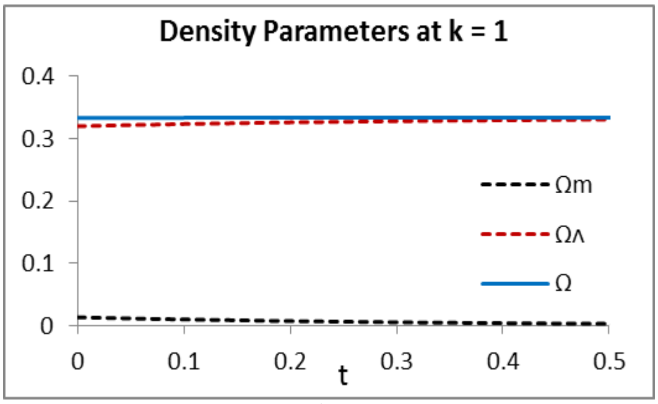

Figure 25: The plot of density parameters vs. t for $\alpha = \rho_0 = h = 1,\ c_2 = 25$, and $\omega_m = \delta = 0.5$ in interacting two-fluid model with exponential volumetric expansion.

In case of interacting fluids the values of density parameters Ω_m and Ω_Λ corresponding to the barotropic matter and DE components are slightly increased and decreased respectively than their values in non-interacting case.

Also, the expressions for $\rho_{(eff)}$, $(\rho - p)_{(eff)}$, $(\rho + p)_{(eff)}$ and $(\rho + 3p)_{(eff)}$ are same as in the non-interacting case, and hence follow the same energy conditions.

7. Physical stability of the models

For the stability of the solutions obtained in non-interacting and interacting cases of both power law and exponential law models, we should check whether our models are physically acceptable or not. For this it is required that the velocity of sound (ϑ_s) should be less than the velocity of light (c), i.e., within the range $0 < \vartheta_s (= \dot{p}/\dot{\rho}) \leq 1$.

(1) Power law models:

In our non-interacting and interacting two-fluid models with power law volumetric expansion, we obtained the sound speeds as

$$\vartheta_s = \left[\frac{\dfrac{2\alpha m\left[3m(\alpha+1)-(3\alpha+2)\right]}{(\alpha+1)(2\alpha+1)t^3} + \dfrac{3m(3\alpha+1)h^2}{(2\alpha+1)c_1^2\, t^{6m+1}} + \dfrac{3m\,\rho_0\,\omega_m\,(1+\omega_m)}{c_1^{(1+\omega_m)}\, t^{3m(1+\omega_m)+1}}}{\dfrac{-2\alpha m\left[3m(\alpha+1)+\alpha\right]}{(\alpha+1)(2\alpha+1)t^3} + \dfrac{3m(\alpha+1)h^2}{(2\alpha+1)c_1^2\, t^{6m+1}} + \dfrac{3m\,\rho_0\,(1+\omega_m)}{c_1^{(1+\omega_m)}\, t^{3m(1+\omega_m)+1}}}\right] \quad (61)$$

and

$$\vartheta_s = \left[\frac{\dfrac{2\alpha m\left[3m(\alpha+1)-(3\alpha+2)\right]}{(\alpha+1)(2\alpha+1)t^3} + \dfrac{3m(3\alpha+1)h^2}{(2\alpha+1)c_1^2\, t^{6m+1}} + \dfrac{3m\,\rho_0\,\omega_m\,(1+\omega_m-\delta)}{c_1^{(1+\omega_m-\delta)}\, t^{3m(1+\omega_m-\delta)+1}}}{\dfrac{-2\alpha m\left[3m(\alpha+1)+\alpha\right]}{(\alpha+1)(2\alpha+1)t^3} + \dfrac{3m(\alpha+1)h^2}{(2\alpha+1)c_1^2\, t^{6m+1}} + \dfrac{3m\,\rho_0\,(1+\omega_m-\delta)}{c_1^{(1+\omega_m-\delta)}\, t^{3m(1+\omega_m-\delta)+1}}}\right] \quad (62)$$

The graphical behavior of the sound speeds given by (61) and (62) are shown in the Figure 26 and Figure 27, respectively.

In both non-interacting and interacting cases for $m = 0.4$ ($0 < m < 1$, case of decelerating expansion); the sound speed $\vartheta_s \approx 0.8$, i.e., $0 < \vartheta_s < 1$, throughout the evolution of the universe.

In non-interacting case for $m \geq 1$ (case of accelerating expansion); the sound speed ϑ_s vary rapidly from 5 to – 4 in the initial epoch, and then it start increasing and assumes constant negative value near to zero. Thus, we obtain $\vartheta_s < 0$ throughout the evolution of the universe.

In interacting case for $m \geq 1$ (case of accelerating expansion); the sound speed ϑ_s vary rapidly from 3 to – 42 in the initial epoch, and then it increases rapidly and assumes the constant negative value near to zero. In this case also, we obtain $\vartheta_s < 0$ throughout the evolution of the universe.

Thus, our non-interacting and interacting two-fluid model with power law volumetric expansion is stable in decelerating case (for $0 < m < 1$, i.e., $q > 0$), and unstable in accelerating case (for $m \geq 1$, i.e., $q \leq 0$).

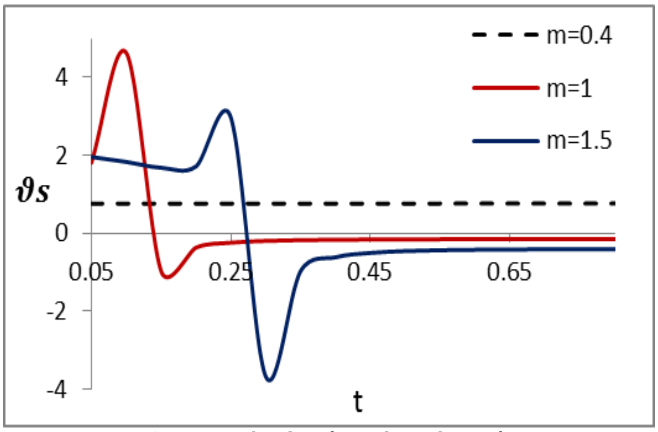

Figure 26: The plot of sound speed Vs. t for $\alpha = \rho_0 = h = 1,\ c_1 = 100$, $\omega_m = 0.5$ in non-interacting two-fluid model with power law volumetric expansion.

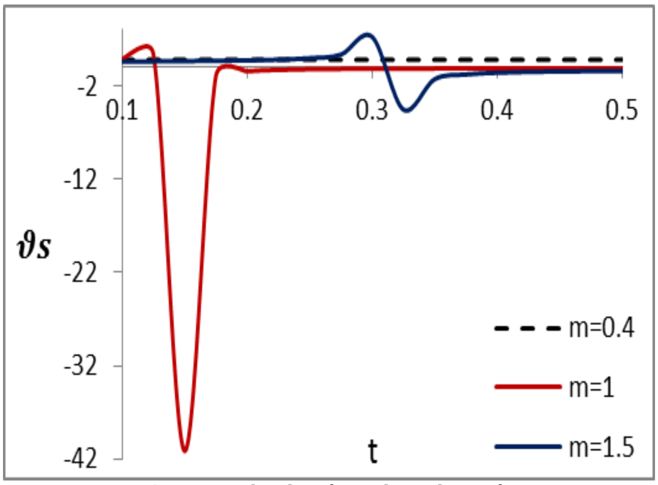

Figure 27: The plot of sound speed Vs. *t* for $\alpha=\rho_0=h=1,\ c_1=100$, $\omega_m=0.5,\ \delta=0.8$ in interacting two-fluid model with power law volumetric expansion.

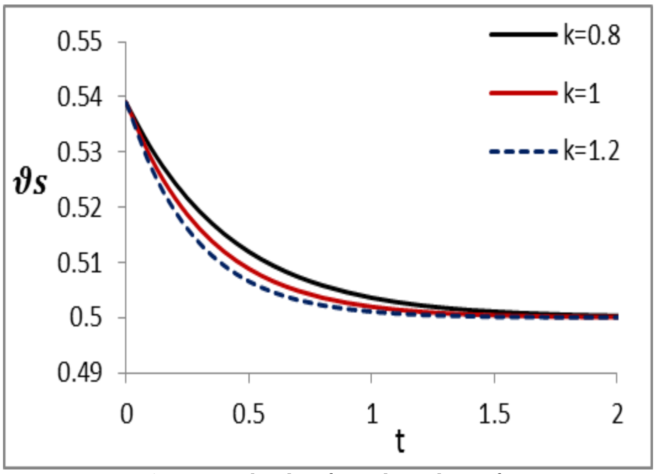

Figure 29: The plot of sound speed Vs. *t* for $\alpha=\rho_0=h=1,\ c_2=25$, and $\omega_m=\delta=0.5$ in interacting two-fluid model with exponential volumetric expansion.

(2) Exponential models:

In our non-interacting and interacting two-fluid models with exponential volumetric expansion, we obtained the sound speeds as

$$\vartheta_s=\left[\frac{\dfrac{3k(3\alpha+1)h^2}{(2\alpha+1)c_2^2e^{6kt}}+\dfrac{3k\rho_0\,\omega_m(1+\omega_m)}{c_2^{1+\omega_m}e^{3k(1+\omega_m)t}}}{\dfrac{3k(\alpha+1)h^2}{(2\alpha+1)c_2^2e^{6kt}}+\dfrac{3k\rho_0(1+\omega_m)}{c_2^{1+\omega_m}e^{3k(1+\omega_m)t}}}\right], \tag{63}$$

and

$$\vartheta_s=\left[\frac{\dfrac{3k(3\alpha+1)h^2}{(2\alpha+1)c_2^2e^{6kt}}+\dfrac{3k\rho_0\,\omega_m(1+\omega_m-\delta)}{c_2^{1+\omega_m-\delta}e^{3k(1+\omega_m-\delta)t}}}{\dfrac{3k(\alpha+1)h^2}{(2\alpha+1)c_2^2e^{6kt}}+\dfrac{3k\rho_0(1+\omega_m-\delta)}{c_2^{1+\omega_m-\delta}e^{3k(1+\omega_m-\delta)t}}}\right]. \tag{64}$$

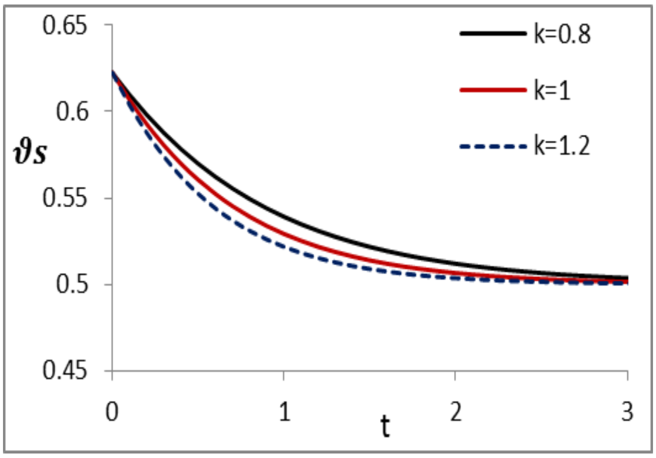

Figure 28: The plot of sound speed Vs. *t* for $\alpha=\rho_0=h=1,\ c_2=25$ and $\omega_m=0.5$ in non-interacting two-fluid model with exponential volumetric expansion.

Figures 28 and 29 depict the graphical behavior of sound speeds given by (63) and (64), respectively.

It is observed from the figures that in case of interacting fluids, the sound speed is slightly decreased than in case of non-interacting fluids. But, in both non-interacting and interacting cases the sound speed ϑ_s satisfies the condition: $0<\vartheta_s<1$, throughout the evolution of the universe. Thus, our non-interacting and interacting two-fluid models with exponential volumetric expansion are physically stable.

8. Discussion and Conclusions

In this paper we have studied the two-fluid scenario coupled with zero-mass scalar fields in the *f* (*R*, *T*) theory of gravity for isotropic flat FLRW space-time. The non-interacting and interacting two-fluids models have been considered and discussed by assuming power law ($V=c_1t^{3m}$) and exponential law ($V=c_2e^{3kt}$) of volumetric expansion. The derived power law cosmological models exhibit both decelerating as well as accelerating phase of expansion, and shows physical stability and instability, depending on the values of *m*. The models with exponential law of volumetric expansion are accelerating and physically stable for all $k\geq 0$. Both the models are shear free and isotropic throughout the evolution of the universe. In both non-interacting and interacting two-fluid models with power law of volumetric expansion, the parameters such as ρ_Λ , p_Λ , ω_Λ , ρ_m , p_m , ϑ_s, etc., behave in a same way but with only a slight increase/decrease in their values, while the energy conditions and overall density parameters are the same. The same things are observed are in case of both non-interacting and interacting two-fluid models with exponential law of volumetric expansion.

- The power law model has initial singularity, i.e., the universe starts evolving with a big-bang, and it expands with decreasing rate; and it is free from a big rip. The power law of volumetric expansion yields a constant deceleration parameter whose value depends on *m*. The values $0 < m < 1$, and $m = 1$, and $m < 0$, $m > 1$ correspond to the universe's decelerating expansion, expansion with constant rate, and accelerating expansion, respectively.

For $0 < m < 1$ (case of decelerating expansion), the EoS parameter (ω_Λ) of DE has a constant positive value 0.78 (approx.) showing the matter dominance of the universe. In this case, all type of energy conditions are satisfied and the speed of sound $\vartheta_s \approx 0.8$, i.e., $0 < \vartheta_s < 1$ throughout the evolution of the universe, which shows the model is physically stable and hence acceptable.

For $m \geq 1$ (case of accelerating and constant expansion), at early stage of evolution of the universe, there is a rapid transition from matter dominated region $(\omega_\Lambda > 0)$ to quintessence region $(-1 < \omega_\Lambda < 0)$ and then to a phantom region $(\omega_\Lambda < -1)$, which is a Quintom DE scenario stated by Zhang [85]; and later on it remains present in the quintessence region $(-1 < \omega_\Lambda < 0)$ throughout the evolution, which is acceptable as per the SNe Ia observational data. In this case, all the energy conditions are satisfied except SEC and it is due to an accelerating cosmic expansion. The speed of sound, $\vartheta_s < 0$ throughout the evolution of the universe, shows the model is physically unstable.

The value of overall density parameter Ω is lying between 0.35 and 0.5 (i.e., $\Omega < 1$), showing the universe is open, which is not strictly compatible with the present-day observations of a flat universe, but resembles with the results favouring a universe with spatial curvature [6-7].

- The model with exponential law of expansion is free from any kind of singularity, starts expanding with some nonzero fixed volume (c_2), and it expands exponentially with the time. The model is shear free and isotropic throughout the evolution of the universe. An assumed exponential law of expansion yields constant negative value (-1) of a deceleration parameter which indicates an accelerating expansion of the universe. The EoS parameter (ω_Λ) of DE component has a fixed negative value less than -1 initially, it increases with time and approaches towards -1, but does not cross the phantom divide or cosmological constant ($\omega_\Lambda = -1$) region, and it remains present in the phantom region throughout the evolution of universe. This is found compatible with the cosmological tests based on present data, including SNe Ia data as well as CMB anisotropy and mass power spectrum. Thus our derived model represents early stage evolution as well as the present universe. It is observed that the DE energy density dominates the evolution of universe throughout the time, which may be the probable cause for accelerating expansion of the present universe. The value of the total density parameter is less than 1, and hence our derived model predicts an open universe. This is not strictly compatible with the observational results as the present day universe is very close to the flat universe. It is observed that only the weak energy condition is satisfied in this model. The sound speed ϑ_s satisfies the condition: $0 < \vartheta_s < 1$ throughout the evolution of the universe. Thus, the model with exponential volumetric expansion is physically stable.

In summary, we found that the models with exponential volumetric expansion are open, have accelerating expansion and physically stable; while, the models with power law volumetric expansion are open in both accelerating and decelerating cases, but physically stable and unstable in decelerating and accelerating case respectively. So, our results are in fair resemblance with the relevant observations found in [1-11, 66, 72-77] with the exception that the present-day observational results shows universe is very close to flat. Thus, the solutions obtained in this paper may be useful for exploration and understanding of various characteristics of DE models in the evolution of universe within the scope of *f*(*R*, *T*) theory of gravitation.

Conflict of Interest

Authors declare that they do not have any conflict of interest.

Funding Source

This research work has been carried out without any support from funding agencies.

Acknowledgements

Authors are grateful to the reviewers and editors for their constructive comments and suggestions which have helped in the improvement of the manuscript.

References

[1] Garnavich, Peter M., Robert P. Kirshner, Peter Challis, John Tonry, Ron L. Gilliland, R. Chris Smith, Alejandro Clocchiatti et al. "Constraints on cosmological models from Hubble Space Telescope observations of high-z supernovae." *The Astrophysical Journal* **493**, no. **2**, L**53, 1998**.

[2] Garnavich, Peter M., Saurabh Jha, Peter Challis, Alejandro Clocchiatti, Alan Diercks, Alexei V. Filippenko, Ron L. Gilliland et al. "Supernova limits on the cosmic equation of state." *The Astrophysical Journal* **509**, no. **1**, **74, 1998**.

[3] Riess, Adam G., Alexei V. Filippenko, Peter Challis, Alejandro Clocchiatti, Alan Diercks, Peter M. Garnavich, Ron L. Gilliland et al. "Observational evidence from supernovae for an accelerating universe and a cosmological constant." *The astronomical journal* **116**, no. **3**, **1009, 1998**.

[4] Riess, Adam G., Louis-Gregory Strolger, John Tonry, Stefano Casertano, Henry C. Ferguson, Bahram Mobasher, Peter Challis et al. "Type Ia supernova discoveries at z> 1 from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution." *The Astrophysical Journal* **607**, no. **2**, **665, 2004**.

[5] Perlmutter, Saul, Goldhaber Aldering, Gerson Goldhaber, R. A. Knop, Peter Nugent, Patricia G. Castro, Susana Deustua et al. "Measurements of Ω and Λ from 42 high-redshift supernovae." *The Astrophysical Journal* **517**, no. **2**, **565, 1999**.

[6] Bennett, C. L. "First year Wilkinson microwave anisotropy probe (WMAP) observations: Preliminary maps and basic results." *Astrophys. J. Suppl* **148**, no. **1, 2003**.

[7] Spergel, David N., Licia Verde, Hiranya V. Peiris, Eiichiro Komatsu, M. R. Nolta, Charles L. Bennett, Mark Halpern et al. "First-year Wilkinson Microwave Anisotropy Probe (WMAP)* observations: determination of cosmological parameters." *The Astrophysical Journal Supplement Series* **148**, no. **1**, **175, 2003**.

[8] Caldwell, Robert R., and Michael Doran. "Cosmic microwave background and supernova constraints on quintessence: concordance regions and target models." *Physical Review D* **69**, no. **10**, **103517, 2004**.

[9] Huang, Zhuo-Yi, Bin Wang, Elcio Abdalla, and Ru-Keng Su. "Holographic explanation of wide-angle power correlation suppression in the cosmic microwave background radiation." *Journal of Cosmology and Astroparticle Physics* **2006**, no. **05**, **013, 2006**.

[10] Tegmark, Max, Michael A. Strauss, Michael R. Blanton, Kevork Abazajian, Scott Dodelson, Havard Sandvik, Xiaomin Wang et al. "Cosmological parameters from SDSS and WMAP." *Physical review D* **69**, no. **10, 103501, 2004**.

[11] Tegmark, Max, Michael R. Blanton, Michael A. Strauss, Fiona Hoyle, David Schlegel, Roman Scoccimarro, Michael S. Vogeley et al. "The three-dimensional power spectrum of galaxies from the sloan digital sky survey." *The Astrophysical Journal* **606**, no. **2** (**2004**): **702**.

[12]Weinberg, Steven. "The cosmological constant problem." *Reviews of modern physics* **61**, no. **1** (**1989**): **1**.

[13] Carroll, Sean M. "The cosmological constant." *Living reviews in relativity* **4**, no. **1** (**2001**): **1-56**.

[14] Peebles, P. James E., and Bharat Ratra. "The cosmological constant and dark energy." *Reviews of modern physics* **75**, no. **2** (**2003**): **559**.

[15] Padmanabhan, Thanu. "Cosmological constant—the weight of the vacuum." *Physics Reports* **380**, no. **5-6** (**2003**): **235-320**.

[16] Eisenstein, Daniel J., Idit Zehavi, David W. Hogg, Roman Scoccimarro, Michael R. Blanton, Robert C. Nichol, Ryan Scranton et al. "Detection of the baryon acoustic peak in the large-scale correlation function of SDSS luminous red galaxies." *The Astrophysical Journal* **633**, no. **2** (**2005**): **560**.

[17] Nojiri, Shin'ichi, and Sergei D. Odintsov. "Introduction to modified gravity and gravitational alternative for dark energy." *International Journal of Geometric Methods in Modern Physics* **4**, no. **01** (**2007**): **115-145**.

[18] Aghanim, Nabila, Yashar Akrami, Mark Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini, A. J. Banday et al. "Planck 2018 results-VI. Cosmological parameters." *Astronomy & Astrophysics* **641** (**2020**): **A6**.

[19] Babourova, O. G., and B. Frolov. "The solution of the cosmological constant problem: the cosmological constant exponential decrease in the super-early Universe." *Universe* **6**, no. **12** (**2020**): **230**.

[20] Geng, C. Q., C. C. Lee, and L. Yin. "Constraints on a special running vacuum model." *The European Physical Journal C* **80**, no. **1** (**2020**): **69**.

[21] Wetterich, Christof. "Cosmology and the fate of dilatation symmetry." *Nuclear Physics B* **302**, no. **4** (**1988**): **668-696**.

[22] Ratra, Bharat, and Philip JE Peebles. "Cosmological consequences of a rolling homogeneous scalar field." *Physical Review D* **37**, no. **12**, **3406, 1988**.

[23] Cicoli, Michele, Francisco G. Pedro, and Gianmassimo Tasinato. "Natural quintessence in string theory." *Journal of Cosmology and Astroparticle Physics* **2012**, no. **07**, **044, 2012**.

[24] Wetterich, Christof. "Inflation, quintessence, and the origin of mass." *Nuclear Physics B* **897** (**2015**): **111-178**.

[25] Tsujikawa, Shinji. "Quintessence: a review." *Classical and Quantum Gravity* **30**, no. **21** (**2013**): **214003**.

[26] Turner, **M. S.**, and M. White. "CDM models with a smooth component." *Physical Review D* **56**, no. 8 (**1997**): **R4439**.

[27] Xu, L., Y. Wang, M. Tong, and H. Noh. "CMB temperature and matter power spectrum in a decay vacuum dark energy model." *Physical Review D* **84**, no. 12 (**2011**): **123004**.

[28] Armendariz-Picon, C., T. Damour, and V. Mukhanov. "k-Inflation." *Physics Letters B* **458**, no. **2-3** (**1999**): **209-218**.

[29] Chiba, Takeshi, Takahiro Okabe, and Masahide Yamaguchi. "Kinetically driven quintessence." *Physical Review D* **62**, no. **2** (**2000**): **023511**.

[30] Varshney, Gunjan, Umesh Kumar Sharma, and Anirudh Pradhan. "Reconstructing the k-essence and the dilation field models of the THDE in f (R, T) gravity." *The European Physical Journal Plus* **135**, no. **7** (**2020**): **541**.

[31] Caldwell, Robert R. "A phantom menace? Cosmological consequences of a dark energy component with super-negative equation of state." *Physics Letters B* **545**, no. **1-2** (**2002**): **23-29**.

[32] Caldwell, Robert R., Marc Kamionkowski, and Nevin N. Weinberg. "Phantom energy: dark energy with w<− 1 causes a cosmic doomsday." *Physical review letters* **91**, no. **7** (**2003**): **071301**.

[33] Elizalde, Emilio, Shin'ichi Nojiri, and Sergei D. Odintsov. "Late-time cosmology in a (phantom) scalar-tensor theory: dark energy and the cosmic speed-up." *Physical Review D* **70**, no. **4** (**2004**): **043539**.

[34] Anisimov, Alexey, Eugeny Babichev, and Alexander Vikman. "B-inflation." *Journal of Cosmology and Astroparticle Physics* **2005**, no. **06** (**2005**): **006**.

[35] Sen, Ashoke. "Rolling tachyon." *Journal of High Energy Physics* **2002**, no. **04** (**2002**): **048**.

[36] Padmanabhan, T., and T. Roy Choudhury. "Can the clustered dark matter and the smooth dark energy arise from the same scalar field?." *Physical Review D* **66**, no. **8** (**2002**): **081301**.

[37] Khoury, Justin, and Amanda Weltman. "Chameleon fields: Awaiting surprises for tests of gravity in space." *Physical review letters* **93**, no. **17** (**2004**): **171104**.

[38] Nojiri, Shin'ichi, Sergei D. Odintsov, V. K. Oikonomou, and Tanmoy Paul. "Unifying holographic inflation with holographic dark energy: A covariant approach." *Physical Review D* **102**, no. **2** (**2020**): **023540**.

[39] Shekh, S. H. "Models of holographic dark energy in f (Q) gravity." *Physics of the dark Universe* **33** (**2021**): **100850**.

[40] Wang, Shuang, Yi Wang, and Miao Li. "Holographic dark energy." *Physics reports* **696** (**2017**): **1-57**.

[41] S. D. Katore, S. V. Gore, and A. Y. Shaikh, "Holographic Dark Energy Cosmological Models in f(G) Theory," *New Astron.*, vol. **80**, p. **101420**, Oct. **2020**, doi: 10.1016/j.newast.2020.101420.

[42] Pawar, D. D., R. V. Mapari, and P. K. Agrawal. "A modified holographic Ricci dark energy model in f (R, T) theory of gravity." *Journal of Astrophysics and Astronomy* **40**, pp.**1-8, 2019**.

[43] Zhang, Jing-Fei, Yun-He Li, and Xin Zhang. "A global fit study on the new agegraphic dark energy model." *The European Physical Journal C* **73**, no. **1**, **2280, 2013**.

[44] Pourbagher, A., and Alireza Amani. "Thermodynamics of the viscous f (T, B) gravity in the new agegraphic dark energy model." *Modern Physics Letters A* **35**, no. **20**, **2050166, 2020**.

[45] Pourhassan, Behnam. "Viscous modified cosmic Chaplygin gas cosmology." *International Journal of Modern Physics D* **22**, no. **09**, **1350061, 2013**.

[46] Pourhassan, Behnam, and E. O. Kahya. "Extended Chaplygin gas model." *Results in Physics* **4**, **101, 2014**.

[47] Kahya, Emre Onur, and B. Pourhassan. "The universe dominated by the extended Chaplygin gas." *Modern Physics Letters A* **30**, no. **13**, **1550070, 2015**.

[48] Campos, Juliano P., Júlio C. Fabris, Rafael Perez, Oliver F. Piattella, and Hermano Velten. "Does Chaplygin gas have salvation?." *The European Physical Journal C* **73**, **1-15, 2013**.

[49] Avelino, P. P., K. Bolejko, and G. F. Lewis. "Nonlinear Chaplygin gas cosmologies." *Physical Review D* **89**, no. **10** (**2014**): **103004**.

[50] Buchdahl, Hans A. "Non-linear Lagrangians and cosmological theory." *Monthly Notices of the Royal Astronomical Society* **150**, no. **1** (**1970**): **1-8**.

[51] Ferraro, R., and F. Fiorini. "Modified teleparallel gravity: Inflation without an inflaton." *Physical Review D* **75**, no. **8** (**2007**): **084031**.

[52] Bengochea, G. R., and R. Ferraro. "Dark torsion as the cosmic speed-up." *Physical Review D* **79**, no. **12** (**2009**): **124019**.

[53] Harko, Tiberiu, Francisco SN Lobo, Shin'ichi Nojiri, and Sergei D. Odintsov. "f (R, T) gravity." *Physical Review D* **84**, no. **2** (**2011**): **024020**.

[54] Felice, A. De, and S. Tsujikawa. "Construction of cosmologically viable f (G) gravity models." *Physics Letters B* **675**, no. **1**, (**2009**): **1-8**.

[55] Bamba, K., M. Ilyas, M. Z. Bhatti, and Z. Yousaf. "Energy conditions in modified f (G) gravity." *General Relativity and Gravitation* **49** (**2017**): **1-17**.

[56] Goheer, N., R. Goswami, P. KS Dunsby, and K. Ananda. "Coexistence of matter dominated and accelerating solutions in f (G) gravity." *Physical Review D* **79**, no. **12** (**2009**): **121301**.

[57] Elizalde, E., R. Myrzakulov, V. V. Obukhov, and D. Sáez-Gómez. "ΛCDM epoch reconstruction from F (R, G) and modified Gauss–Bonnet gravities." *Classical and Quantum Gravity* **27**, no. **9** (**2010**): **095007**.

[58] Bamba, K., S. D. Odintsov, L. Sebastiani, and S. Zerbini. "Finite-time future singularities in modified Gauss–Bonnet and $\mathcal{F}$ (R, G) gravity and singularity avoidance." *The European Physical Journal C* **67**(**2010**): **295-310**.

[59] Bahamonde, S., M. Marciu, and P. Rudra. "Generalised teleparallel quintom dark energy non-minimally coupled with the scalar torsion and a boundary term." *Journal of Cosmology and Astroparticle Physics* **2018**, no. **04** (**2018**): **056**.

[60] Jiménez, J. B., L. Heisenberg, and T. Koivisto. "Coincident general relativity." *Physical Review D* **98**, no. **4** (**2018**): **044048**.

[61] Jiménez, J. B., L. Heisenberg, T. Koivisto.and S. Pekar. "Cosmology in f (Q) geometry." *Physical Review D* **101**, no. **10** **(**2020**): 103507**.

[62] Xu, Y., G. Li, T. Harko, and S. D. Liang. "f (Q, T) gravity." *The European Physical Journal C* **79** (**2019**): **1-19**.

[63] Xu, Y., T. Harko, S. Shahidi, and S. D. Liang. "Weyl type f (Q, T) gravity, and its cosmological implications." *The European Physical Journal C* **80**, no. **5** (2020): **1-22**.

[64] Miranda, V., S. E. Jorás, L. Waga, and M. Quartin. "Viable singularity-free f (R) gravity without a cosmological constant." *Physical Review Letters* **102**, no. **22** (**2009**): **221101**.

[65] Sharif, M., and Z. Yousaf. "Instability of a dissipative restricted non-static axial collapse with shear viscosity in f (R) gravity." *Journal of Cosmology and Astroparticle Physics* **2014**, no. **06**, **019, 2014**.

[66] Chirde, V. R., and S. H. Shekh. "Isotropic background for interacting two fluid scenario coupled with zero mass scalar field in modified gravity." *Bulg. J. Phys* **43** (2016): **156-169**.

[67] Pawar, K., N. T. Katre, and S. H. Shekh. "Accelerating expansion of two fluid Universe coupled with zero mass scalar field in f(R) gravity". *J. of Emerging Tech. and Research*, **9**, no. **12**, pp. **e294-e300, 2022**.

[68] Houndjo, M. J. S., and O. F. Piattella. "Reconstructing f (R, T) gravity from holographic dark energy." *International Journal of Modern Physics D* **21**, no. **03**, **1250024, 2012**.

[69] Samanta, G. C. "Universe filled with dark energy (DE) from a wet dark fluid (WDF) in f (R, T) gravity." *International Journal of Theoretical Physics* **52**, pp.**2303-2315, 2013**.

[70] Singh, J. K., A. Singh, G. K. Goswami, and J. Jena. "Dynamics of a parametrized dark energy model in f (R, T) gravity." *Annals of Physics* **443** (**2022**): **168958**.

[71] Houndjo, M. J. S. "Reconstruction of f (R, T) gravity describing matter dominated and accelerated phases." *International Journal of Modern Physics D* **21**, no. **01** (**2012**): **1250003**.

[72] Amirhashchi, H., A. Pradhan, and B. Saha. "An interacting two-fluid scenario for dark energy in an FRW universe." *Chinese Physics Letters* **28**, no. **3** (**2011**): **039801**.

[73] Amirhashchi, H., A. Pradhan, and H. Zainuddin. "An interacting and non-interacting two-fluid dark energy models in FRW universe with time dependent deceleration parameter." *International Journal of Theoretical Physics* **50** (**2011**): **3529-3543**.

[74] Amirhashchi, H., A. Pradhan, and H. Zainuddin. "Interacting two-fluid viscous dark energy models in a non-flat universe." *Research in Astronomy and Astrophysics* **13**, no. **2** (**2013**): **129**.

[75] Pradhan, A., H. Amirhashchi, and B. Saha. "An interacting and non-interacting two-fluid scenario for dark energy in FRW universe with constant deceleration parameter." *Astrophysics and Space Science* **333**, pp.**343-350, 2011**.

[76] Saha, B., H. Amirhashchi, and A. Pradhan. "Two-fluid scenario for dark energy models in an FRW universe-revisited." *Astrophysics and Space Science* **342**, no. **1**, **257-267, 2012**.

[77] Adhav, K. S., G. B. Tayade, and A. S. Bansod. "Interacting dark matter and holographic dark energy in an anisotropic universe." *Astrophysics and space science* **353**, pp.**249-257, 2014**.

[78] Rao, V. U. M., M. Vijaya Santhi, T. Vinutha, and Y. Aditya. "Two-Fluid Scenario for Higher Dimensional Dark Energy Cosmological Model in Saez-Ballester Theory of Gravitation." *Prespacetime Journal* **7**, no. **2**, **293-308, 2016**.

[79] Reddy, D.R.K., Santikumar, R. and Naidu, R.L. Bianchi Type III Cosmological Models in f(R, T) Theory of Gravity. Astrophysics and Space Science, **342**, **249-252, 2012**.

[80] Reddy, D. R. K., R. L. Naidu, and B. Satyanarayana. "Kaluza-Klein Cosmological Model in f (R, T) Gravity." *International Journal of Theoretical Physics* **51**, pp.**3222-3227, 2012**.

[81] Singh, C. P., and V. Singh. "Reconstruction of modified f (R, T) gravity with perfect fluid cosmological models." *General Relativity and Gravitation* **46**, pp.**1-14, 2014**.

[82] Sahoo, P. K., B. Mishra, and G. Chakradhar Reddy. "Axially symmetric cosmological model in f (R, T) gravity." *The European Physical Journal Plus* **129**, pp.**1-8, 2014**.

[83] Kumar Yadav, Anil. "Bianchi-V string cosmology with power law expansion in f (R, T) gravity." *The European Physical Journal Plus* **129**, no. **9**, **194, 2014**.

[84] Brahma, Bishnu Prasad, and Mukunda Dewri. "Bulk Viscous Bianchi Type-V Cosmological Model in f (R, T) Theory of Gravity." *Frontiers in Astronomy and Space Sciences* **9, 11, 2022**.

[85] Zhang, Xin. "An interacting two-fluid scenario for quintom dark energy." *Communications in Theoretical Physics* **44**, no. **4, 762, 2005**.

AUTHORS PROFILE

Dr. Kalpana N. Pawar pursued M.Sc. and Ph.D. in Mathematics from the Institute of Science, Nagpur, in 1997 & 2003 respectively. She started her teaching career as an Assistant Professor at Institute of Science, Nagpur, and currently working as Professor & Head, Department of Mathematics, Shivaji Science College, Nagpur (M.S.). She is a Ph.D. supervisor of RTM Nagpur University, Nagpur since 2008 and SGB Amravati University, Amravati since 2019. She had guided five research scholars for award of M. Phil and Ph. D. and five research scholars are presently working under her supervision for the award of Ph.D. She had published various research papers in reputed national and international journals. Her main research area is General Theory of Relativity and Cosmology, Alternating Theories of Gravitation, Modified Theories of Gravitation, and Fuzzy Controllers. She had in all 25 years of teaching experience and 24 years of research experience.

Mr. N. T. Katre (M.Sc., B.Ed., SET) is currently pursuing a Ph.D. and working as an Assistant Professor and Head, Department of Mathematics, Nabira Mahavidyalaya, Katol, Dist. Nagpur (M.S.), India. He has 23 years of teaching experience at U.G., 12 years of teaching experience at P.G., and 8 years of Research Experience. He is a life member of the Indian Mathematical Society since 2007. His main research work focuses on Gravitation Theories/ General Relativity/ Cosmology. He has published more than 17 Books and 13 research papers in reputed national/international journals.

