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Research Article

Application of Laplace Transform in Physics for Solving Ordinary Differntial Equations

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Abstract**—** The Laplace transform is a crucial tool in solving complex physical problems in physics by converting differential equations into algebraic ones. This paper explores its applications in electromagnetism, classical mechanics, and thermodynamics. It demonstrates how the Laplace transform simplifies the analysis of electrical circuits, wave propagation, and heat conduction. The research highlights its effectiveness through examples, providing a framework for researchers to address complex challenges and advance understanding in various physical domains.

Keywords**—** Laplace transform, Lagrangian Mechanics, Integral transform, Simple pendulum, compound pendulum etc.

1. Introduction

Physics is a vast subject due the topics we have studied in physics, it is because physics rises due to a simple question How nature works? For most of the persons it is definition of the subject in itself. This question seems easier but it's not, yet! there is a lot to answer for answer the perfect question, every new research in this field is a further step to add something to the answer of the question. Likewise, this paper may will add something to the answer.

In the several branches of Physics, Mathematical physics is one on them which is used for developing a mathematical frame work for any field of the physics respectively. This branch of physics can directly relate with any other branches of physics, and the purpose of this paper is to highlight this statement because in this paper we are trying to simplify or answer the given physical problem using Laplace transformation, Laplace inverse transformation.

Laplace transform is purely an integral transform satisfying the well-known definition of integral transform that is
Integral transform If $\emptyset(p,t)f(t)$ is defined under the limits a and b then $T\{f(t)\}_p$ is output of the input function $f(t)$ and output can be obtained by

$$T\{f(t)\}_p = \int_a^b \emptyset(t,p)f(t)dt$$

Where $\emptyset(t,p)$ is termed as kernel or nucleus of the transformation, it is the function of two variables p and t. While $f(t)$ is a function of one variable here it is t [1].

For different kernel and limits we have different integral transform as for Mellin transform $\emptyset(t,p) = t^{p-1}$ and limits from 0 to ∞. Similarly for Fourier transform $\emptyset(t,p) = e^{-2i\pi pt}$ and limits $-\infty$ to ∞.
Laplace transform is an integral transform whose $\emptyset(p,t) = e^{-pt}$ and limits from 0 to ∞. The output of this transform is denoted by $\mathcal{L}$ [2].

Mathematically,

$$\mathcal{L}\{f(t)\}_p = \int_0^\infty e^{-pt}f(t)dt$$

With the help of above expression
We have

$$\mathcal{L}\{1\}_p = \frac{1}{p}$$
$$\mathcal{L}\{t^n\}_p = \frac{\Gamma(n+1)}{p^{n+1}}$$
$$\mathcal{L}\{e^{at}\}_p = \frac{1}{p-a}$$
$$\mathcal{L}\{\sin at\}_p = \frac{a}{p^2+a^2}$$
$$\mathcal{L}\{\cos at\}_p = \frac{p}{p^2+a^2}$$
$$\mathcal{L}\{\sinh at\}_p = \frac{a}{p^2-a^2}$$
$$\mathcal{L}\{\cosh at\}_p = \frac{p}{p^2-a^2}$$

And the properties of Laplace transform are as follows

1. **Linearity property**

$$\mathcal{L}\{af(t)+bg(t)\}_p = a\mathcal{L}\{f(t)\}_p + b\mathcal{L}\{g(t)\}_p$$

2. **First Shifting property**

$$\mathcal{L}\{e^{at}f(t)\}_p = \mathcal{L}\{f(t)\}_{p=p-a}$$

3. **Second Shifting property**

If $\mathcal{L}\{f(t)\} = F(p)$ and

$$g(t) = \begin{cases} F(t-a), t > a \\ 0, t < a \end{cases}$$

Then

$$\mathcal{L}\{G(t)\} = e^{-ap}F(p)$$

4. **Change of scale**

If $\mathcal{L}\{f(t)\} = F(p)$ then

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{p}{a}\right)$$

5. **Multiple by t**

If $\mathcal{L}\{f(t)\} = F(p)$ then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(p)}{dp^n}$$

6. **Division by t**

If $\mathcal{L}\{f(t)\} = F(p)$ then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(p)dp$$

7. **n^th^ order derivative**

If $\mathcal{L}\{f(t)\} = F(p)$ then

$$\mathcal{L}\left\{\frac{d^n}{dt^n}f(t)\right\} = p^n F(p) - \sum_{k=1}^{n} p^{n-k} \left|\frac{d^{k-1}f(t)}{dt^{k-1}}\right|_{t=0}$$

These all properties are helpful for solving given problem easier or in other words for simplifying the given problem we have used these properties respectively.
Now, coming on the inverse integral transform, which is defined as

Inverse Integral transform

$$f(t) = \int_{u_1}^{u_2} \emptyset^{-1}(p,t)T\{f(t)\}_p dp$$

One can easily found the inverse Laplace transform from Laplace transform of the given problem that's why generally we want require to solve another integral for inverse transform when we already got its transform. But in case of Fourier and other transform we must have to find both integral's value.
If $\mathcal{L}$ denoted Laplace transform then $\mathcal{L}^{-1}$ denotes inverse Laplace transform. For instance, we use

$$\mathcal{L}^{-1}\left(\frac{1}{p}\right) = 1$$

$$\mathcal{L}^{-1}\left(\frac{\Gamma(n+1)}{p^{n+1}}\right) = t^n$$

$$\mathcal{L}^{-1}\left(\frac{1}{p-a}\right) = e^{at}$$

$$\mathcal{L}^{-1}\left(\frac{a}{p^2+a^2}\right) = \sin at$$

$$\mathcal{L}^{-1}\left(\frac{p}{p^2+a^2}\right) = \cos at$$

$$\mathcal{L}^{-1}\left(\frac{a}{p^2-a^2}\right) = \sinh at$$

$$\mathcal{L}^{-1}\left(\frac{p}{p^2-a^2}\right) = \cosh at$$

It is a notable point that Laplace transform with its inverse is very useful for solving differential equations whether its linear differential equation with constant coefficients or either variable coefficient. In other words, Laplace transform combinedly with its inverse is a mathematical tool useful in physics too! As in every third or fourth topic in physics we have to deals with differential equations that's increase the necessity of integral transformations.

Let's take an example, suppose we have to solve

$$\frac{d^2x}{dt^2} + a^2x = 0$$

With $x(0) = 2$ and $x'(0) = 0$. Now, Let's solve the equation

$$\frac{d^2x}{dt^2} + a^2x = 0$$

On taking Lapace transform both sides,

$$L[x"(t)] + a^2L[x(t)] = 0$$

$$p^2L[x(t)] - pL[x(0)] - x'(0) + a^2L[x(t)] = 0$$

$$(p^2+a^2)L[x(t)] - pL[x(0)] - x'(0) = 0$$

$$(p^2+a^2)L[x(t)] = pL[x(0)] + x'(0)$$

$$L[x(t)] = \frac{pL[x(0)] + x'(0)}{(p^2+a^2)}$$

$$L[x(t)] = \frac{p}{(p^2+a^2)}L[x(0)] + \frac{1}{(p^2+a^2)}x'(0)$$

Since it was given $x(0) = 2$ and $x'(0) = 0$. Therefore

$$L[x(t)] = 2\frac{p}{(p^2+a^2)}$$

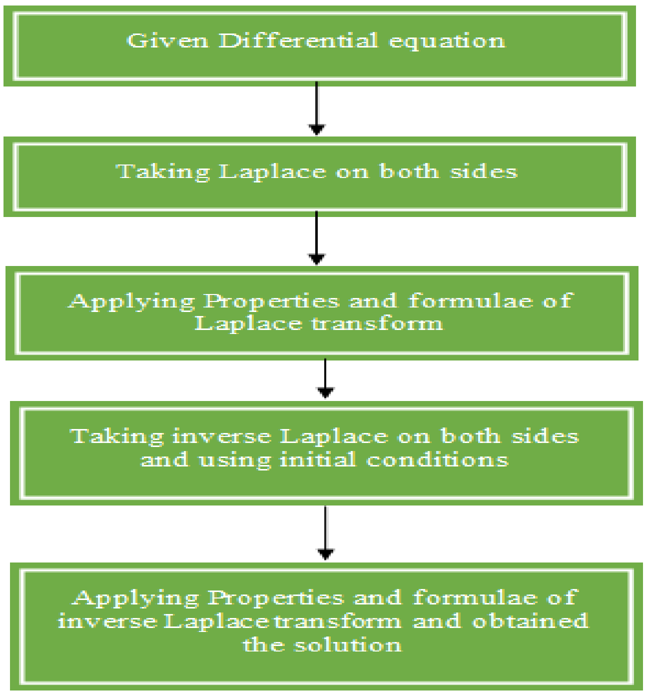
On taking inverse Laplace transform, we have

$$L^{-1}L[x(t)] = L^{-1}\left[2\frac{p}{(p^2+a^2)}\right]$$

$$x(t) = 2L^{-1}\left[\frac{p}{p^2+a^2}\right]$$

$$x(t) = 2\cos at$$

This is the method for solving a given differential equation.



This paper contains various sections as abstract, keywords, introduction, related work, applications, conclusion, references.

2. Related Work

The Laplace transform is an essential tool for developing mathematical models to solve equations. It converts a function f(t) from its time domain into the frequency domain F(s). The inverse Laplace transform then translates this frequency domain F(s) back into the time domain. Essentially, the Laplace transform simplifies differential or integral equations into algebraic equations, making it a powerful analytical tool for engineering problems [3].

This transformation technique is widely employed in solving higher-order differential equations and has a multitude of applications across mathematics, applied sciences, and engineering. It is particularly useful in calibrating integral and differential systems, circuit systems, mechanical systems, avionics systems, and image processing, among other areas. The following section will discuss the application of the Laplace transform in various fields.

For parabolic and hyperbolic heat conduction equations, the classical variational principle is not applicable. P. Szymczyk and M. Szymczyk (2015) [4] explored and explained the principles governing these equations. Their study focuses on models such as the Cattaneo-Vernotte model, Jeffrey model, and two-temperature models, among others. Laplace transformations are utilized to derive classical variational principles. Sumit Gupta et al. (2015) [5] developed a method for solving linear and nonlinear convection-diffusion problems encountered in physical phenomena where energy is transferred through diffusion and convection. The Homotopy Perturbation Transformation Method (HPTM) is introduced, which combines the Laplace transform with homotopy perturbation, thereby simplifying the solution of convection-diffusion equations.

WK Zahra et al. (2017) [6] investigated fractional linear electrical systems and introduced new parameters for the generalization of RL and RC circuits. The study compares classical electrical systems with fractional electrical systems. Fractional linear electrical systems are addressed using fractional calculus, offering a more accurate representation of real inductors and capacitors. Solutions for fractional models of RL and RC circuits are derived using the Laplace transform. Throughout this paper, we will further discuss several other cases from various branches of physics where the Laplace transform can be applied.

3. Applications

In this section of the paper, we are considering several applications that simply means physical problems that we'll handle using Laplace transform:

3.1. Radioactive decay

With reference to the law of radioactive decay which states that, "*For a particular time, the rate of radioactive decay of an atom is directly proportional to the number of nuclei of the elements present at that time*." [7]

Mathematically,

$$\frac{-dN}{dt} \propto N$$

Or

$$\frac{dN}{dt} = -\lambda N$$

$$\Rightarrow \frac{dN}{dt} + \lambda N = 0$$

On taking Lapace transform both sides, we have

$$L\left[\frac{dN}{dt} + \lambda N\right] = 0$$

$$\Rightarrow L\left[\frac{dN}{dt}\right] + L[\lambda N] = 0$$

$$\Rightarrow L\left[\frac{dN}{dt}\right] + \lambda L[N] = 0$$

$$\Rightarrow pL[N(t)] - N'(0) + \lambda L[N(t)] = 0$$

$$\Rightarrow (p+\lambda)L[N(t)] - N'(0) = 0$$

$$\Rightarrow (p+\lambda)L[N(t)] = N(0)$$

$$\Rightarrow L[N(t)] = \frac{N(0)}{(p+\lambda)}$$

Now, take Laplace inverse both sides,

$$N(t) = N(0)L^{-1}\left(\frac{1}{p+\lambda}\right)$$

$$N(t) = N(0)e^{-\lambda t}$$

Where, $N(0)$ is the number of nuclei initially while number of nuclei present at time t is N and λ is constant of proportionality.

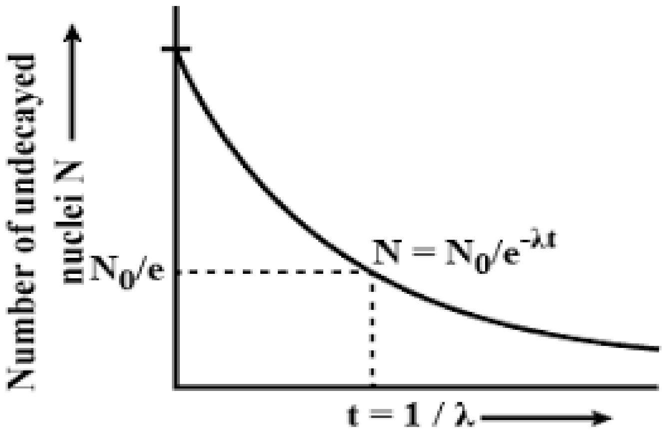


Figure 1 Radioactive decay

In this figure 1, radioactive decay has been shown as t approaches to infinite number of nuclei becomes zero. N(0) is the initial number of nucleus and these are in under exponential decay.

3.2. Spring Mass system

At the equilibrium position the spring is relaxed. *When the block is displaced through a distance x towards right, it experiences a net restoring force F = -kx towards left.*

The negative sign shows that the restoring force is always opposite to the displacement. That is, when x is positive, F is negative, the force is directed to the left. When x is negative, F is positive, the force is directed to the right. Thus, the force always tends to restore the block to its equilibrium position x = 0.

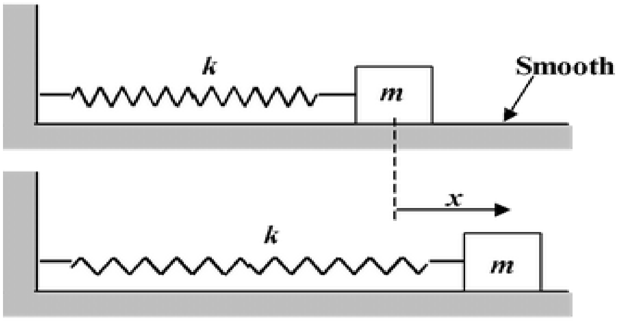


Figure 2: Spring-Mass system

$$F = -kx$$

Applying Newton's Second law of motion and $a = \frac{dv}{dt}, v = \frac{dx}{dt}$

$$ma = -kx$$
$$\Rightarrow m\frac{dv}{dt} = -kx$$
$$\Rightarrow m\frac{d}{dt}\left(\frac{dx}{dt}\right) = -kx$$
$$\Rightarrow m\frac{d^2x}{dt^2} = -kx$$
$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Let $w^2 = \frac{k}{m}$,

$$\Rightarrow \frac{d^2x}{dt^2} = -w^2x$$
$$\Rightarrow \frac{d^2x}{dt^2} + w^2x = 0$$
$$\Rightarrow x"(t) + w^2x(t) = 0$$

Taking Lapace transform on both sides

$$L[x"(t) + w^2x(t)] = 0$$
$$\Rightarrow L[x"(t)] + w^2L[x(t)] = 0$$
$$\Rightarrow p^2L[x(t)] - pL[x(0)] - x'(0) + w^2L[x(t)] = 0$$
$$\Rightarrow (p^2 + w^2)L[x(t)] - pL[x(0)] - x'(0) = 0$$
$$\Rightarrow (p^2 + w^2)L[x(t)] = pL[x(0)] + x'(0)$$
$$\Rightarrow L[x(t)] = \frac{pL[x(0)] + x'(0)}{(p^2 + w^2)}$$
$$\Rightarrow L[x(t)] = \frac{p}{(p^2 + w^2)}L[x(0)] + \frac{1}{(p^2 + w^2)}x'(0)$$

Let $x'(0) = 0$ and take Laplace inverse both sides,

$$x(t) = x(0)\cos wt$$

Where, $x(0) = A$ i.e. amplitude

$x(t) = A\cos wt$.

3.3. Simple pendulum

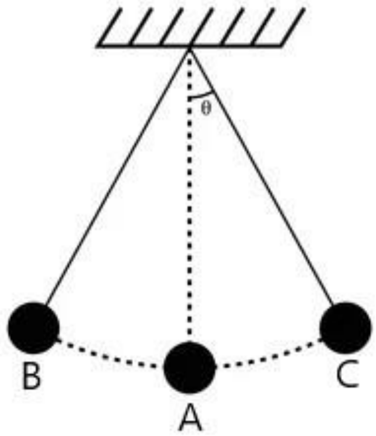


Figure 3 Oscillation of a simple pendulum

A simple pendulum is a mechanical arrangement that demonstrates periodic motion. The simple pendulum comprises a small bob of mass 'm' suspended by a thin string secured to a platform at its upper end of length L.

The simple pendulum is a mechanical system that sways or moves in an oscillatory motion. This motion occurs in a vertical plane and is mainly driven by gravitational force. Interestingly, the bob that is suspended at the end of a thread is very light; somewhat, we can say it is even massless. The assumptions we are considering are as follows: -

- There is negligible friction from the air and the system
- The arm of the pendulum does not bend or compress and is massless
- The pendulum swings in a perfect plane
- Gravity remains constant

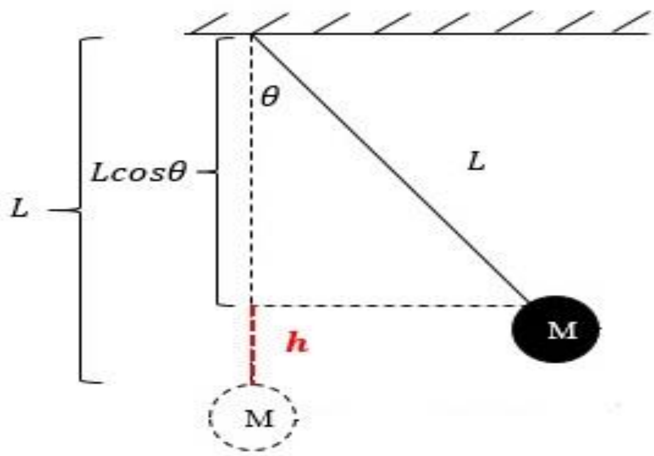


Figure 4: Geomerty of Simple Pendulum

As the distance taken by an arc from C to A therefore $\theta = \frac{AC}{l} \Rightarrow AC = l\theta$. With differentiate it with respect to time we have $v = l\dot{\theta}$.

For point C, **kinetic energy** (T) can be obtained by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

While for the same point, **potential energy** (V) can be obtained by

$$V = mgh = mgl(1 - \cos\theta)$$

With reference to the Lagrange mechanics, Lagrangian (L) [8] can be obtained by

$$L = T - V$$
$$= \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

Now,

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta, \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

Using Lagrange equation, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\Rightarrow \frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta = 0$$
$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

For small θ, $\sin\theta \sim \theta$. So,

$$ml^2\ddot{\theta} + mgl\theta = 0$$
$$\Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

Using $w = \sqrt{\frac{g}{l}}$, we can rewrite the above equation as $\ddot{\theta} + w^2\theta = 0$

On taking Lapace transform on both sides

$$L[\theta"(t)] + w^2L[\theta(t)] = 0$$
$$\Rightarrow p^2L[\theta(t)] - pL[\theta(0)] - \theta'(0) + w^2L[\theta(t)] = 0$$
$$\Rightarrow (p^2 + w^2)L[\theta(t)] = pL[\theta(0)] + \theta'(0)$$
$$\Rightarrow L[\theta(t)] = \frac{p}{(p^2 + w^2)}\theta(0) + \frac{1}{(p^2 + w^2)}\theta'(0)$$

Let $\theta(0) =$constant and take Laplace inverse both sides,

$$\theta(t) = \theta(0)L^{-1}\left[\frac{p}{(p^2 + w^2)}\right] + \theta'(0)L^{-1}\left[\frac{1}{(p^2 + w^2)}\right]$$
$$\Rightarrow \theta(t) = \theta(0)\cos wt + \frac{\theta'(0)}{w}\sin wt$$

Suppose $\theta(0) = A\sin\emptyset, \frac{\theta'(0)}{w} = A\cos\emptyset$

$$\theta(t) = A\sin\emptyset\ \cos wt + A\cos\emptyset\sin wt$$
$$\Rightarrow \theta(t) = A\sin(wt + \emptyset)$$

A is generally the amplitude then Time period can be obtained by

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{l}{g}}$$

Where, $w = \sqrt{\frac{g}{l}}$

3.4. Compound pendulum

In case of compound pendulum, we can obtain Lagrangian (L) in terms of momentum of inertia [9]:

$$L = \overbrace{\frac{1}{2}I\dot{\theta}^2}^{T} + \overbrace{Mgl\cos\theta}^{-V}$$

Where I is moment of inertia. Now,

$$\frac{\partial L}{\partial \theta} = -Mgl\sin\theta, \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

Using Lagrange equation, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\Rightarrow \frac{d}{dt}\left(I\dot{\theta}\right) + Mgl\sin\theta = 0$$
$$\Rightarrow I\ddot{\theta} + Mgl\sin\theta = 0$$

Again, for small θ, $\sin\theta \sim \theta$. So,

$$I\ddot{\theta} + Mgl\theta = 0$$
$$\Rightarrow \ddot{\theta} + \frac{Mgl}{I}\theta = 0$$

Using $\frac{Mgl}{I} = w^2$. Then

$$\ddot{\theta} + w^2\theta = 0$$

On taking Lapace transform on both sides

$$L[\theta"(t)] + w^2L[\theta(t)] = 0$$
$$\Rightarrow p^2L[\theta(t)] - pL[\theta(0)] - \theta'(0) + w^2L[\theta(t)] = 0$$
$$\Rightarrow (p^2 + w^2)L[\theta(t)] = pL[\theta(0)] + \theta'(0)$$
$$\Rightarrow L[\theta(t)] = \frac{p}{(p^2 + w^2)}\theta(0) + \frac{1}{(p^2 + w^2)}\theta'(0)$$

Let $\theta(0) =$constant and take Laplace inverse both sides,

$$\theta(t) = \theta(0)L^{-1}\left[\frac{p}{(p^2 + w^2)}\right] + \theta'(0)L^{-1}\left[\frac{1}{(p^2 + w^2)}\right]$$
$$\Rightarrow \theta(t) = \theta(0)\cos wt + \frac{\theta'(0)}{w}\sin wt$$

Suppose $\theta(0) = A\sin\emptyset, \frac{\theta'(0)}{w} = A\cos\emptyset$

$$\theta(t) = A\sin\emptyset\ \cos wt + A\cos\emptyset\sin wt$$
$$\Rightarrow \theta(t) = A\sin(wt + \emptyset)$$

A is generally the amplitude then Time period can be obtained by

$$T = \frac{2\pi}{w}$$

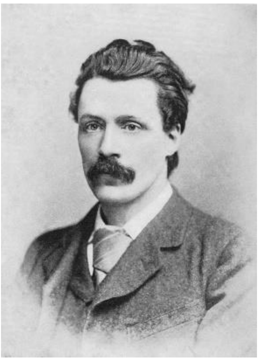
Where, $w = \sqrt{\frac{g}{l}}$

$$T = \frac{2\pi}{\sqrt{\frac{Mgl}{I}}} = 2\pi\sqrt{\frac{I}{Mgl}}$$

Moment of inertial of the pendulum about the axis of rotation gives $I = M(K^2 + l^2)$. Then on substituting it

$$T = 2\pi\sqrt{\frac{M(K^2 + l^2)}{Mgl}} = 2\pi\sqrt{\frac{K^2 + l^2}{gl}}$$

3.5. Atwood Machine



In 1784, the Rev. **George Atwood (1745-1807)**, tutor at Trinity College, Cambridge, came up with a great demo for finding g. It's still with us. The traditional Newtonian solution of this problem is to write F=ma for the two masses, then eliminate the tension T. (To keep things simple, we'll neglect the rotational inertia of the top pulley.)

The Lagrangian approach [10][11] is, of course, to write down the Lagrangian, and derive the equation of motion.

Measuring gravitational potential energy from the top wheel axle, the potential energy is

$$U = -m_1gx - m_2g(l - x)$$

While kinetic energy be

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

Then the Lagrangian for this system be

$$L = T - U$$

$$\Rightarrow L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1gx + m_2g(l - x)$$

Now,

$$\frac{\partial L}{\partial x} = m_1g - m_2g = g(m_1 - m_2), \frac{\partial L}{\partial \dot{\theta}} = (m_1 + m_2)\dot{x}$$

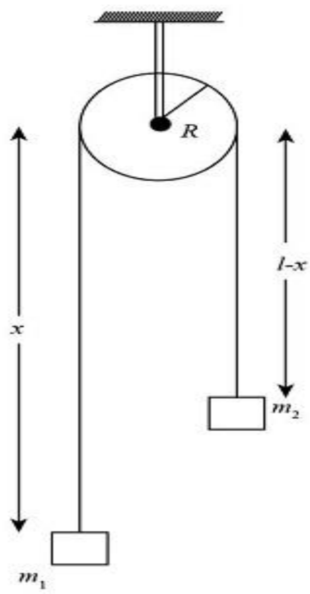


Figure 5 A two-dimensional figure showing Atwood machine contains two masses m_1 and m_2

Using Lagrange equation, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}[(m_1 + m_2)\dot{x}] - g(m_1 - m_2) = 0$$

$$\Rightarrow (m_1 + m_2)\ddot{x} + -g(m_1 - m_2) = 0$$

$$\Rightarrow x" = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$$

Taking Lapace transform on both sides,

$$L[x"(t)] = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)gL[1]$$

$$P^2L[x(t)] - PL[x(0)] - x'(0) = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g\left(\frac{1}{P}\right)$$

$$P^2L[x(t)] = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g\left(\frac{1}{P}\right) + PL[x(0)] + x'(0)$$

$$L[x(t)] = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g\left(\frac{1}{P^3}\right) + \frac{1}{P}L[x(0)] + \frac{1}{P^2}x'(0)$$

Now, take Laplace inverse both sides,

$$L^{-1}L[x(t)] = \begin{Bmatrix} g\left(\frac{m_1 - m_2}{m_1 + m_2}\right)L^{-1}\left(\frac{1}{P^3}\right) + L[x(0)]L^{-1}\left(\frac{1}{P}\right) \\ +x'(0)L^{-1}\left(\frac{1}{P^2}\right) \end{Bmatrix}$$

Since $L^{-1}\left(\frac{1}{P^3}\right) = \frac{1}{2}L^{-1}\left(\frac{2!}{P^3}\right) = \frac{1}{2}t^2$ and $L^{-1}\left(\frac{1}{P^2}\right) = t, L^{-1}\left(\frac{1}{P}\right) = 1$. Thus

$$\Rightarrow x(t) = \frac{1}{2}\left(\frac{m_1 - m_2}{m_1 + m_2}\right)gt^2 + x(0) + tx'(0)$$

3.6. Newton's law of cooling

According to Newton's law of cooling, the rate of loss of heat from a body is directly proportional to the difference in the temperature of the body and its surroundings [12].

Let, the rate of loss of heat be $-\frac{dT}{dt}$

And the difference in the temperature of the body and its surroundings be $T - T_s$. Then

$$-\frac{dT}{dt} \propto (T - T_s) \text{ Or}$$

$$\frac{dT}{dt} = -k(T - T_s)$$

Negative sign used for the loss.

Taking Lapace transform on both sides

$$L[T'(t)] = -kL[T(t)] + kL[T_s]$$

$$pL[T(t)] - T(0) = -kL[T(t)] + kL[Ts]$$

$$(p + k)L[T(t)] = kL[T_s] + T(0)$$

$$L[T(t)] = \frac{kL[T_s] + T(0)}{(p + k)}$$

Now, take Laplace inverse both sides,

$$T(t) = ke^{-kt}L[T_s] + e^{-kt}T(0)$$

$$T(t) = e^{-kt}[kL[T_s] + T(0)]$$

If we considered T_s not varies with time (t) then

$$(p + k)L[T(t)] = kL[T_s] + T(0)$$

$$\Rightarrow (p + k)L[T(t)] = kT_sL[1] + T(0)$$

$$\Rightarrow (p+k)L[T(t)] = T_s\frac{k}{p} + T(0)$$

$$\Rightarrow L[T(t)] = T_s\frac{k}{p(p+k)} + \frac{T(0)}{p+k}$$

$$\Rightarrow L[T(t)] = T_s\frac{(p+k)-p}{p(p+k)} + \frac{T(0)}{p+k}$$

$$\Rightarrow L[T(t)] = T_s\left(\frac{1}{p} - \frac{1}{p+k}\right) + \frac{T(0)}{p+k}$$

$$\Rightarrow L[T(t)] = T_s\frac{1}{p} + (T_0 - T_s)\frac{1}{p+k}$$

Now, take Laplace inverse both sides,

$$T(t) = T_s L^{-1}\left(\frac{1}{p}\right) + (T_0 - T_s)L^{-1}\left(\frac{1}{p+k}\right)$$

$$\Rightarrow T(t) = T_s + (T_0 - T_s)e^{-kt}$$

Since $L^{-1}\left(\frac{1}{p}\right) = 1$ and $L^{-1}\left(\frac{1}{p+k}\right) = e^{-kt}$.

3.7. LC Oscillations

Let us consider an electric circuit, containing inductance L and capacitance C. When the charge on the condenser is q and the flowing in the circuit is i.

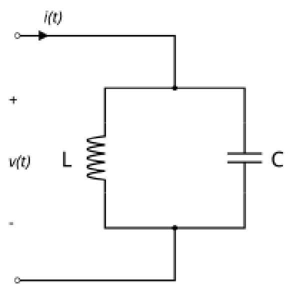


Figure 6
LC Circuit, contains a capacitor of capacitance C and inductor of inductance L.

The magnetic energy $\frac{1}{2}Li^2$ in an electric circuit is analogues to the kinetic energy $\frac{1}{2}mv^2$ in a mechanical system, where we can think inductance L as charge inertia similar to mass inertia and $i = dq/dt$ as $v = dx/dt$; charge q is playing the role of displacement. The electrical potential energy of the circuit is $V = q^2/2C$. Hence the Lagrange of L-C circuit [13][14] is

$$L = T - V = \frac{1}{2}Li^2 - \frac{q^2}{2C}$$

$$= \frac{1}{2}L\left(\frac{dq}{dt}\right)^2 - \frac{q^2}{2C} = \frac{1}{2}L\dot{q}^2 - \frac{q^2}{2C}$$

Taking q as the generalized coordinate, the Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Here

$$\frac{\partial L}{\partial \dot{q}} = L\dot{q}\,, \qquad \frac{\partial L}{\partial q} = -\frac{q}{C}$$

On substituting both

$$\Rightarrow \frac{d}{dt}[L\dot{q}\,] + \frac{q}{C} = 0$$

$$\Rightarrow L\ddot{q} + \frac{q}{C} = 0$$

$$\Rightarrow \ddot{q} + \omega^2 q = 0$$

On taking Laplace on both sides

$$\mathcal{L}[\ddot{q} + \omega^2 q] = 0$$

$$\Rightarrow \mathcal{L}[\ddot{q}] + \mathcal{L}[\omega^2 q] = 0$$

$$\Rightarrow \mathcal{L}[\ddot{q}] + \omega^2\mathcal{L}[q] = 0$$

$$\Rightarrow p^2\mathcal{L}[q] - pq(0) - q'(0) + \omega^2\mathcal{L}[q] = 0$$

$$\Rightarrow (p^2 + \omega^2)\mathcal{L}[q] - pq(0) - q'(0) = 0$$

$$\Rightarrow (p^2 + \omega^2)\mathcal{L}[q] = pq(0) + q'(0)$$

$$\Rightarrow \mathcal{L}[q] = \frac{p}{(p^2+\omega^2)}q(0) + \frac{\omega}{(p^2+\omega^2)}\frac{q'(0)}{\omega}$$

On taking Inverse Laplace on both sides

$$q = q(0)\mathcal{L}^{-1}\left[\frac{p}{(p^2+\omega^2)}\right] + \frac{q'(0)}{\omega}\mathcal{L}^{-1}\left[\frac{\omega}{(p^2+\omega^2)}\right]$$

$$\Rightarrow q = q(0)\cos\omega t + \frac{q'(0)}{\omega}\sin\omega t$$

Sine and cosine function of time shows the oscillating behaviour of the system.

With assuming $q'(0) = 0$

$$q = q(0)\cos\omega t$$

Where,

$$\omega = \frac{1}{\sqrt{LC}}$$

For time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$$

For frequency

$$f = \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}}$$

3.8. Damped oscillation

Damped oscillation refers to the motion of an oscillating system, like a pendulum or a spring, in which the amplitude of oscillation gradually decreases over time due to the presence of a damping force, such as friction or air resistance [15][16]. This damping force dissipates the system's energy, causing the oscillations to lose intensity and eventually come to a stop. The rate at which the amplitude decreases depend on the strength of the damping force, with heavier damping leading to quicker cessation of oscillations.

We will investigate the effect of damping on the harmonic oscillations of a simple system having one degree of freedom. One such system is shown in the figure. When the system is displaced from its equilibrium state and released, it begins to move. The forces acting on the system are given below:

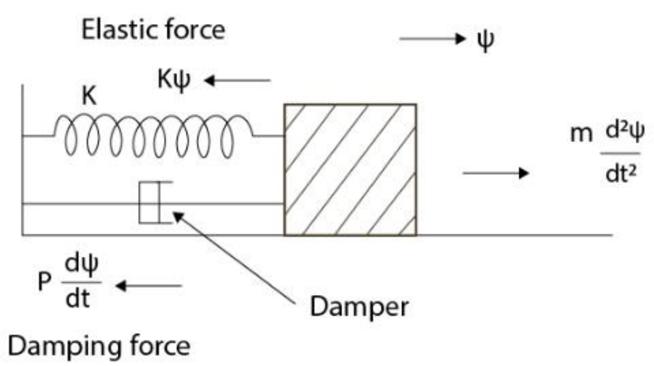


Figure 7 Shows spring block system

- A restoring force $-Kx$, where K is the coefficient of the restoring force, and x is the displacement
- A damping force -p(dx/dt), where p is the coefficient of the damping force and (dx/dt) is the velocity of the moving part of the system. From Newton's law for a rigid body in translation, these forces must balance with Newton's force $m(d^2x/dt^2)$, where m is the mass of the oscillator and (d^2x/dt^2) is its acceleration. Since the restoring force and the damping force acts in a direction opposite to Newton's force [17] , we have

$$m\frac{d^2x}{dt^2} = -kx - p\frac{dx}{dt}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx + p\frac{dx}{dt} = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{p}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Let $\frac{p}{m} = 2\alpha$ and $\frac{k}{m} = \omega^2$. Then

$$\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0$$

It is a second order linear differential equation, on taking Laplace

$$\mathcal{L}[\ddot{x} + 2\alpha\dot{x} + \omega^2 x] = 0$$

$$\Rightarrow \mathcal{L}[\ddot{x}] + 2\alpha\mathcal{L}[\dot{x}] + \omega^2\mathcal{L}[x] = 0$$

$$\Rightarrow \left(p^2\mathcal{L}[x] - px(0) - x'(0)\right) + 2\alpha\left(pL[x] - x(0)\right) + \omega^2\mathcal{L}[x] = 0$$

$$\Rightarrow (p^2 + 2\alpha p + \omega^2)\mathcal{L}[x] - (p + 2\alpha)x(0) - x'(0) = 0$$

$$\Rightarrow (p^2 + 2\alpha p + \alpha^2 + \omega^2 - \alpha^2)\mathcal{L}[x] = (p + 2\alpha)x(0) + x'(0)$$

$$\Rightarrow \left((p+\alpha)^2 + \left(\sqrt{\omega^2 - \alpha^2}\right)^2\right)\mathcal{L}[x] = (p + 2\alpha)x(0) + x'(0)$$

$$\Rightarrow \mathcal{L}[x] = \frac{(p+2\alpha)}{\left((p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2\right)}x(0) + \frac{1}{\left((p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2\right)}x'(0)$$

$$\Rightarrow \mathcal{L}[x] = \frac{(p+\alpha)}{(p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2}x(0) + \frac{\alpha}{\sqrt{\omega^2-\alpha^2}}\frac{\sqrt{\omega^2-\alpha^2}}{(p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2}x(0) + \frac{1}{\sqrt{\omega^2-\alpha^2}}\frac{\sqrt{\omega^2-\alpha^2}}{(p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2}x'(0)$$

On taking inverse Laplace on both sides

$$\Rightarrow x(t) = x(0)\mathcal{L}^{-1}\left[\frac{(p+\alpha)}{(p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2}\right] + x(0)\frac{\alpha}{\sqrt{\omega^2-\alpha^2}}\mathcal{L}^{-1}\left[\frac{\sqrt{\omega^2-\alpha^2}}{(p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2}\right] + x'(0)\frac{1}{\sqrt{\omega^2-\alpha^2}}\mathcal{L}^{-1}\left[\frac{\sqrt{\omega^2-\alpha^2}}{(p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2}\right]$$

Using first shifting of Laplace transform we can say that

$$\mathcal{L}^{-1}\left[\frac{(p+\alpha)}{(p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2}\right] = e^{-\alpha t}\cos\left(t\sqrt{\omega^2-\alpha^2}\right),$$

$$\mathcal{L}^{-1}\left[\frac{\sqrt{\omega^2-\alpha^2}}{(p+\alpha)^2 + \left(\sqrt{\omega^2-\alpha^2}\right)^2}\right] = e^{-\alpha t}\sin\left(t\sqrt{\omega^2-\alpha^2}\right)$$

On substituting both

$$x(t) = x(0)e^{-\alpha t}\cos\left(t\sqrt{\omega^2-\alpha^2}\right) + x(0)\frac{\alpha}{\sqrt{\omega^2-\alpha^2}}e^{-\alpha t}\sin\left(t\sqrt{\omega^2-\alpha^2}\right) + x'(0)\frac{1}{\sqrt{\omega^2-\alpha^2}}e^{-\alpha t}\sin\left(t\sqrt{\omega^2-\alpha^2}\right)$$

$$x(t) = e^{-\alpha t}\left(x(0)\cos\left(t\sqrt{\omega^2-\alpha^2}\right) + \frac{x'(0)\sin\left(t\sqrt{\omega^2-\alpha^2}\right) + \alpha x(0)\sin\left(t\sqrt{\omega^2-\alpha^2}\right)}{\sqrt{\omega^2-\alpha^2}}\right)$$

Using $\alpha = \frac{p}{2m}$ and $\omega^2 = \frac{k}{m}$

$$x(t) = e^{-\frac{p}{2m}t}\left(x(0)\cos\left(t\sqrt{\frac{k}{m}-\frac{p^2}{4m^2}}\right) + \frac{x'(0)\sin\left(t\sqrt{\frac{k}{m}-\frac{p^2}{4m^2}}\right)+\alpha x(0)\sin\left(t\sqrt{\frac{k}{m}-\frac{p^2}{4m^2}}\right)}{\sqrt{\frac{k}{m}-\frac{p^2}{4m^2}}}\right)$$

And now for different cases of $\sqrt{\frac{k}{m}-\frac{p^2}{4m^2}}$ we can obtain different cases but we can't consider the case where $\frac{k}{m}=\frac{p^2}{4m^2}$ in the above equation.

4. Conclusion and Future Scope

This paper has demonstrated the significant utility of the Laplace transform in solving complex physical problems across various domains, particularly in electromagnetism, classical mechanics, and thermodynamics. By converting differential equations into algebraic equations, the Laplace transform streamlines the analytical process, allowing for more straightforward solutions to problems that would otherwise be cumbersome and difficult to solve.

In the analysis of LC circuits, the Laplace transform proves invaluable in simplifying the study of transient and steady-state behaviors. By transforming the circuit's differential equations into algebraic forms, the transform makes it easier to determine the system's response to different inputs, such as step functions or sinusoidal sources. This approach not only enhances the understanding of circuit dynamics but also provides a powerful method for predicting the behavior of more complex electrical networks.

In the context of classical mechanics, the Laplace transform is applied effectively to the Atwood machine—a system traditionally studied through Newtonian mechanics. The transformation simplifies the equations of motion, making it easier to explore the machine's dynamics under various conditions, including different masses and pulley systems. Similarly, for the simple and compound pendulums, the Laplace transform offers a means to analyze oscillatory motion and damping effects. It converts the complex, second-order differential equations governing these systems into more manageable algebraic forms, providing clear insights into the behavior of the pendulums under various initial conditions and external forces.

The effectiveness of the Laplace transform in these specific applications suggests a broader potential for its use in other areas of physics and engineering. Future research could extend these techniques to more complex systems, such as multi-degree-of-freedom mechanical systems, non-linear electrical circuits, and advanced thermodynamic processes. Moreover, the Laplace transform could be combined with numerical methods to tackle problems where analytical solutions are not feasible, offering a hybrid approach that leverages both symbolic and numerical computation.

In conclusion, the Laplace transform stands as a powerful tool that not only simplifies the analysis of complex physical systems but also enhances the accuracy and efficiency of solutions. Its application across various domains underscores its versatility and potential, paving the way for future advancements in both theoretical and applied physics. As researchers continue to explore and expand upon these techniques, the Laplace transform will likely remain a cornerstone in the study and resolution of complex physical phenomena.

Data Availability
All data generated or analysed during this study are included in this article.

Conflict of Interest
The authors declare that they have no known competing financial interest or personal relationship that could have appeared to influence the work reported in this paper.

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Authors' Contributions
Author-1 researched literature, and conceived the study, compile the manuscript. Author-2 involved in deriving first 6 results.

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