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Analysis of the Null Geodesics in Schwarzschild De-Sitter Space-Time

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Abstract— The gravitational field of a spherically symmetric mass in a universe with cosmological constant (ʌ) is described in terms of Schwarzschild de-Sitter (SdS) space-time. Based on this space-time we investigated the geodesics of a test particle- photon. We performed our study on the space-time in between the cosmological event horizon and black hole event horizon i.e. $r_H < r < r_C$ and also imposed $E^2 > V_{eff}$ for the physical acceptance. We detected two types of orbits, namely periodic bound orbit and terminating bound orbit. The mathematical conditions for the existence of these orbits have been discussed. We found that there will no longer be any periodic bound orbits for a large positive cosmological constant. The analysis of effective potential inferred us that only the peak of the curve changes for change in angular momentum (L). These peaks correspond to circular orbits in null geodesics and a circular orbit of radius 3M becomes an allowed null geodesic.

Keywords— SdS space-time, Photon, Effective potential, Orbits, Cosmological constant

I. INTRODUCTION

Squeezing of matter into a tiny space engenders strong gravity, thereby precluding the possibility of escaping of particles or radiations from what is called a black hole. It is considered that three types of black holes exist in the universe: (a) Black holes of stellar masses, which were developed after the death of massive stars, (b) Supermassive black holes having masses up to 10^9 $M_\odot$, where $M_\odot = 2\times10^{33}$ gm is the Solar mass, and more at centers of galaxies, and (c) Primordial black holes whose presence are attributed to the large-scale inhomogeneities at the nascent expansion of the universe [1]. The black holes' concept burgeoned with Albert Einstein's Theory of General Relativity proposed in 1915. The German physicist Karl Schwarzschild substantiated that black holes are a solution to Einstein's equation in spherical symmetry. It was an exact solution for a stationary black hole which could define the gravitational radius, r_s = 2M (called Schwarzschild radius), the radius below which the gravitation attraction must cause a particle to undergo irreversible gravitational collapse.

Energy and momentum carried by gravitational fields contribute to their own source which implies that gravitational/Einstein field equations are nonlinear partial differential equations [2]. The Einstein field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$ explains all gravitational effects in Solar system explicitly. The Supernova Cosmology Project [3], however, rejuvenated the cosmological constant in theoretical descriptions. The observations, for instance, the fluctuations in the cosmic microwave background, structure formation etc are invariably accentuated by an additional energy–momentum term appearing in the Einstein field equation with cosmological constant, $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$, as an additional cosmological term. The incorporation of ʌ in the field equation is indispensable to explain the observation of an accelerated expanding universe as well [4]. In such universe we discuss the gravitational field of a spherically symmetric mass which is provided by Schwarzschild de-Sitter (SdS) metric. We scrutinize the effective potential for photons' motion and the nature of orbits within null geodesics. In addition, we explore the influences of cosmological constant on the nature of orbits.

The remainder of the paper follows the following structure: in sections II and III the mathematical descriptions of SdS space-time and the analytical solutions of polynomial equation are described, respectively. The effective potential of the test particle is provided in section IV. Subsequently, in section V we emphasize the locations of event horizons. Next, in sections VI and VII we describe the photons' orbits in the background of the SdS black hole and the influences of cosmological constant on those orbits, respectively. Finally, in section VIII the findings of the study are concluded.

II. SCHWARZSCHILD DE-SITTER METRIC

The line element for the SdS space-time [5] is given by

$$ds^2 = -B dt^2 + B^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2, \quad (1)$$

where B(r) is known as the lapse function:

$$B(r)=\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right), \tag{2}$$

and M represents the mass of the black hole. The coordinates are circumscribed within: $t\in(-\infty,\infty), r\in(0,\infty), \theta\in[0,\pi]$, and $\phi\in[0,2\pi]$.

The geodesics equations for a space-time are obtained from the Lagrangian

$$\mathcal{L}=\frac{1}{2}g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}.$$

For the SdS space-time the Lagrangian is calculated as

$$\mathcal{L}=\frac{1}{2}\left[B(r)\dot{t}^2-B^{-1}(r)\dot{r}^2-r^2\dot{\theta}^2-r^2\sin^2\theta\,\dot{\phi}^2\right], \tag{3}$$

where the derivatives are calculated with respect to τ. The canonical momenta are then calculated as

$$p_t=\frac{\partial\mathcal{L}}{\partial\dot{t}}=B(r)\dot{t}=E,$$

$$p_r=-\frac{\partial\mathcal{L}}{\partial\dot{r}}=B^{-1}(r)\dot{r},$$

$$p_\phi=-\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=r^2\sin^2\theta\,\dot{\phi},$$

and

$$p_\theta=-\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=r^2\dot{\theta}.$$

The resulting Hamiltonian is

$$H=p_t\dot{t}-\left(p_r\dot{r}+p_\theta\dot{\theta}+p_\phi\dot{\phi}\right)-\mathcal{L}=\mathcal{L}. \tag{4}$$

Since the Hamiltonian and the Lagrangian are equal, the potential energy is absent in the problem. The energy is solely the kinetic energy as is indeed manifested by the expression for the Lagrangian.

The geodesic is expressed in the equatorial plane which is recognized by $\theta=\frac{\pi}{2}$. Thus,

$$p_\phi=r^2\frac{d\phi}{d\tau}=L=\text{constant}, \tag{5}$$

where L is the angular momentum in an axis normal to the equatorial plane. Thus, the Lagrangian becomes

$$2\mathcal{L}=B^{-1}(r)E^2-B^{-1}(r)\dot{r}^2-\frac{L^2}{r^2}.$$

Using $2\mathcal{L}=0$ for null geodesics [5], we get

$$\dot{r}^2+\frac{L^2}{r^2}\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)=E^2. \tag{6}$$

Subsequently, the equations representing the nature of photon in the background of the Schwarzschild de-Sitter black hole are obtained

$$\frac{\dot{t}}{\dot{r}}=\frac{\dfrac{E}{\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)}}{\sqrt{E^2-\frac{L^2}{r^2}\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)}} \tag{7}$$

and

$$\frac{\dot{\phi}}{\dot{r}}=\frac{\dfrac{L}{r^2}}{\sqrt{E^2-\frac{L^2}{r^2}\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)}} \tag{8}$$

III. ANALYTICAL SOLUTION

For simplicity replacing the affine parameter τ by s, equation (6) becomes

$$\left(\frac{dr}{ds}\right)^2=E^2-\frac{L^2}{r^2}\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right).$$

By introducing the new variables: $u=\frac{r_s}{r}$, $\lambda=\left(\frac{r_s}{L}\right)^2\geq 0$, $\mu=E^2\geq 0$, and $\rho=\frac{\Lambda}{3}r_s^2$, it can be obtained for the null geodesics:

$$\left(\frac{du}{d\phi}\right)^2=\lambda\mu+\rho-u^2+u^3=P_3(u), \tag{9}$$

where $P_3(u)$ is the polynomial of 3rd order. The solutions of the last equation is obtained in terms of the Weierstrass Q function [4] as

$$r(\phi)=\frac{r_s}{4Q(\phi;g_2,g_3)+\frac{1}{3}}, \tag{10}$$

with the Weierstrass invariants:

$$g_2=\frac{1}{12}$$

and

$$g_3=-\frac{1}{8}\left[\frac{1}{27}+\frac{1}{2}\left(u_o^3-u_o^2\right)\right],$$

where $u_o=\frac{r_s}{b}$ and b represents the distance of closest approach [4].

The numbers of positive real zeros of polynomial (P_5 in time-like, P_3 in null geodesics) characterize the form of the resulting orbits. If the positive real zeros are represented by $m_1, ..., m_n$, then the regions which we can accept physically are given by $[0, m_1], [m_2, m_3], ..., [m_n, \infty]$ for even n and by $[m_1, m_2], ..., [m_n, \infty]$ for odd n. We can categorize following classes of orbits according to distance r: (i) the region $[0, m_1]$ corresponds to escape orbits, (ii) the region $[m_n, \infty]$ corresponds to terminating orbits on which the test particles terminate into the singularity, and (iii) the regions $[m_i, m_{i+1}]$ correspond to bound orbits. A particle coming from ∞ falls into the singularity for the devoid of positive real zero [4].

IV. EFFECTIVE POTENTIAL

We can interpret the second term on the left hand side of radial equation (6) as the potential energy in the sense that, together with $\dot{r}^2$ interpreted as kinetic energy, it is a constant of motion. This potential energy term is called the effective potential that takes the form

$$V_{eff}=\frac{L^2}{r^2}\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right). \tag{11}$$

The response of effective potential (V_{eff}) with respect to the different values of angular momentum (L) is presented in Figure (1). The figure exhibits that the peaks of the curves change for change in L. The nature of the curves remains similar for all values of L if the distance (r) increases. It is worth noting that the radial motion (L=0) of photon is independent of the cosmological constant [5]. The peak on the curve corresponds to a circular orbit in null geodesics. The circular geodesic is obtained by differentiating the effective potential and equating with zero. Differentiating the V_{eff},

$$V'_{eff}(r) = \frac{dV_{eff}}{dr} = \frac{L^2}{r^2}\left(\frac{2M}{r^2} - \frac{2\Lambda r}{3}\right) - \frac{2L^2}{r^3}\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) = 0.$$

On simplification, we get

$$V'_{eff}(r) = -\frac{2L^2}{r^3} + \frac{6ML^2}{r^4} = 0,$$

which immediately yields r = 3M. Therefore, a circular orbit of radius r = 3M is an allowed null geodesic even in the Schwarzschild de-Sitter space-time.

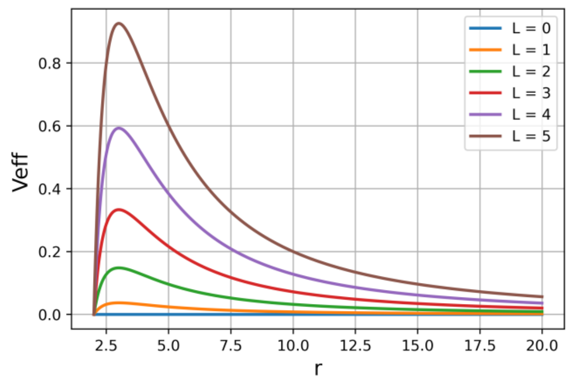


Figure 1: Variation of V_{eff} with angular momentum (L) for $\Lambda = 10^{-6}$.

V. EVENT HORIZONS

The positive cosmological constant repels geodesics. The known exact stationary solutions with $\Lambda > 0$ usually exhibit an external boundary–the cosmological event horizon. The observed small value of Λ sets the length scale of the horizon to be very large in the order of $\frac{1}{\sqrt{\Lambda}}$. It can be speculated that the black hole is situated inside the cosmological horizon and the cosmological event horizon behaves in such a space-time as an outer causal boundary. Outside the cosmological horizon the time-like Killing vector field changes to space-like and communication is occluded along a future directed causal path, which makes the use of asymptotics for an observer located inside the cosmological horizon trivial [6].

Mathematically, the metric coefficient of dt^2 can be equated to zero i.e.

$$\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) = 0$$

On solving,

$$\left(r^3 - \frac{3r}{\Lambda} - \frac{6M}{\Lambda}\right) = 0 \qquad (12)$$

The solutions of equation (12) are [6]:

$$r_H = 2M\left(1 + \frac{4}{3}\Lambda M^2 \ldots\right),$$

$$r_C = \sqrt{\frac{3}{\Lambda}}\left(1 - M\sqrt{\frac{\Lambda}{3}} \ldots\right),$$

and

$$r_O = -(r_H + r_C).$$

Here, r_H and r_C respectively represent the black hole event horizon and the cosmological horizon. It is worth noting that we restrict ourselves on the space-time in between these two horizons i.e. $r_H \le r \le r_C$. In fact, we are interested in the regions of $g_{tt} < 0$, as shown in Figure (2), where both the effects of mass parameter and cosmological constant should be considered.

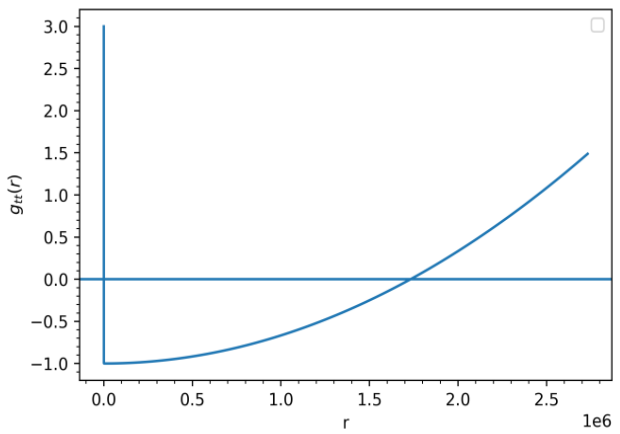


Figure 2: Variation of $g_{tt}(r)$ with r

The parameter r is real and positive, and hence, the physically acceptable regions are given by those r for which $E^2 > V_{eff}$. At the intersection points $E^2 = V_{eff}$, we have $\frac{dr}{d\phi} = 0$. These are called the turning points of the motion. Mathematically,

$$\phi - \phi_o = \int_{r_o}^{r} \frac{L}{r^2} \frac{dr}{\sqrt{E^2 - V_{eff}}} \qquad (13)$$

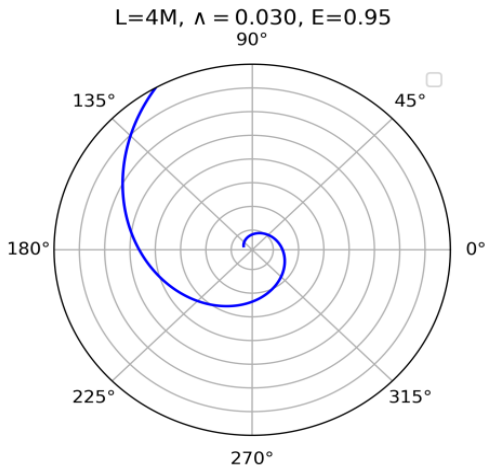
By imposing these two constraints, we work for the trajectories of photons in the background of SdS black hole.

VI. PHOTONS' ORBITS

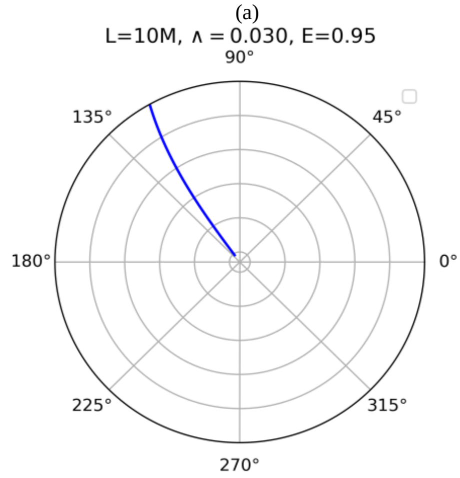
(a) Bound Terminating Orbits

An orbit having finite maximum radius and minimum radius equal to zero is called a bound terminating orbit. We found two conditions for the existence of such an orbit which are enumerated below:

(i)If we choose $\Lambda r_s^2 > \frac{1}{9}$, then for any choice of $\lambda > \frac{1}{3}$ there will be no bound orbits where $\lambda = \left(\frac{r_s}{L}\right)^2$ [4]. In this mathematical constraint, we observed terminating bound orbits presented in Figure (3). To satisfy the above relation, we fixed Λ = 0.030, E = 0.95, and varied the angular momentum (L) only.



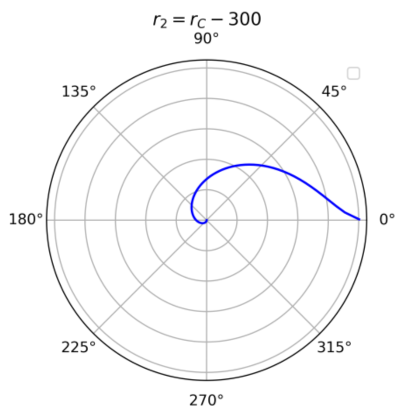
(a)



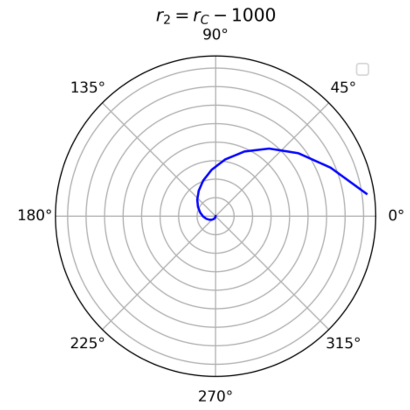
(b)

Figure 3: Plots exhibiting bound terminating orbits: (a)L = 4M (b)L = 10M

(ii)If a test particle starts to move from a point located inside the cosmological event horizon i.e. $r_2 < r_C$, then under such condition the test particle exhibits a terminating bound orbit [7]. This is due to the fact that the metric coefficient $g_{tt}'(r)$ is less than zero in that region. Such type of situation has been also described by Dymnikova, Poszwa, and Soltysek [7]. We fixed E = 0.95, L = 3.7M, and varied the r_2 only. The plots exhibiting this constraint are shown in Figure (4).



(a)

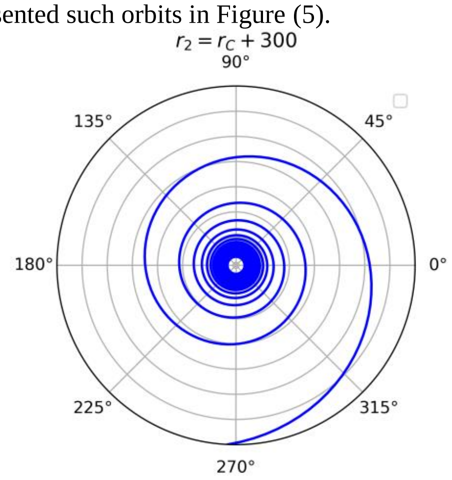


(b)

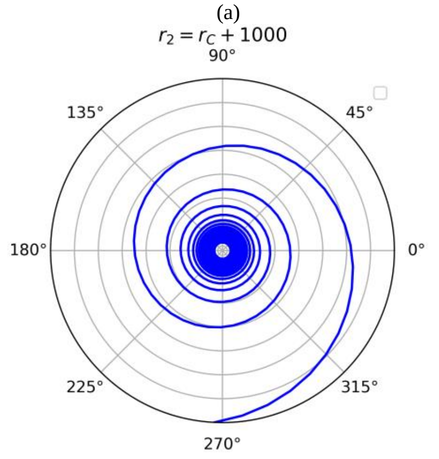
Figure 4: Bound terminating orbits for a photon inside the cosmological event horizon: (a) $r_2 = r_C - 300$ (b) $r_2 = r_C - 1000$.

(b) Periodic Bound Orbits

If a test particle-photon is allowed to start its motion from the space-time outside the cosmological event horizon, then the photon becomes bound around the black hole event horizon. The value of $g_{tt}'(r)$ becomes positive in this region. In this situation, photon exhibits the orbits similar to the planetary orbits. We fixed E = 0.95, L = 3.7M, and varied the r_2 on the space-time outside the r_C i.e. $r_2 > r_C$. We presented such orbits in Figure (5).



(a)

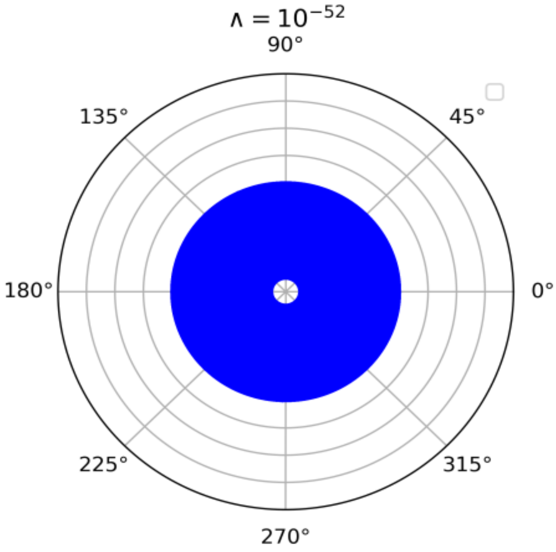


(b)

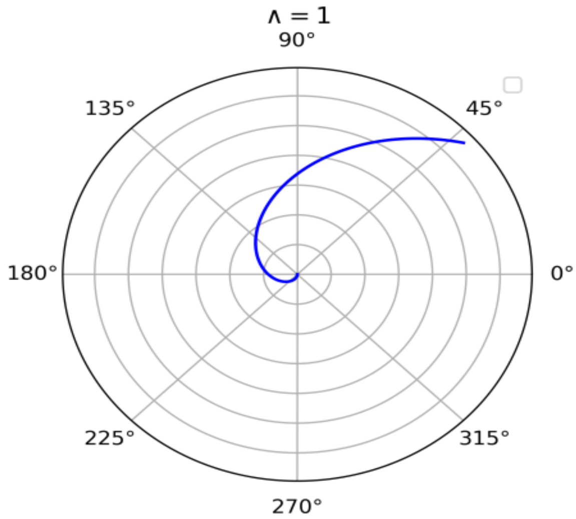
Figure 5: Periodic bound orbits for a photon outside the cosmological event horizon: (a) $r_2 = r_C + 300$ (b) $r_2 = r_C + 1000$.

VII. INFLUENCES OF Λ ON THE ORBITS

The Figure (6) exhibits the effects of cosmological constant (Λ) on the shapes of orbit. Here, we fixed the enregy parameter, E = 0.95, and the angular momentum parameter, L = 3.7M, to investigate the influences merely of cosmological constant. Observational data indicate that the cosmological constant Λ has a value of $\Lambda_o \sim 10^{-52}\, m^{-2}$ [8, 9]. So, it is reasonable to start from that value. We observed the densed bound orbits of test particle around the black hole event horizon. Here, the bound orbits represent the interval bound orbits. The interval bound orbits have a minimal and a maximal finite radius and orbits in between two radii. For $\Lambda = 10^{-5}\, m^{-2}$, the photon came inside from the cosmological event horizon and became bound around the black hole event horizon (not shown here in the plots). In the case of $\Lambda = 10^{-2}\, m^{-2}$, the photon moved inside from the cosmological event horizon same as in the previous case but the geodesic terminated at r ≠ 0 after completing just a few loops (not shown here in the plots). However, on further increasing the Λ i.e. at $\Lambda = 1\, m^{-2}$, we observed the dramatic variation in the shape of orbit. The photon came inside not from the infinity but from some finite radius. The geodesic vanished at the singularity point (r = 0) without completing a single loop. Such an orbit characterizes the terminating bound orbit. Hence, the bound orbit at $\Lambda_o \sim 10^{-52}\, m^{-2}$ converted into the terminating bound orbit at $\Lambda = 1\, m^{-2}$. Thus, for a large positive cosmological constant, the bound orbits do not exist [10, 11]. This is in consistent with the results of Hackmann and Lämmerzahl [4] as well.



(a)



(b)

Figure 6: Variation in the shapes of the orbit with respect to increasing cosmological constant (Λ): (a) $\Lambda = 10^{-52}\, m^{-2}$ (b) $\Lambda = 1\, m^{-2}$.

VIII. CONCLUSIONS

The observation of an accelerated expanding universe is described by the Einstein's field equation with cosmological constant. The spherically symmetric Schwarzschild de-Sitter solution is considered as a perspicuous solution of an isolated mass. In this paper, we investigated the null geodesics in Schwarzschild de-Sitter space-time. Our key findings are organized in the following paragraph.

We detected two types of orbits, namely periodic bound orbit and terminating bound orbit. The mathematical conditions for the existence of these orbits have been discussed. We found that there will no longer be any periodic bound orbits for a large positive cosmological constant. The analysis of effective potential inferred us that only the peak of the curve changes for change in angular momentum (L). These peaks correspond to circular orbits in null geodesics and a circular orbit of radius 3M becomes an allowed null geodesic. The results of our study are consistent with the study of Hackmann and Lämmerzahl [4].

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