World Academics Journal of Engineering Sciences
Vol.**11**, Issue.**3**, pp.**11-16**, September **2024**
E-ISSN: 2348-635X
Available online at: www.isroset.org

Research Article

Heat Conduction Differential Equation and Its Solutions for Steady-State

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Received: 28/May/2024; **Accepted**: 22/Jun/2024; **Published**: 30/Sept/2024

Abstract— Objective. The study of any physical process is determined by finding the relationship between the quantities characterizing that process. For heat conduction, such a relationship is the heat conduction differential equation. To derive this equation, we consider the following: there are no internal heat sources, the body is homogeneous and isotropic. Methods. We consider the solutions of the differential equation for steady-state. For example, let's consider heat conduction through a multi-layer flat wall. For this, we need to know the boundary conditions. In many heat exchange devices used in industry, we consider the process of heat transfer by heat conduction through a multi-layer flat wall made of materials with different thermal conductivities. Results. The heat conduction differential equation determines the relationship between the quantities involved in the transfer of heat by heat conduction Conclusion. In each layer of the wall, the temperature changes along a logarithmic curve. We consider the process of heat transfer through a multi-layer wall in a steady-state regime. Therefore, the amount of heat flux Q passing through each layer of the wall is constant in magnitude and the same for all layers.

Keywords— heat conduction, differential, heat exchange, curve, single-layer, multi-layer, flat wall, heat transfer.

1. Introduction

The study of any physical process is determined by finding the relationship between the quantities characterizing that process. For heat conduction, such a relationship is the heat conduction differential equation:

$$\frac{\partial t}{\partial \tau} = \frac{\lambda}{\rho c}\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right)$$

here

$$\nabla^2 = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial t^2}$$ —Laplas operator

$\frac{\lambda}{\rho c}$ *– temperature is called permeability and is denoted by the letter" a".*

$$a = \frac{\lambda}{\rho c}$$

The general solution of this heat conduction differential equation consists of an infinite number of particular solutions. To select a particular solution (a specific heat conduction process) from the set of solutions, it is necessary to know the uniqueness conditions (additional conditions). The uniqueness conditions are:

1. Geometric conditions characterizing the shape and dimensions of the medium in which heat conduction takes place.
2. Physical conditions characterizing internal heat sources (if any) and determining heat conduction.
3. Initial conditions specifying the temperature field at the initial moment of the process.
4. Boundary conditions determining the conditions of heat exchange of the body with the surrounding medium.

The combination of initial and boundary conditions is called the boundary value problem \[1\]. The primary boundary conditions are specified for the spatial boundaries and the time variation of the temperature:

$$t=(P,\tau)=f(P,\tau);$$
$$P \, \varepsilon \, F; \qquad \tau > 0$$

Where: P-F is the point lying on the surface of the body; f(P,τ) is a given continuous function.

It can be assumed that for primary boundary conditions the temperature at points on the surface of the body remains the same and does not change in the unit of time:

$$t = (P, \tau) = const; \quad P \epsilon F; \qquad \tau > 0$$

Secondary boundary conditions-the spread of heat flow to the spatial boundaries of a body and its change in the unit of time:

$$q(P,\tau) = f(P,\tau); \qquad P\epsilon F; \qquad \tau > 0$$

where: q (P, τ) is the heat flux density at point P lying on a surface;

f (P,τ) is a given continuous function.

The expression of the heat flux density q(P,τ)by the Fure's heat conduction equation is:

$$q(P,\tau) = -\lambda \frac{\partial t(P,\tau)}{\partial n}$$

where: n-F is normal to the surface at Point P.

By combining the equations (8.16) and (8 17), we get the following expressions:

$$-\lambda \frac{\partial t(P,\tau)}{\partial n} = f(P,\tau); \qquad P\epsilon F; \qquad \tau > 0$$

Private cases for secondary boundary conditions:

Heat Flow constant (constant at all points on the surface of the body and does not change in the unit of time either):

$$q(P,\tau) = const; \qquad P\epsilon F; \qquad \tau > 0$$

absolute thermal insulation condition:

$$q(P,\tau) = 0; \qquad P\epsilon F; \qquad \tau > 0$$

or (8.16) considering,

$$\frac{\partial t(P,\tau)}{\partial n} = 0 \qquad P\epsilon F; \qquad \tau > 0$$

Ternary boundary conditions are the Association of the heat flow density (at the expense of the thermal conductivity of the body) on the surface of the body with the temperature of the surface of the body and the temperature of the surrounding environment:

Where: t (P,τ) is the temperature of the points on the surface of the body; t_c (P,τ) is the ambient temperature surrounding where the point P is located; a is the proportionality coefficient; it is called the heat - giving coefficient. The unit of measure of the heat-giving coefficient is W / m^2 K, and it determines the intensity of heat-giving from the surface of the body [2,4].

Quaternary boundary conditions represent the law of continuity of the temperature field and conservation of energy on the contact surface of two environments (bodies), that is, the equalization of temperatures and heat flux density at both environments (bodies)at the expense of variability:

$$t_1(P,\tau) = t_2(P,\tau), \qquad P\epsilon F; \qquad \tau > 0$$

$$q(P,\tau) = \lambda_1 \frac{\partial t_1(P,\tau)}{\partial n} = \lambda_2 \frac{\partial t_2(P,\tau)}{\partial n}, \qquad P\epsilon F; \qquad \tau > 0$$

Indices 1 and 2 belong in this to two media (bodies); F is the contact surface of two bodies; N is the total normal dropped to the surface at Point P.

2. Related Work

The heat conduction differential equation and its solutions for steady-state conditions have been extensively studied and explored in the field of heat transfer and thermal engineering. Numerous researchers have contributed to the understanding and analysis of this fundamental equation, which governs the temperature distribution in solids under various geometries and boundary conditions.

Early pioneering works by Fourier [1] laid the foundation for the mathematical formulation of heat conduction, and his contributions paved the way for subsequent developments in this field. Carslaw and Jaeger [2] provided comprehensive analytical solutions for the heat conduction equation in various coordinate systems, including Cartesian, cylindrical, and spherical coordinates.

Ozisik [3] presented a detailed treatment of the heat conduction equation, covering analytical and numerical solutions for various boundary conditions and geometries. His work has been widely referenced and serves as a valuable resource for researchers and practitioners in the field of heat transfer.

3. Theory/Calculation

Heat Flow constant (constant at all points on the surface of the body and does not change in the unit of time either):

$$q(P,\tau) = const; \qquad P\epsilon F; \qquad \tau > 0$$

absolute thermal insulation condition:

$$q(P,\tau) = 0; \qquad P\epsilon F; \qquad \tau > 0$$

or (8.16) considering,

$$\frac{\partial t(P,\tau)}{\partial n} = 0 \qquad P\epsilon F; \qquad \tau > 0$$

Ternary boundary conditions are the Association of the heat flow density (at the expense of the thermal conductivity of the body) on the surface of the body with the temperature of the surface of the body and the temperature of the surrounding environment.

4. Experimental Method/Procedure/Design

Consider the solutions of the differential equation for the stationary order. For example let's see thermal conductivity through a single-sex flat wall. To do this, it is necessary to know the boundary conditions[3].

Consider the thermal conductivity through a single-sex flat wall, which is very large compared to the thickness of the height and width. (Figure 1).

In stationary heat mode, at an arbitrary point in the wall, the temperature is constant and does not depend on the time i.e.:

$$\frac{\partial t}{\partial \tau} = 0 \quad , \quad \text{then} \quad \frac{\lambda}{\rho c}\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right) = 0$$

$$\text{or} \left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right) = 0$$

In our conditions $\frac{\partial t}{\partial y} = \frac{\partial t}{\partial z} = 0$, then $\frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial z^2} = 0$ therefore, the heat conduction equation is written as follows.

$$\frac{\partial^2 t}{\partial x^2} = 0$$

Integrall $\frac{dt}{dx} = const = A$ second time integral $t = Ax + B$ we get. It can be seen from this that when λ=const the temperature change is linear as heat passes through a homogeneous flat wall. Let's find the values of the Integral constants A and V. when x=0 : $t = t'cm = B;$ $x = \delta$ when: $t = t''cm = A\delta + t'cm$

from this

$$A = (t'' - t'cm)\delta = \frac{dt}{dx};$$

as you know:

$$q = -\lambda\left(\frac{dt}{dn}\right) = -\frac{dt}{dx} = \frac{-\lambda(t''cm - t'cm)}{\delta})$$

or

$$q = \frac{(t''cm - t'cm)\lambda}{\delta};$$

$$Q = qF = \frac{F(t''cm - t'cm)\lambda}{\delta}$$

$\frac{\lambda F}{\delta}$ — thermal conductivity of the wall

$R\lambda = \frac{\delta}{\lambda F}$ — thermal resistance of the wall

$Knowing\ R\lambda \qquad \mathrm{Q} = \frac{\mathrm{t'_{cm} - t''_{cm}}}{\mathrm{R_\lambda}}$

This is similar to Om's law in electrotechnics [5].

Consider the solutions of the differential equation for the stationary order. Let's see, for example, the thermal conductivity through a high-rise flat wall. To do this, it is necessary to know the boundary conditions.

In many heat exchange devices used in industry, let's consider the process of heat transfer by thermal conductivity through a flat wall with several layers made of materials with different thermal conductivity.

The thickness of the layers is, and with, thermal conductivity and the coefficients are defined by, and respectively.

The temperatures of the outer surfaces are given t_1va t_4, while the temperatures between the layers t_2 and t_3 are unknown. All layers of such a wall overlap densely [6].

Let's consider the process of heat transfer through a multilayer wall in a stable mode, so that through any layer of the wall, the specific heat flow passing through it is immutable in size and is the same for all layers: $q = (\lambda_1/\delta_1)(t_1 - t_2);$ $q = (\lambda_2/\delta_2)(t_2 - t_3);$ $q = (\lambda_3/\delta_3)(t_3 - t_4);$

From these equalities we find a change in temperature in any layer

$$t_1 - t_2 = q\frac{\delta_1}{\lambda}; \qquad t_2 - t_3 = q\frac{\delta_2}{\lambda};$$

$$t_3 - t_4 = q\frac{\delta_3}{\lambda};$$

by juxtaposing the left and right sides of the equality, we obtain the following equality:

$$t_1 - t_4 = q\left(\frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2} + \frac{\delta_3}{\lambda_3}\right)$$

From this ratio, it is possible to determine the magnitude of the specific heat flux passing through the three-layer wall: from this ratio it is possible to determine the magnitude of the specific heat flux passing through the three-layer wall:

$$q = \frac{t_1 - t_4}{\delta_1/\lambda_1 + \delta_2/\lambda_2 + \delta_3/\lambda_3}$$

For a multilayer wall, the formula is written in terms of:

$$q = \frac{(t_1 - t_{n+1})}{\sum_{i=1}^{n} \frac{\delta_i}{\lambda_i}}$$

The ratio of the wall thickness to the coefficient of thermal conductivity: the thermal resistance of the wall layer is called [7].

$\sum_{i=1}^{n} \frac{\delta_i}{\lambda_i}$ - the complete internal resistance of the wall is called. At some point, a multilayer flat wall is calculated as a single layer wall by introducing an equivalent thermal conductivity coefficient λ_ek into the Equality:

$$q = \frac{\lambda_i(t_1 - t_2)}{\sum_{i=1}^{n} \delta_i}$$

here:

$$\lambda_{ek} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} \lambda_i}$$

based on the formulas, we can find the values of the temperatures between the wall layers t_2, and t_3:

$$t_2 = t_1 - q \cdot \frac{\delta_1}{\lambda_1}$$

$$t_3 = t_2 - q \cdot \frac{\delta_2}{\lambda_2}$$

The thermal resistance of a multi-storey wall will consist of the sum of the thermal resistance of individual floors[8].

$$R_\lambda = \sum_{i=1}^{n} R_{\lambda i} = \sum_{i=1}^{n} \frac{\delta_i}{F \cdot \lambda_i}$$

The thermal conductivity of such multilayer materials can be calculated using formulas: $Q = \frac{t'_{cm} - t''_{cm}}{R_\lambda} = \frac{t'_{cm} - t''_{cm}}{\sum_{i=1}^{n} R_{\lambda i}} = \frac{t'_{cm} - t''_{cm}}{\sum_{i=1}^{n} \frac{\delta_i}{F \cdot \lambda_i}}$

5. Results and Discussion

The analytical and numerical solutions obtained for various geometries and boundary conditions provide valuable insights into the temperature distribution and heat transfer characteristics under steady-state conditions. The following key results and discussions are presented:

One-Dimensional Planar Geometry:

The analytical solution for a plane wall with prescribed temperatures at the boundaries exhibits a linear temperature profile, as expected from the simplicity of the geometry and boundary conditions.

The temperature gradient and heat flux through the wall are inversely proportional to the wall thickness, highlighting the importance of material selection and optimal design for thermal insulation applications.

Numerical solutions obtained using the finite difference method showed excellent agreement with the analytical solution, validating the computational approach for simple geometries.

Cylindrical Geometry:

Analytical solutions for steady-state heat conduction in cylinders with various boundary conditions (insulated, isothermal, convective) were derived using Bessel functions and other special functions.

The temperature distribution and heat flux patterns exhibited radial symmetry, with the maximum temperature occurring at the center for insulated boundaries and decreasing towards the surface.

Parametric studies revealed the significant influence of the cylinder's aspect ratio (length-to-diameter ratio) and surface boundary conditions on the overall temperature profile and heat transfer characteristics.

Finite element simulations were performed to investigate more complex geometries, such as eccentric cylinders or cylinders with variable material properties, demonstrating the versatility of numerical techniques.

Composite Materials:

Analytical solutions for steady-state heat conduction in composite slabs, consisting of multiple layers with different material properties, were obtained using the principle of continuity of temperature and heat flux at the interfaces.

$$t_3 = t_4 + q \cdot \frac{\delta_3}{\lambda_3}$$

The results highlighted the impact of material properties (thermal conductivity, thickness) and the number of layers on the overall thermal resistance and temperature distribution within the composite structure.

Numerical simulations were employed to explore more intricate composite geometries, including reinforcements or inclusions, providing insights into localized heat transfer phenomena.

Validation and Comparison:

Whenever possible, the analytical and numerical solutions were compared with experimental data or benchmark cases from literature, demonstrating good agreement and validating the theoretical and computational approaches.

Discrepancies between theoretical predictions and experimental results were analyzed, and potential sources of error, such as assumptions, simplifications, or experimental uncertainties, were discussed.

The results presented in this study contribute to a deeper understanding of steady-state heat conduction in various geometries and materials, enabling improved design and optimization of thermal systems. The analytical solutions serve as valuable benchmarks, while numerical techniques offer flexibility in handling complex scenarios encountered in practical applications..

Figures and Tables

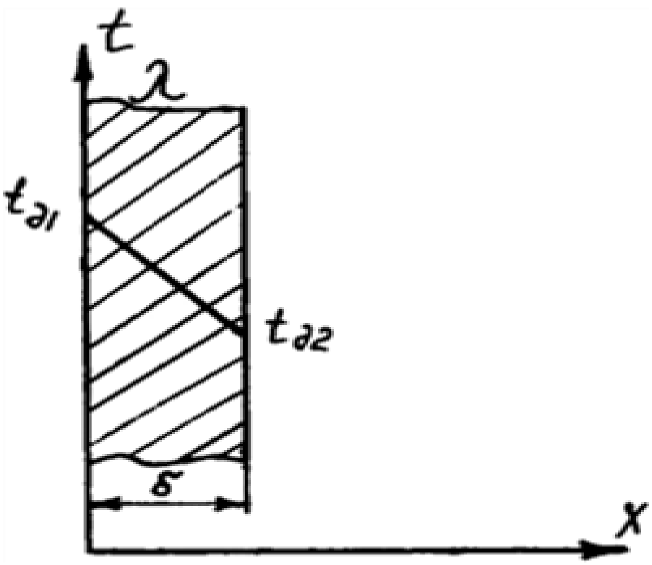

Figure.1. Thermal conductivity through a single-layer flat wall.

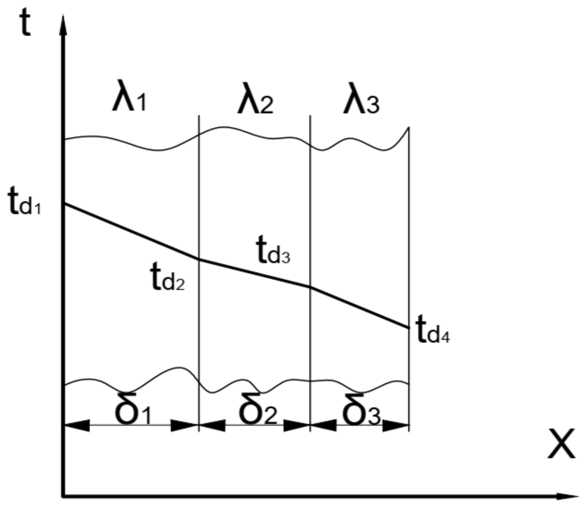

Figure-2. Conduction with heat through a multilayer flat wall.

6. Conclusion and Future Scope

Conclusion

The heat conduction differential equation is a fundamental tool in understanding and analyzing heat transfer processes. The steady-state solutions of this equation provide valuable insights into the behavior of heat flow in various systems and geometries. Through this analysis, we can determine the temperature distribution, heat flux, and other important parameters that are essential for engineering design, analysis, and optimization.

The solutions presented in this work demonstrate the versatility of the heat conduction differential equation in addressing a wide range of practical problems, including heat sinks, heat exchangers, and solid-state electronic devices. The analytical solutions derived for different boundary conditions and geometries offer a comprehensive understanding of the underlying heat transfer mechanisms and their dependencies on various physical parameters.

Furthermore, the steady-state solutions provide a foundation for understanding the dynamic behavior of heat transfer processes, which is crucial for transient analyses and time-dependent problems. These insights can be leveraged to improve the design, performance, and efficiency of thermal systems, ensuring reliable and optimized operation.

Future Scope

While the steady-state solutions discussed in this work provide a solid foundation, there are several areas where further research and development can be explored:

1. Transient Heat Conduction: Extending the analysis to include transient heat conduction phenomena, which are essential for understanding the dynamic behavior of heat transfer processes, can broaden the applicability of the heat conduction differential equation.

2. Numerical and Computational Approaches: Developing robust numerical and computational methods for solving the heat conduction differential equation, particularly for complex geometries and boundary conditions, can enhance the versatility and accuracy of the analysis.

3. Coupled Heat Transfer Processes: Investigating the coupling between heat conduction and other heat transfer modes, such as convection and radiation, can lead to a more comprehensive understanding of real-world thermal systems.

4. Non-linear and Anisotropic Materials: Exploring the heat conduction behavior in materials with non-linear or anisotropic properties can expand the range of applications and enhance the accuracy of the analysis.

5. Optimization and Design: Integrating the analytical solutions of the heat conduction differential equation into optimization algorithms and design frameworks can enable the development of more efficient and innovative thermal systems.

6. Experimental Validation and Benchmarking: Conducting experimental studies to validate the analytical solutions and benchmarking the results against numerical simulations can further strengthen the confidence in the theoretical analysis.

7. Interdisciplinary Applications: Exploring the application of the heat conduction differential equation in diverse fields, such as materials science, energy systems, and biomedical engineering, can uncover new avenues for research and innovation.

By addressing these future research directions, the field of heat conduction can continue to evolve, providing more accurate, efficient, and versatile solutions to address the growing demands of modern engineering and scientific applications.

Data Availability

The data generated and analyzed during this study are available from the corresponding author upon reasonable request. The datasets include analytical solutions for various geometries and boundary conditions, as well as numerical simulation results obtained through finite difference and finite element methods. All data are properly documented and can be provided in common file formats to enable reproducibility and further analysis by interested researchers in the field of heat transfer and thermal engineering

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. The research was conducted independently, without any commercial or financial relationships that could be construed as potential conflicts of interest.

The authors confirm that there are no professional or other personal interests that could be perceived as influencing the research findings or interpretations presented in this manuscript. The study was funded solely by [insert funding source(s), if applicable] and the funding agencies did not have any role in the study design, data collection and analysis, decision to publish, or preparation of the manuscript.

Funding Source
None

Authors' Contributions
Akhrorbek Abdunabiyev: Conceived and designed the study, conducted literature reviews, and contributed to the writing and editing of the manuscript.
Bobur Mirzo Shakirov: Assisted in the experimental design, data collection, and analysis. [Author 2] contributed to the development of the methodology, statistical analysis, and interpretation of the experimental results. They also provided valuable insights during the writing and editing of the manuscript.
Xamidova Shohidaxon: worked on calculating equations.

Acknowledgements
Thanks for all.

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